LONGMANS' ALGEBRA

for

Higher Secondary Schools

Revised by

T. RAJAGOPALAN,

Principal,

M. E. A. Higher Secondary School,

Pusa Road, Karolbagh, New Delhi

and

R. P. GOEL,

Head of the Department of Mathematics,

M. E. A. Higher Secondary School,

Lodi Estate, New Delhi

ORIENT LONGMANS

BOMBAY CALCUTTA MADRAS NEW DELHI

ORIENT LONGMANS LIMITED
Regd. Office: 3/5, ASAF ALI ROAD, NEW DELHI 1
NICOL ROAD, BALLARD ESTATE, BOMBAY 1
17, CHITTARANJAN AVENUE, CALCUTTA 13
36A, MOUNT ROAD, MADRAS 2
1/24, ASAF ALI ROAD, NEW DELHI 1

LONGMANS, GREEN & CO. LTD.
48, GROSVENOR STREET, LONDON, W.1

Associated companies, branches and representatives throughout the world

First Edition 1933
Second Edition 1935
Third Edition 1936
Fourth Edition 1938
New Impression 1940
Fifth Edition 1944
New Impression 1946, 1948 (twice), 1950 (twice), 1955, 1958, 1959, 1960, 1962, 1965 (twice)
Revised Edition 1966

© *Orient Longmans Limited, 1966*

Price Rs. 9·00

PRINTED IN INDIA
AT HOOGHLY PRINTING COMPANY LIMITED
41, CHOWRINGHEE ROAD, CALCUTTA 16

PREFACE

Longmans' Algebra for Higher Secondary Schools is a revised edition of what has hitherto been popularly known as "Longmans' Matriculation Algebra". The need for revision arose on account of the changes that have been introduced in the syllabuses.

While the text of the original has been retained in most parts of the book, it has undergone a thorough change for the better. The unwieldy exercises have been split into workable units within the class periods besides the home task for practice.

The Remainder Theorem has been assigned its proper place in the early stages of the book. The exercises have been graded in order of importance and those involving cumbersome work have been eliminated.

Chapters on the Theory of Quadratic Equations and the three Progressions have been introduced so as to make the book adaptable to the Lower Mathematics and Geometrical Drawing Group. Objective-type questions and the selected questions from public examination papers are new innovations.

By virtue of the improvements, it is hoped that this volume will be found more useful and will aim greater appreciation from its users than the original.

—Author

CONTENTS

PART I

Chapter		*Page*
I.	**Literal Arithmetic** ...	1
II.	**Directed Numbers, Fundamental Laws, First Four Rules** ...	21
III.	**Simple Equations** ...	55
IV.	**Formulae** ...	93
V.	**Factors** ...	111
VI.	**Remainder Theorem, Applications** ...	129
VII.	**H.C.F. and L.C.M.** ...	144
VIII.	**Fractions** ...	154
IX.	**Simultaneous Equations of the First Degrees and Problems** ...	167
X.	**Linear Graphs** ...	190
XI.	**Statistical Graphs** ...	208

PART II

XII.	**Indices** ...	218
XIII.	**Ratio** ...	231
XIV.	**Proportion** ...	238
XV.	**Fractional Equations and with Literal Co-efficients Equations** ...	256
XVI.	**Simultaneous Equations — Three Unknowns** ...	275
XVII.	**Involution and Evolution** ...	295
XVIII.	**Surds** ...	310
XIX.	**Elimination** ...	327

Chapter		Page
XX.	**Integral Function, Detached Co-efficients, Indeterminate Co-efficients**	... 339
XXI.	**Homogeneity, Symmetry, Cyclic Order**	... 347
XXII.	**Conditional Identities**	... 361
XXIII.	**Fractions with Denominators in Cyclic Order**	... 366
XXIV.	**Miscellaneous Factors**	... 372

PART III

XXV.	**Quadratic Equations and Problems**	... 389
XXVI.	**Imaginary Quantities**	... 398
XXVII.	**Theory of Quadratic Equations**	... 402
XXVIII.	**Arithmetical Progression**	... 413
XXIX.	**Geometrical Progression**	... 427
XXX.	**Harmonical Progression**	... 441

PART I

CHAPTER I

LITERAL ARITHMETIC

1. Algebra is a method of calculation in which letters such as $a, b, c, \ldots\ldots, x, y, z$ are used in addition to the numbers 0, 1, 2, ..., 9 ..... So Algebra can be called **Literal Arithmetic.** The letters used in Algebra may be assigned any values. When we speak of x pencils, we may be thinking of 5 or 7 or in fact *any number* of pencils. The letters do *not* stand for quantities. They stand only for *numbers*. But one letter can have only one particular value in the same problem.

The letters and numbers are connected by the signs $+, -, \times, \div, =, >, <$, all of which have the *same* meaning as in Arithmetic.

In Literal Arithmetic the symbol $\sim$ is used as the sign of *difference*. $a \sim b$ stands for $a-b$ if a is greater than b, or $b-a$ if b is greater than a.

In Literal Arithmetic, the sign '$\times$' is often omitted. $5 \times a$ is written as $5a$; $4 \times a \times b$ is written as $4ab$. But in Arithmetic the sign '$\times$' cannot be omitted without changing the meaning. $3 \times 4 = 12$ but $34 = 3 \times 10 + 4$.

NOTE : The products $5 \times a$, $a \times b \times 4$ are written as $5a$ and $4ab$ and not $a5$ or $ab4$. The number is always written first.

In $5a$ and $4ab$, the numbers 5 and 4 are called the **numerical coefficients** of a and ab respectively. When the numerical coefficient is 1, it is not expressed. ab really means $1 \times ab$, the numerical coefficient being 1.

Just as in Arithmetic $4 \times 0 = 0 \times 4 = 0$,
in Literal Arithmetic also, $0 \times a = a \times 0 = 0$.

Just as in Arithmetic 5×5 is written as 5^2, in Algebra,

$a \times a$ is written as a^2 (Read 'a squared')
$b \times b \times b$ is written as b^3 (Read 'b cubed')
$c \times c \times c \times c$ is written as c^4 (Read 'c to the fourth')
and so on.

If a factor is repeated a certain number of times, the product is called the *power* of that factor and the small number placed above, on the right hand top corner, is called the *index*. It denotes the number of times a factor is repeated in that product. In 7^2, 6^4, a^5, the indices are 2, 4, 5 respectively.

2. Algebraic Language or Symbolic Writing.

(*i*) If 5 is added to 7 or subtracted from 7, we write $7+5$ or $7-5$. Likewise, if y is added to x, or subtracted from x, we write $x+y$ or $x-y$.

(*ii*) The statement, "three times x added to 7 equals 25" is written as $3x+7=25$.

(*iii*) The statement, "If twice a quantity x is increased by 9 and the result is divided by 3, the result is 15" is written as $\frac{2x+9}{3}=15$.

EXERCISE I.

1. Write in symbols:

(i) 5 added to x.
(ii) x increased by 6.
(iii) x increased by y.
(iv) 5 added to x, and y added to the result.

2. Express in words:

(i) $x+7$.
(ii) $p+q$.
(iii) $x+11+y$.
(iv) $a+b+c$.

3. Write in symbols:

(i) 9 subtracted from 15.
(ii) 9 subtracted from m.
(iii) 11 diminished by x.
(iv) a taken from b.
(v) The difference between m and n.
(vi) The sum of 11 and b diminished by c.
(vii) y subtracted from x and z added to the result.

4. Express in words:

(i) $p-7$.
(ii) $m \sim n$.
(iii) $l+m-n$.
(iv) $p-q+r$.
(v) $x-y-z$.
(vi) $a+b-c-d$.

5. (i) How many centimetres are there in 6 metres?
(ii) ,, ,, paise ,, ,, in x rupees?
(iii) ,, ,, minutes ,, ,, in t hours?

(iv) How many metres are there in l kilometres?
(v) ,, ,, milligrams ,, ,, in w grams?

6. If 1 horse costs Rs. r, what will n horses cost?

7. If a person travels at the rate of m kilometres an hour, how far will he go in t hours?

8. Write in symbols:
(i) 5 multiplied by x. (ii) m multiplied by 5.
(iii) The product of m and n.
(iv) 7 multiplied by a and the product by b.

9. Write in symbols:
(i) 15 divided by 7. (ii) 15 divided by x.
(iii) m divided by 7. (iv) p divided by q.
(v) The sum of p and q divided by m.
(vi) The difference between a and b divided by c.

10. If the price of 7 chairs is Rs. $28m$, what is the price of 1 chair? the price of x chairs?

11. Write in symbols:
(i) Five added to 6 is equal to 11.
(ii) Four and five make nine.
(iii) Two and two make four.
(iv) Four times three equals three times four.
(v) Twenty divided by five is equal to eighty divided by twenty.
(vi) Half of p is equal to q.
(vii) Since five times x is equal to forty, therefore x is equal to eight.
(viii) Since one-third of y is equal to fifteen, therefore y is equal to forty-five.
(ix) Since seven times m is greater than thirty-five, therefore m is greater than five.
(x) Since six times p is less than eighteen, therefore p is less than three.

12. Express in words:
(i) $x+7=16$. (ii) $x-11=5$.
(iii) $5x=30$. (iv) $\frac{2x}{9}=6$.

(v) $2a+3b = 5$. (vi) $\frac{2a+3b}{7} = 10$.

(vii) $\frac{3a-2b}{5} = 12$. (viii) $\because\ 5x-3 = 12,\ \therefore\ x = 3$.

(ix) $\because\ 7x-2 > 12,\ \therefore\ x > 2$.

(x) $\because\ 2x+5 < 15,\ \therefore\ x < 5$.

3. Definitions, Fundamental Operation.

An **expression** is a collection of symbols—that is, of letters, figures and signs. Thus $3a$, $7a^2b$, $4a+5b$, $3a^2-2ab-b^2$ are all expressions.

The parts of an expression beginning with one + or − sign upto the next + or − sign is called a **term.**

$3a$, $7a^2b$ are *expressions of one term* or **monomials.**

$4a+5b$ is an *expression of two terms* or a **binomial.**

$3a^2-2ab-b^2$ is an *expression of three terms* or a **trinomial.**

Terms having the *same* letters are called **like terms** and terms having *different* letters or *different* powers of the same letters are called **unlike terms.**

Just as $3\times4 = 4\times3$, $a\times b = b\times a$; ab and ba are like terms. Similarly, abc, bca, cab, bac are all like terms. But a^2b and ab^2 are unlike terms.

An expression containing only *one* term is called a **simple expression.** If it contains more than one term, it is a **compound expression.**

If the values of the letters are known, the value of an expression can be found by *substitution.*

Example 1. Distinguish between $3x$ and x^3.
What is the difference if $x = 2$ or 3 ?

Solution : $3x = 3\times x = x+x+x$ (x written 3 times and added together).

$x^3 = x\times x\times x$ (x written 3 times and multiplied together.)

When $x = 2$, $3x = 3\times2 = 6$
$x^3 = 2\times2\times2 = 8$; $\therefore$ difference $= 2$.

When $x = 3$, $3x = 3\times3 = 9$
$x^3 = 3\times3\times3 = 27$; $\therefore$ difference $= 18$.

Example 2. If $a = 4$, $b = 3$ and $c = 2$, find the value of:

(i) $5ab$. (ii) a^2b^3. (iii) $3a^b$. (iv) $4a^3 - 3b + 5^c$.

Solution: (i) $5ab = 5 \times a \times b = 5 \times 4 \times 3 = 60$.

(ii) $a^2b^3 = a \times a \times b \times b \times b = 4 \times 4 \times 3 \times 3 \times 3 = 432$.

(iii) $3a^b = 3 \times a^3 = 3 \times a \times a \times a = 3 \times 4 \times 4 \times 4 = 192$.

(iv) $4a^3 - 3b + 5^c = 4 \times a \times a \times a - 3 \times b + 5^2$
$= 4 \times 4 \times 4 \times 4 - 3 \times 3 + 5 \times 5$
$= 256 - 9 + 25 = 272$.

Example 3. Simplify $3a \times 5b$.

Solution $3a \times 5b = 3 \times a \times 5 \times b$
$= 3 \times 5 \times a \times b$
$= 15ab$.

Example 4. If $a = 3$, $b = \frac{1}{2}$, $c = 2\frac{1}{4}$, find the value of:

(i) $a - b + c$, (ii) $\dfrac{2ab - bc}{3ac}$.

Solution: (i) $a - b + c = 3 - \frac{1}{2} + 2\frac{1}{4} = 4\frac{3}{4}$.

(ii) $\dfrac{2ab - bc}{3ac} = \dfrac{2 \times 3 \times \frac{1}{2} - \frac{1}{2} \times \frac{9}{4}}{3 \times 3 \times \frac{9}{4}}$

$= \dfrac{3 - \frac{9}{8}}{8\frac{1}{4}} = \dfrac{\frac{15}{8}}{\frac{81}{4}}$

$= \frac{15}{8} \times \frac{4}{81} = \frac{5}{54}$.

EXERCISE 2.

1. Write briefly:

(i) $7 \times 7 \times 7 \times 7$. (ii) $a \times a \times a \times a \times a$.

(iii) $a \times p \times p \times p$. (iv) $1 \times a \times b \times b$.

(v) $\cdot 01 \times \cdot 01 \times \cdot 01$. (vi) $\dfrac{1}{a} \times \dfrac{1}{a} \times \dfrac{1}{a} \times \dfrac{1}{a}$

(vii) $3 \times m \times m + 4 \times n \times n$. (viii) $a \times a \times n \times n \times n$.

(ix) $pppp + ppp$. (x) $a \times x \times x \times x + b \times y \times y \times y$.

2. Read the following:

(i) p^3. (ii) x^4. (iii) r^6. (iv) n^{15}.

3. State the value of:

(i) 5^4. (ii) 7^2. (iii) 2^5. (iv) 3^4.

4. Distinguish between:

(i) $5a$ and a^5. (ii) p^4 and $4p$.

(iii) Three times 5 and 5 cubed.

5. Find the value of $x+y+z$

(i) when $x=3$, $y=2$, $z=4$;
(ii) when $x=\frac{2}{3}$, $y=\frac{3}{4}$, $z=\frac{1}{6}$;
(iii) when $x=\cdot24$, $y=\cdot16$, $z=\cdot15$;
(iv) when $x=0$, $y=5$. $z=13$.

6. Find the value of $a+b-c-d$

(i) when $a=2$, $b=7$, $c=1$, $d=3$;
(ii) when $a=\frac{1}{2}$, $b=\frac{3}{4}$, $c=\frac{1}{4}$, $d=\frac{1}{6}$;
(iii) when $a=\cdot15$, $b=\cdot8$, $c=\cdot02$, $d=\cdot46$.

7. Find the value of $5mn+3pq$

(i) when $m=6$, $n=2$, $p=4$, $q=1$;
(ii) when $m=\cdot7$, $n=\cdot4$, $p=\cdot5$, $q=\cdot24$;
(iii) when $m=\frac{1}{2}$, $n=\frac{7}{10}$, $p=\frac{5}{6}$, $p=\frac{2}{3}$;
(iv) when $m=7$, $n=1$, $p=0$. $q=2$.

8. Find the value of $\dfrac{3ab+4cd}{2ad}$,

(i) if $a=2$, $b=5$, $c=1$, $d=4$;
(ii) if $a=\cdot4$, $b=\cdot5$, $c=\cdot16$, $d=2\cdot5$;
(iii) if $a=\frac{1}{3}$, $b=\frac{3}{4}$, $c=\frac{2}{3}$, $d=\frac{1}{2}$.

9. If $a=3$, $b=5$, $c=2$, $x=1$, $y=4$, find the value of:

(i) $2a^3+3b^2$.
(ii) $5a^4-2b^2+4x^3$.
(iii) $11a^3+6b^3+2x^c+y^3$.
(iv) $2ab^3+3a^2c^2+4xy^3$.
(v) a^c+b^x.
(vi) $y^a+x^b+c^x$.

4. Laws of Indices.

(i) $a^3\times a^2=(a\times a\times a)\times(a\times a)$
$=a\times a\times a\times a\times a=a^5=a^{3+2}$.

(ii) $a^3\times a^4=(a\times a\times a)\times(a\times a\times a\times a)$
$=a\times a\times a\times a\times a\times a\times a$
$=a^7=a^{3+4}$.

Similarly, $a^4\times a^5=a^{4+5}=a^9$
$a^5\times a^7=a^{5+7}=a^{12}$.

or in general, $a^m\times a^n=a^{m+n}$ where m and n are ordinary arithmetical numbers.

This is the **First Law of Indices.**

(i) $a^5\div a^3=\dfrac{a^5}{a^3}=\dfrac{a\times a\times a\times a\times a}{a\times a\times a}=a^2=a^{5-3}$.

(ii) $a^7\div a^4=\dfrac{a^7}{a^4}=\dfrac{a\times a\times a\times a\times a\times a\times a}{a\times a\times a\times a}=a^3=a^{7-4}$.

Similarly $p^9 \div p^5 = p^{9-5}$ and $x^{11} \div x^6 = x^{11-6}$ or in general, $a^m \div a^n = a^{m-n}$ where m and n are ordinary numbers and $m > n$.

This is the **Second Law of Indices.**

(i) $(a^3)^2 = a^3 \times a^3 = a^6 = a^{3\times 2}$.
(ii) $(x^2)^3 = x^2 \times x^2 \times x^2 = x^6 = x^{2\times 3}$
(iii) $(p^2)^5 = p^2 \times p^2 \times p^2 \times p^2 \times p^2 = p^{10} = p^{2\times 5}$.

In general, we may say $(x^m)^n = x^{m\times n}$ where m and n are ordinary arithmetical numbers.

This is the **Third Law of Indices.**

Example 5. Multiply $5a^3b^2 \times 3a^2b^4$.

Solution : $5a^3b^2 \times 3a^2b^4 = 5 \times a^3 \times b^2 \times 3 \times a^2 \times b^4$
$= 5 \times 3 \times a^3 \times a^2 \times b^2 \times b^4$
$= 15a^{3+2}b^{2+4}$
$= 15a^5b^6$.

Example 6. Simplify $\frac{12xy}{4y}$.

Solution : $\frac{12xy}{4y} = \frac{12 \times x \times y}{4 \times y} = 3x$.

Example 7. Simplify $\frac{18x^5y^3}{6x^4y}$.

Solution : $\frac{18x^5y^3}{6x^4y} = \frac{18 \times x^5 \times y^3}{6 \times x^4 \times y}$
$= 3x^{5-4}y^{3-1}$
$= 3xy^2$.

EXERCISE 3.

1. Prove that :
(i) $3^4 \times 3^2 = 3^6$. (ii) $6^4 \times 6^7 = 6^{11}$.

2. Write down the values of the following :
(i) $5^3 \times 5^2 \times 5^2$. (ii) $7^2 \times 7^3 \times 7$.
(iii) $a^4 \times a^2 \times a^7$. (iv) $x^6 \times x^3 \times x^4$.

3. Find the product :
(i) $2ab \times 3cd$. (ii) $6x^3y \times 2xy^2$.
(iii) $a^2bc \times ab^2c$. (iv) $a^2b \times ab \times ac^2$.
(v) $3a^2b^3c^2 \times 5a^3b^4c$. (vi) $2p^3q \times pq^2 \times 3pq$.

4. Prove that:
(i) $5^7 \div 5^3 = 5^{7-3}$. (ii) $7^{12} \div 7^4 = 7^{12-4}$.

5. Write down the values of:
(i) $a^{17} \div a^{12}$. (ii) $p^{24} \div p^{15}$.
(iii) $m^{13} \div m^7$. (iv) $x^{24} \div x^{12}$.

6. Divide:
(i) $12x^4y^6$ by $6x^2y^2$. (ii) $10m^7n^5$ by $2m^4n^4$.
(iii) $8p^5q^4r^3$ by $4p^3q^2r^3$. (iv) $4m^6n^3p^5$ by $2m^2n^3p^4$.
(v) $15l^6m^2n^9$ by $5l^4m^2n^2$.

7. Write down the values of the following:
(i) $(2^3)^2$. (ii) $(3^2)^3$. (iii) $(a^m)^3$.
(iv) $(x^3)^m$. (v) $(p^4)^n$. (vi) $(y^3)^5$.

5. Power and Root.

When a number or letter is multiplied by itself, we get a **power** of that number or letter. The number or letter is then called the **root** of that power.

Corresponding to the square, cube, fourth power, etc., we have the square root, cube root, fourth root and so on.

The symbol $\sqrt{\ }$ stands for '*the root of*'. $\sqrt{x}$ is read as '*the square root of x*'.

9 is the square of 3. $\therefore$ 3 is the square root of 9.
It is written thus: $3 = \sqrt{9}$.
64 is the cube of 4. $\therefore$ 4 is the cube root of 64.
It is written thus: $4 = \sqrt[3]{64}$.
Similarly, a is the 4th root of a^4 or $a = \sqrt[4]{a^4}$.

Example 8. Prove that:
(i) $\sqrt{a^{12}} = a^6$. (ii) $\sqrt[3]{a^{12}} = a^4$. (iii) $\sqrt[4]{a^{12}} = a^3$.

Sol. (i) Since $a^{12} = a^6 \times a^6 = (a^6)^2$,

$$\therefore \quad \sqrt{a^{12}} = \sqrt{(a^6)^2} = a^6.$$

(ii) Since $a^{12} = a^4 \times a^4 \times a^4 = (a^4)^3$,

$$\therefore \quad \sqrt[3]{a^{12}} = \sqrt[3]{(a^4)} = a^4.$$

(iii) Since $a^{12} = a^3 \times a^3 \times a^3 \times a^3 = (a^3)^4$,

$$\therefore \quad \sqrt[4]{a^{12}} = \sqrt{(a^3)^4} = a^3.$$

EXERCISE 4.

1. Prove that:

(i) $\sqrt{10^4} = 10^2$. (ii) $\sqrt{x^{10}} = x^5$.

(iii) $\sqrt{9a^2b^2} = 3ab$. (iv) $\sqrt{36^3} = 6^3$.

(v) $\sqrt[3]{a^6} = a^2$. (vi) $\sqrt[3]{8x^3y^3} = 2xy$.

(vii) $\sqrt[4]{16} = 2$. (viii) $\sqrt[4]{81} = 3$.

2. Write down the square root of:

(i) 5^4. (ii) 4^6. (iii) x^{22}.

(iv) $4x^6$. (v) $9y^8$. (vi) $25a^2b^4$.

3. Write down the cube root of:

(i) $64a^9$. (ii) $216x^{12}$. (iii) $125y^{27}$.

4. Write down the fourth root of:

(i) $16a^8$. (ii) $81b^{12}$. (iii) $256c^{20}$.

6. Addition and Subtraction.

In Arithmetic,

8 apples + 5 apples + 7 apples = (8+5+7) apples
= 20 apples.

In Literal Arithmetic, $8a+5a+7a = (8+5+7)\,a = 20a$

$8b-5b = (8-5)\,b = 3b$.

But, if $5b$ is added to $8a$, we get $8a+5b$. This *cannot* be simplified.

Example 9. Add $3xy$, $4xy$, $2xy$.

Solution: $3xy+4xy+2xy = (3+4+2)xy = 9xy$.

Example 10. Simplify $3a+4a+b+5b$.

Solution: Since $3a+4a = 7a$ and $b+5b = 6b$,

$\therefore \quad 3a+4a+b+5b = 7a+6b$.

Example 11. Simplify $5x-2x+3y-y+1$.

Solution: Taking the like terms, we have

$5x-2x = 3x$ and $3y-y = 2y$,

$\therefore \quad 5x-2x+3y-y+1 = 3x+2y+1$.

Example 12. Simplify $4xy+3x+2y-x-y$.
Solution: Taking the like terms together, we have
$$4xy+3x+2y-x-y = 4xy+\underline{3x-x}+\underline{2y-y} = 4xy+2x+y.$$

Example 13. Simplify $5a^2+a-7+3a-2a^2+6$.
Solution: Taking the like terms together, we have
$$5a^2+a-7+3a-2a^2+6 = \underline{5a^2-2a^2}+\underline{a+3a}-\underline{7+6}$$
$$= 3a^2+4a-1.$$

EXERCISE 5(a).

Write down the value of:

1. $4a^2+a^2+6a^2$.
2. $3ab+4ab+2ab$.
3. $p+p+p+p+$...to 15 terms
4. $9xy+7xy+12xy$.
5. $13ab-7ab$.
6. $17abc-12abc$.
7. $9a^2-4a^2$.
8. $15x^2-11x^2$.

Simplify:

9. $x+x+y+y+y$.
10. $a+b+a+b$.
11. $x+y+x+y+x$.
12. $x+5+x+2+x$.
13. $5x+7y-2x+4$.
14. $a+b-c+a-b+c$.
15. $5a+b-2c+b-2a$.
16. $4xy+3xz-2xy$.
17. a^2+5a^2-3a.
18. $x^3+2x^2+4x-x^2$.
19. $p^2-3p+2+p^2+4p+1$.
20. $7x^3+2x^2+2x+4-5x^2-4x$.

7. Arrangement of the Terms of an Expression.

An expression like $3x^5+2x+7x^3-x^2-5x^4+1$ cannot be simplified, for, all its terms are unlike. But it is useful to arrange the terms either (*i*) in *descending* powers of x or (*ii*) in *ascending* powers of x.

In descending powers of x, it becomes $3x^5-5x^4+7x^3-x^2+2x+1$ and in ascending powers of x, it becomes $1+2x-x^2+7x^3-5x^4+3x^5$.

EXERCISE 5 (b).

1. Arrange $4+5x^2+x^3+7x^4+3x$ in (i) descending powers of x, (ii) ascending powers of x.

2. Arrange $a^2+a^7+5a^3+1+4a^6$ in (i) descending powers of a, (ii) ascending powers of a.

3. Simplify $x^4+3x^2+4x+5-x+2x^3-x^2$ and arrange the answer in ascending powers of x.

4. Simplify $8+x^2+3x^4+2x^2+7x-4x^3+5x^2-3x+1$ and arrange the answer in descending powers of x.

5. The number 435 can be written $4\times10^2+3\times10+5$. Write similarly 374 and in the answer substitute x for 10 and x^2 for 10^2 or 100.

Can you simplify $3x^2+7x+4$?

6. The number 3048 can be written $3\times10^3+4\times10+8$. Write similarly 5704 and in the answer substitute x^2 for 10^2 and x^3 for 10^3.

Can you simplify $5x^3+7x^2+4$?

8. Highest Common Factor, Lowest Common Multiple, Fractions.

The term 'factor' and 'multiple' are already known in Arithmetic. 2, 3, 7, etc. are contained in 84 an *exact* number of times. 2, 3, 7, etc. are factors of 84. They divide 84 exactly *i.e.* leaving *no* remainder. 84 is a multiple of 2, 3, 7, 12, etc.

Of the two expressions $6a^3b^2$, $8a^2b^3$, there are several common factors, viz., 2, $2a$, $2a^2$, $2b$, $2b^2$, $2ab$ and $2a^2b^2$. But $2a^2b^2$ is the highest of all these factors in power, and contains in it every other common factor.

$2a^2b^2$ is called the **Highest Common Factor** (H. C. F.) of $6a^3b^2$ and $8a^2b^3$.

The **Lowest Common Multiple** (L. C. M.) of $6a^3b^2$ and $8a^2b^3$ is the lowest expression which is exactly divisible by $6a^3b^2$ as well as $8a^2b^3$; therefore it must contain the L.C.M. of 6 and 8, *i.e.*, 24 as a factor and also it must be divisible by a^3b^2 and a^2b^3, *i.e.*, it must contain a^3b^3 as a factor, for it is the lowest in power of all the common multiples like a^3b^4, a^4b^3, a^4b^4 &c. Hence $24a^3b^3$ is the L.C.M. of the two given expressions.

Example 14. Find (i) the H. C. F. and (ii) the L. C. M. of $9x^2y^3$ and $12xy^2z$.

Solution : (i) The H. C. F. of 9 and 12 is 3.

The H. C. F. of x^2 and x is x.
The H. C. F. of y^3 and y^2 is y^2.
z is not a common letter. It is ignored.
$\therefore$ The H. C. F. of $9x^2y^3$ and $12xy^2z$ is $3xy^2$.

(ii) The L. C. M. of 9 and 12 is 36.
The L. C. M. of x^2 and x is x^2.
The L. C. M. of y^3 and y^2 is y^3.
z is an extra letter and it is included.
$\therefore$ The L. C. M. of $9x^2y^3$ and $12xy^2z$ is $36x^2y^3z$.

NOTE: In H. C. F., the extra letter is dropped out. In L. C. M., the extra letter is included in the result.

Example 15. Simplify $\frac{6x^3y^2}{4xy^3z}$.

Solution: The H. C. F. of the numerator and the denominator is $2xy^2$. Divide the numerator and the denominator by $2xy^2$.

$$\therefore \quad \frac{6x^3y^2}{4xy^3z} = \frac{3x^2}{2yz}.$$

Example 16. Express $\frac{4x}{3y}$ as an equivalent fraction with denominator $6ay$.

Solution: If $6ay$ be divided by $3y$, the quotient is $2a$.

Multiplying the numerator and the denominator by $2a$, we have

$$\frac{4x}{3y} = \frac{2a \times 4x}{2a \times 3y} = \frac{8ax}{6ay}.$$

Example 17. Express $\frac{2a^2x^2}{3b^2y^2}$ as an equivalent fraction with numerator $12a^2x^2z^2$.

Solution: If $12a^2x^2z^2$ be divided by $2a^2x^2$, the quotient is $6z^2$.
Multiplying the numerator and the denominator by $6z^2$, we have

$$\frac{2a^2x^2}{3b^2y^2} = \frac{6z^2 \times 2a^2x^2}{6z^2 \times 3b^2y^2} = \frac{12a^2x^2z^2}{18b^2y^2z^2}.$$

Example 18. Reduce $\frac{5}{3x}$ and $\frac{7}{4x}$ to fractions having the least common denominator.

Solution: The L.C.M. of $3x$ and $4x = 12x$.

$$\therefore \quad \frac{5}{3x} = \frac{4 \times 5}{4 \times 3x} = \frac{20}{12x} \text{ and } \frac{7}{4x} = \frac{3 \times 7}{3 \times 4x} = \frac{21}{12x}.$$

Example 19. (i) Add $\frac{3}{4x}$ and $\frac{1}{10x}$.

(ii) Subtract $\frac{1}{10x}$ from $\frac{3}{4x}$.

Solution: The L.C.M. of the denominators is $20x$.

Expressing each fraction with denominator $20x$, we have

(*i*) $\frac{3}{4x}+\frac{1}{10x}=\frac{15}{20x}+\frac{2}{20x}$

$=\frac{15+2}{20x}$

$=\frac{17}{20x}$.

(*ii*) $\frac{3}{4x}-\frac{1}{10x}=\frac{15}{20x}-\frac{2}{20x}$

$=\frac{15-2}{20x}$

$=\frac{13}{20x}$.

EXERCISE 6.

1. Of the expressions $3a$, $6ab$, $4a^2$, $2ab$, $12a^3b^3$, $12a^2b^2$, $6a^2b$, $16a^4$, ab, $3a^2b$, $2ac$, $6a^2c$, $6abc^2$, which are factors of: (*i*) $6a^2$, (*ii*) factors of $3ab$, (*iii*) common factors of $6ab$ and $4a^2$, (*iv*) multiples of $4a^2$, (*v*) multiples of $3ab$ (*vi*) common multiples of $2a^2$ and $3ab$?

2. Find the H.C.F. and the L.C.M. of:

(i) $4a^2b^2$, $6a^3b^3$. (ii) $12a^3b$, $18ab^3$.

(iii) $12ab$, $3a^2b$. (iv) $39a^3b^4$, $52a^2b^5$.

(v) $4a^2b$, $6a^3b^3$, $3abc^2$. (vi) $6ab^2$, $8a^2b^2$, $12a^2bc$, $18abc^2$,

3. Find by factors the H.C.F. and the L.C.M. of:

(i) $2^3\times7^2$, $7^3\times2$. (ii) $3^2\times5^3$, $3^3\times5^2$.

4. Simplify:

(i) $\frac{6ab^3c^4}{10a^2b^2c^3}$. (ii) $\frac{15a^2x^2y}{5axy^2}$. (iii) $\frac{48a^2x^2y^5}{12a^2x^3y^4}$.

(iv) $\frac{13a^2b^2c^4}{39a^3b^3c^5}$. (v) $\frac{51x^5y^3z^2}{17x^4y^2z}$. (vi) $\frac{21a^7x^7y^3}{7a^8x^6y^5}$.

5. Fill in the blanks in the following :—

(i) $\frac{5x}{2y}=\frac{*}{14y^2}$. (ii) $\frac{4a^2b}{6ab^2}=\frac{*}{18a^2b^2}=\frac{*}{3b}$.

(iii) $a^3=\frac{*}{b^2}$. (iv) $\frac{1}{a}=\frac{b^2}{*}$. (v) $\frac{m^2}{a}=\frac{m^3n^3}{*}$.

6. Reduce the following to fractions having the least common denominator:

(i) $\frac{5}{6a}, \frac{4}{3x}, \frac{3c}{ax}$. (ii) $\frac{3}{4x}, \frac{x}{6y^2}, \frac{y}{3x^2}$.

(iii) $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$. (iv) $\frac{x^2y}{a^3}, \frac{2x^3}{3a^2b}, \frac{3y^3}{4ab^2}, \frac{4xy^2}{5b^3}$.

7. Simplify the following fractions:

(i) $\frac{2x}{3}+\frac{3x}{4}$. (ii) $\frac{3a}{2x}+\frac{b}{x}$. (iii) $\frac{1}{2x}+\frac{1}{3x}$.

(iv) $\frac{x}{y}+\frac{y}{x}$. (v) $1+\frac{x}{y}$. (vi) $\frac{1}{x}+3$.

(vii) $\frac{1}{x}-\frac{1}{3x}$. (viii) $\frac{a}{b}-\frac{2a}{3b}$. (ix) $1-\frac{a}{b}$.

(x) $1-\frac{1}{xy}$. (xi) $\frac{2}{xy}-\frac{xy}{z}$. (xii) $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.

(xiii) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$. (xiv) $\frac{2x}{3}-\frac{x}{4}+\frac{3x}{5}$. (xv) $\frac{a}{2x}+\frac{a}{3x}+\frac{a}{4x}$.

(xvi) $\frac{2a}{3x}+\frac{5b}{12y}+\frac{3c}{4z}$. (xvii) $\frac{x^2}{12}-\frac{3xy}{15}+\frac{2y^2}{18}$.

(xviii) $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$. (xix) $\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab}$.

8. Simplify the following:

(i) $\frac{a^2}{bc}\times\frac{b^2}{ac}$. (ii) $xz\div\frac{x}{z}$. (iii) $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{a}$.

(iv) $\frac{a^2b^2}{c}\times\frac{c^2}{ab}$. (v) $\frac{12bc}{35a^2}\times\frac{21ab^2}{30b^3}$. (vi) $\frac{36x^2y^2}{15z^3}\div\frac{27xy^2}{10z^2}$.

9. What fraction is $4m$ centimetres of m metres?

10. ABC is a straight line; its part AB measures x units and BC measures y units. What fraction is (i) AB of AC? (ii) AC of BC?

9. **Generalisation.**

(i) We know that
$$0\times 3 = 0$$
$$0\times 4 = 0$$
$$0\times 5 = 0$$
$$0\times \text{ any number } = 0;$$

∴ the general statement is $0 \times x = 0$ where x stands for any number.

(ii) We know that 2×1 is an even number.
2×2 ,, ,,
2×3 ,, ,,
2×4 ,, ,,
$2 \times$ any number is even ;

∴ the general form of an even number is $2x$.

(iii) We know that $5 \times 4 = 4 \times 5$
$6 \times 9 = 9 \times 6$
$12 \times 8 = 8 \times 12$.
The general form is : $a \times b = b \times a$.

(iv) In the case of rectangles
if length $= 10m$, and breadth $= 5m$,
perimeter $= 2\ (10+5)m$;
if length $= 15m$, and breadth $= 8m$,
perimeter $= 2\ (15+8)m$;
if length $= 22m$, and breadth $= 14m$,
perimeter $= 2\ (22+14)m$;
i.e., if length is l metres and breadth is b metres,
perimeter is $2\ (l+b)$ metres.
The generalised statement is $p = 2\ (l+b)$.

EXERCISE 7.

Generalise :

1. $4+5 = 5+4$
$7+3 = 3+7$
$9+4 = 4+9$.

2. $7^2 \times 7 = 7^3$
$9^2 \times 9 = 9^3$
$11^2 \times 11 = 11^3$.

3. $2 \times 1+1$ is an odd number
$2 \times 2+1$,, ,,
$2 \times 3+1$,, ,,

4. The sum of the first 2 odd numbers is 2^2
,, ,, 3 ,, ,, 3^2
,, ,, 4 ,, ,, 4^2

5. 4 times $6+6 = 5$ times 6
4 times $7+7 = 5$ times 7
4 times $9+9 = 5$ times 9.

6. In a rectangle, if

length = 16m, breadth = 12m, area = 16×12 sq. m.
,, = 24m, ,, = 18m, ,, = 24×18 ,,
,, = 15m, ,, = 14m, ,, = 15×14 ,,

7. Give a general form for numbers ending in 5, by observing the following statements :

$$15 = 10 \times 1 + 5$$
$$25 = 10 \times 2 + 5$$
$$35 = 10 \times 3 + 5$$
$$75 = 10 \times 7 + 5.$$

8. Considering 23, 33, 43, 63,.....obtain a general form for numbers ending in 3.

9. Generalise the following :
In a circle,

if the radius is 5 units, area of the circle = $\pi \times 5^2$ sq. units
,, ,, 6 units, ,, ,, = $\pi \times 6^2$,,
,, ,, 7 units, ,, ,, = $\pi \times 7^2$,,

10. Given the following data, obtain a general form or formula for the area of a trapezium.
In a trapezium :

Parallel sides in cm	Distance between them in cm	Area of a trapezium in sq. cm
4, 3	2	$\frac{1}{2}(4+3) \times 2$
10, 7	4	$\frac{1}{2}(10+7) \times 4$
7, 8	3	$\frac{1}{2}(7+8) \times 3$

10. It is important to remember that the *letters* stand for *numbers* and not for *quantities* or *number of things*. 'x rupees' is a quantity,but x is only a number. It is wrong to say, "Let the length of a rectangle be l". It should be, "Let the length of a rectangle be l metres". Similarly, the weight of a box is not 'w' but 'w' kilograms or 'w' units of weight.

In solving simple problems, in Algebra we adopt the unitary method, just as we do in Arithmetic.

Example 20. For x kilometres the railway fare is Rs. y. How much will the fare be for (i) $8x$ kilometres (ii) l kilometres ?

Sol. (i) For x km the fare is Rs. y

For 1 km the fare is Rs. $\frac{y}{x}$.

$\therefore$ For $8x$ km the fare is Rs. $\frac{y}{x} \times 8x$

$= \text{Rs. } 8y.$

(ii) As above, for 1 km the fare is Rs. $\frac{y}{x}$

$\therefore$ for l km the fare is Rs. $\frac{y}{x} \times l$

$= \text{Rs. } \frac{ly}{x}$

Example 21. If x men can do a piece of work in 14 days, how many days will y men take to do it ?

Sol. x men can do a work in 14 days

$\therefore$ 1 man ,, ,, ,, $14 \times x$ days

$\therefore$ y men ,, ,, ,, $\frac{14x}{y}$ days.

EXERCISE 8.

1. Reduce Rs. x, y P. to paise.

2. How many hours does it take to walk
(i) x km at the rate of 5 km an hour ?
(ii) x km at the rate of y km an hour ?

3. In t hours a man walks m km.
(i) At what rate does he walk in km per hour ?
(ii) How many metres does he walk in 15 minutes ?

4. A train runs with the speed of m km per hour ; what is its speed in metres per minute ?

5. A clock loses s seconds in 24 hours ; in how many days will it be x minutes too slow ?

6. If three-fourths of the distance between Delhi and Simla is x km, what is the distance between the two towns ?

7. If a metres of silk cost Rs. 15, how much of it can be had for Rs. x ?

8. A motor car runs l km in a hours on the first day, m km in b hours on the second day and n km in c hours on the third day. Find its average speed per hour.

9. What is the result of increasing x by 16 per cent ?

10. How much per cent is x of y ?

11. Express as a percentage (i) $\frac{1}{a}$, (ii) $\frac{a}{b}$.

12. Find the simple interest on Rs. 300 at r% per annum for 5 years.

13. Find the simple interest on Rs. p at r% per annum for 5 years.

14. Find the simple interest on Rs. p at r% per annum for t years.

15. The area of a wall is A sq. m and its height is 8 m ; find its length.

11. If you fail to solve a literal problem, frame for yourself a similar problem using numbers for letters and solve it ; then follow the *same* process for the given problem, as illustrated further in examples 22 and 23.

Example 22. The telegraph posts are fixed d metres apart along a railway line and a train passes n poles per minute. Find the speed of the train per hour.

Sol. Let us frame a similar problem, putting numbers instead of the letters d and n.

The telegraph posts are fixed 70 metres apart along a railway line and a train passes 9 poles per minute. Find the speed of the train per hour.

In one minute the train passes 9 poles or covers (9—1) or 8 distances between two consecutive poles.

$\therefore$ in 1 minute it goes 70×8 metres

$\therefore$ „ 60 minutes „ $70 \times 8 \times 60$ metres

$\therefore$ „ 1 hour „ $\frac{70 \times 8 \times 60}{1000}$ km.

Now we can work out the original problem:

In one minute the train passes n poles or covers $(n-1)$ distances between two consecutive poles.

$\therefore$ in 1 minute it goes $d \times (n-1)$ metres

$\therefore$,, 60 minutes ,, $d \times (n-1) \times 60$ metres

$\therefore$,, 1 hour ,, $\frac{d \times (n-1) \times 60}{1000}$ km

$\therefore$,, 1 ,, ,, $\frac{3d(n-1)}{50}$ km

Example 23. A cow is bought for x rupees and sold at a profit, for y rupees ; find the profit per cent.

Sol. Let us take a similar problem in Arithmetic. A cow is bought for Rs. 57 and sold for Rs. 72 ; find the profit per cent.

The profit is Rs. $(72-57)$,

The cost price is Rs. 57,

$\therefore$ the profit is $\left[\frac{(72-57)}{57} \times 100\right]\%$

Now in the original problem,

the profit is Rs. $(y-x)$,

the cost price is Rs. x,

$\therefore$ the profit is $\left[\frac{(y-x)}{x} \times 100\right]\%$

i.e. the profit $= \frac{100(y-x)}{x}\%$

EXERCISE 9.

1. A merchant bought sugar worth Rs. m and sold it at a profit for Rs. n. Find his profit per rupee.

2. The price of cycles is reduced by 5%. If the old price of a cycle were r rupees, what is its new price ?

3. A bookseller allows x% commission on the published price of his books sold for cash. How many paise in a rupee is this ?

4. The price of wheat has fallen p% in two years. It is now selling @ 1·5 kg per rupee. What was its rate two years ago ?

5. Find the cost in rupees of a carpet for a room x metres long and y metres wide @ Rs. 3 per sq. metre.

6. A piece of cloth is x metres long and y cm wide. How many pieces 50 cm $\times$ 20 cm can be cut out of it ?

7. What is the volume of a brick a cm long, b cm wide and c cm thick ? What is the total area of its surfaces ?

8. A reservoir is l metres long and b metres wide. If its capacity is x cubic metres, find its depth.

9. The thickness of a sheet of paper is $\frac{1}{k}$ cm. How many sheets will be required to form a pile of paper h cm high ?

10. Find the length of a wire whose cross-section is $\frac{1}{t}$ cm^2 and whose volume is 1 cm^3.

CHAPTER II

DIRECTED NUMBERS, FUNDAMENTAL LAWS, FIRST FOUR RULES

1. Illustrations.

(i) If a tradesman invests in business a capital of Rs. 100, and from his first transaction he gains Rs. 5, and from his second transaction he loses Rs. 5, his capital or financial position remains unchanged.

Here the *loss* of Rs. 5 *destroys* the *gain* of Rs. 5 ; thus the loss of Rs. 5 and the gain of Rs. 5 are *opposite in character* but *equal in magnitude.*

(ii) If a man ascends 10 m and then descends 10 m, his ultimate position remains unchanged.

Here the *descent* of 10 m *destroys* the *ascent* of 10 m, thus the ascent of 10 m and the descent of 10 m ; are *opposite in character* but *equal in magnitude.*

(iii) If a man walks 3 km east and then 3 km west, his ultimate position remains unchanged.

Here the effect of *going* 3 km *east* is *destroyed* by the effect of *going* 3 km *west* ; thus going 3 km east and going 3 km west are *opposite in character* but *equal in magnitude.*

These three examples are of *equal* and *opposite quantities* or *movements.*

It is evident that the gain of Rs. 8 does not *completely* destroy the loss of Rs. 12, but the effect of one on the capital is opposite to the effect of the other ; hence *gain* and *loss* are two *opposite quantities.*

Similarly, income and expenditure, rise and fall, the distance to the right-hand side and the distance to the left-hand side, the distance to the north and the distance to the south, the degrees above the freezing-point and the degrees below the freezing-point, the arrival of men and the departure of men, the time before a particular event and the time after that event, are pairs of opposite quantities or movements.

In Arithmetic the operations of addition and subtraction which are opposite to each other, are denoted by the signs + and − respectively ; similarly, it would be *convenient* to denote any pair of opposite quantities or movements by the same signs, + and −. Thus + and − assume another meaning now.

If one quantity or movement be represented by the sign + the opposite one would be represented by the sign −. They form part of the quantities expressed.

2. In general, it is a convention to mark such quantities *positive* as *raise* or *strengthen our position*, and to mark such quantities *negative* as *lower* or *weaken our position*.

Thus, *gain*, *income*, *credit*, *ascent*, *rise*, etc. are taken as positive, and *loss*, *expenditure*, *debit*, *descent*, *fall*, etc. are taken as negative.

Similarly, motion and distance to the right-hand, motion and distance to the east and north, degrees above the freezing point, etc. are taken as positive ; whereas motion and distance to the left-hand, to the west and south, degrees below the freezing-point, etc. are taken as negative.

NOTE: The + sign is generally omitted before a positive quantity. + Rs. 5 and Rs. 5, + 8 and 8 are positive. The − sign must be invariably written before a negative quantity.

EXERCISE 10.

1. Name three pairs of equal and opposite quantities.

2. Name two pairs of equal and opposite movements.

3. If + Rs. 12 denotes Rs. 12 received, what would − Rs. 12 mean ?

4. If 1000 metres height above the sea-level be represented by + 1000 metres, what would be the meaning of − 500 metres ?

5. If the arrival of 5 men be denoted by + 5 men, what would be the meaning of − 5 men ?

6. Write the following symbolically prefixing the *usual* signs :

(i) An income of Rs. 50. (ii) A loss of Rs. 20.
(iii) A debt of Rs. 100. (iv) An expenditure of Rs. 40.
(v) A saving of Rs. 15. (vi) An ascent of 25 metres.

(vii) A promotion of Rs. 5.
(viii) 200 m below the sea level.
(ix) 13 km due east. (x) 15 km due north.
(xi) 17 km due south. (xii) 20 km due west.
(xiii) 8 men arrived. (xiv) 10 men left.
(xv) 16 metres to the left.
(xvi) 24 metres to the right.

7. Express with proper signs each of the following quantities as a *gain*:

(i) gain of Rs. 12 ; (ii) loss of Rs. 9 ;
(iii) gain of Rs. 25 ; (iv) loss of Rs. 15.

8. Express with proper signs each of the quantities in question 7 as a *loss*.

9. Express all the following as *northerly* movements:

(i) 5 km north. (ii) 7 km south.
(iii) 3 km north. (iv) 8 km south.

10. Express all the movements in question 9 as *southerly* movements.

3. **Directed Numbers.**

The pairs of opposite quantities mentioned previously, *i.e.* gain and loss, credit and debit, have reference to the original financial position of a person.

Rise and fall, ascent and descent are measured from the level of the ground.

Similarly, distances to the east and west, to the north and south are measured from a certain point.

In all cases, the magnitudes of opposite quantities are measured from a particular position. This particular position is called the *starting-point* ; the numbers indicating their measures and direction are called *directed numbers*.

Example 1: A barrel floating on the surface of a sea is carried by successive tides as follows:

3 km east, 4 km west, 2 km east, 5 km west, 6 km east. Represent these movements by means of the usual signs.

Sol. Using + sign for east and — sign for west, we have: (+3 km) + (−4 km) + (+2 km) + (−5 km), as representation of the changes in the position of the barrel.

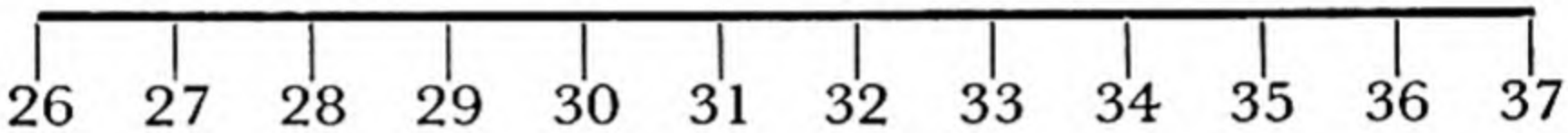
Example 2: A person gains Rs. 30, loses Rs. 25, gains Rs. 15, and then loses Rs. 10. Represent these changes symbolically, using a for Rs. 5.

Sol. Since a is to stand for Rs. 5, the given gains and losses would be represented by $+6a, -5a, +3a$ and $-2a$ respectively. Thus we get the following symbolic representation:

$$(+6a)+(-5a)+(+3a)+(-2a).$$

4. Relative Position.

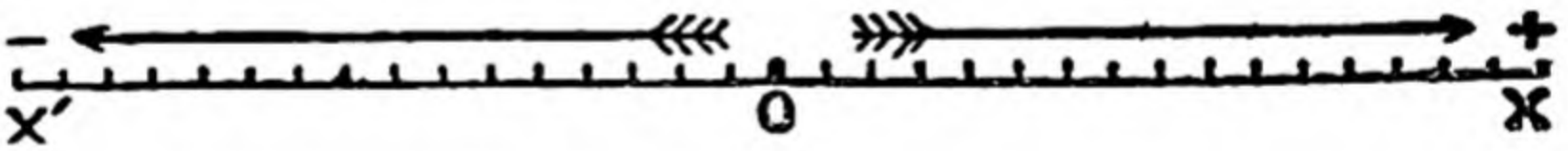
Suppose all measurements are calculated on the following scale from the mark 32.



Since 35 is 3 steps forward from 32, it is said to be $(+3)$ *relative* to 32, and as 30 is 2 steps backward from 32, it is said to be (-2) *relative* to 32.

EXERCISE 11.

1. In the graduated scale XX', take O as the starting-point, and *mark* on it the positions of the following points:



(i) $+3$.	(ii) -3.	(iii) $+5$.
(iv) -8.	(v) -10.	(vi) $+12$.
(vii) -9.	(viii) $+15$.	(ix) $+7$.
(x) 0.	(xi) $+14$.	(xii) -15.

2. *Tell* from the scale the number of divisions between

(i) $+3$ and -3.	(ii) $+5$ and -8.
(iii) -10 and $+10$.	(iv) -5 and -8.
(v) -6 and $+4$.	(vi) 0 and -7.
(vii) -9 and 0.	(viii) $+8$ and 0.

How many divisions do I move, and in what direction

3. if I move from $+3$ to -3 ?

4. if I move from $+5$ to $+8$?

5. if I move from -8 to -5 ?

6. if I move from -5 to $+8$?

7. if I move from $+9$ to $+4$?

8. if I move from $+11$ to -3 ?

9. if I move from -10 to $+2$?

10. if I move from -4 to -10 ?

11. On Monday a merchant gains Rs. 60, on Tuesday he loses Rs. 25, on Wednesday he loses Rs. 15, on Thursday he gains Rs. 12, on Friday he loses Rs. 16, on Saturday he gains Rs. 24, and on Sunday he gains Rs. 40. Represent these changes by means of the usual signs.

12. An aeroplane rises to a height of 850 m, then descends 350 m., it again rises 260 m, and then descends 145 m. Express its ascent and descent symbolically.

13. The changes in the level of a hilly-road are as follows: a rise of 80 m, a drop of 50 m, a rise of 110 m, a drop of 60 m, a rise of 130 m, a drop of 70 m, a rise of 150 m. Represent the level symbolically, using x for 10 m.

14. P, Q, R, S, T are five stations on a railway. From P to Q the level falls 50 m, from Q to R it rises 70 m, from R to S it falls 40 m, and from S to T it rises 80 m. Represent the whole level symbolically, using a for 5 m.

15. The average temperature of a day was 80°. At certain hours of the day the temperature was 75°, 78°, 84°, 88°. State each of these, relative to the average temperature.

16. Reckoning times from 10 a.m., express each of the following times on a certain day by directed numbers:

(i) 4 p.m., (ii) noon, (iii) 7-30 a.m., (iv) 3-20 p.m.

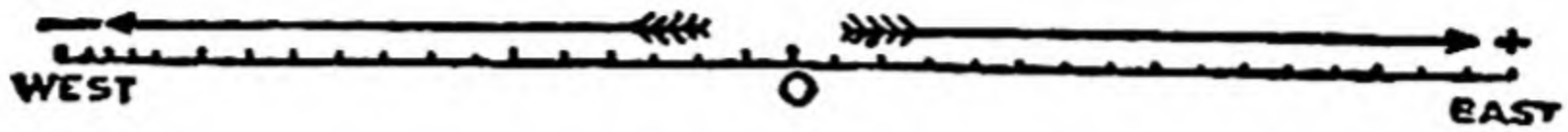
17. The freezing-point of water according to a Fahrenheit thermometer is 32°. Express the following temperatures relative to the freezing-point:

(i) 78°, (ii) 24°, (iii) 32°, (iv) 0°.

5. **Addition.**

The graduated scale below stands for a straight road

running east-west. Each division stands for one kilometre. 0 is the starting-point.



A man is supposed to be walking along this road in both directions as detailed below. His movements are expressed symbolically :

Details of movement	Symbolic expression
(i) When he goes 8 km east, then 5 km east, his final position is 13 km east.	(East is +; West is −) (+8 km)+(+5 km) = +13 km.
(ii) When he goes 8 km west, then 5 km west, his final position is 13 km west.	(−8 km)+(−5 km) = −13 km.
(iii) When he goes 8 km to the east and then 5 km to the west, his final position is 3 km east.	(+8 km)+(−5 km) = +3 km.
(iv) When he goes 8 km to the west and then 5 km to the east, his final position is 3 km west.	(−8 km)+(+5 km) = − 3 km.

In the following illustration, some transactions are represented symbolically, using + sign for gain and — sign for loss.

Transaction	Symbolic expression
(i) Rs. 3 gain + Rs. 4 gain results in Rs. 7 gain.	(+Rs. 3)+(+Rs. 4) = +Rs. 7.
(ii) Rs. 3 loss + Rs. 4 loss results in Rs. 7 loss.	(−Rs. 3)+(−Rs. 4) = −Rs. 7.
(iii) Rs. 3 gain + Rs. 4 loss results in Re. 1 loss.	(+Rs. 3)+(−Rs. 4) = −Re. 1.
(iv) Rs. 3 loss + Rs. 4 gain results in Re. 1 gain.	(−Rs. 3)+(+Rs. 4) = +Re. 1.

Introducing letters in the above illustrations and expressing in algebraic terms, we have

$(+8k)+(+5k) = +13k$	$(+3x)+(+4x) = +7x$
$(-8k)+(-5k) = -13k$	$(-3x)+(-4x) = -7x$
$(+8k)+(-5k) = +\ 3k$	$(+3x)+(-4x) = -\ x$
$(-8k)+(+5k) = -\ 3k$	$(-3x)+(+4x) = +\ x$

If you buy 8 apples and again 5 apples, you will have 13 apples.

Similarly, 3 books + 4 books = 7 books.
or algebraically, $8a+5a = 13a$ (a stands for apples).
$3b+4b = 7b$ (b stands for books).

But the sum of 8 apples and 3 books can neither be 11 apples nor 11 books. They cannot be added. They are of different kinds. Likewise, in Algebra, *only like terms can be added or subtracted.*

Thus $3x+7x = 10x.$
$5ab+3ab+6ab = 14ab$
$2x^2y+5x^2y+3yx^2 = 10x^2y.$

NOTE: x^2y is the same as yx^2, but unlike xy^2.

Example 3: Add together: $+2xy, -3xy, +6xy, -4xy.$

Sol.

(i) See if all the terms are *like terms*	yes, all are take
(ii) Add the coefficients of positive terms	terms $(+2)+(+6) = +8$
(iii) Add the coefficients of negative terms	$(-3)+(-4) = -7$
(iv) Combine these two results	$(+8)+(-7) = +1$
(v) Attach the literal part to the right	Result is $1xy$ *i.e.* xy

It can also be written thus:

$$(+2xy)+(-3xy)+(+6xy)+(-4xy)$$
$$=(+2xy+6xy)+(-3xy-4xy)$$
$$=+8xy+(-7xy)$$
$$=(+8-7)xy$$
$$=+1xy=xy.$$

Example 4: Simplify: $6ab^2c-4cab^2+8acb^2-12b^2ca$.

Sol. Note that all the terms are alike but only have different signs. The literal part in ab^2c.

The expression $=6ab^2c-4ab^2c+8ab^2c-12ab^2c$
$=(6+8-4-12)ab^2c$
$=(14-16)ab^2c$
$=-2ab^2c.$

Example 5: Simplify:

$$5xy-6yz+3zx-4xy+3yz-zx$$

Sol. The given expression

$=5xy-6yz+3zx-4xy+3yz-zx$
$=(5xy-4xy)+(-6yz+3yz)+(3zx-zx)$
$=1xy+(-3yz)+2zx$
$=xy-3yz+2zx.$

EXERCISE 12.

1. Using a graduated scale or a diagram, perform the following additions:

(i) $(+7)+(+12)$ (ii) $(-7)+(-12)$
(iii) $(+7)+(-12)$ (iv) $(-7)+(+12)$
(v) $(+5)+(-8)$ (vi) $(-5)+(+8)$
(vii) $(+8)+(-8)$ (viii) $(-12)+(+12)$
(ix) $+5-7+6$ (x) $-5+7-6.$

Add:

2.
$-7x$
$+4x$

3.
$5ax$
$-7ax$

4.
$+19axy$
$-9axy$

5.
$+8ab$
$-7ab$
$+5ab$

6.
$-4xyz$
$+7xyz$
$-13xyz$

7.
$+9x^2y$
$-5x^2y$
$+12x^2y$

8. $-5y^2z$
$-8y^2z$
$9y^2z$

9. $-2m^2n$
$+17m^2n$
$-5m^2n$

10. $+8p^2q^2$
$-12p^2q^2$
$+4p^2q^2$

Simplify :

11. $4ab+3ab-2ab-7ab.$
12. $13xyz-4xyz+5xyz.$
13. $-5abx^2+8abx^2-9abx^2+4abx^2-2abx^2.$
14. $-7pq^2r-12pq^2r+7pq^2r--3pq^2r+5pq^2r-8pq^2r.$

Simplify by collecting like terms :

15. $a-2b+3c-4a+6b-5c+2a.$
16. $x-4y+5z-3y+4z-5x+2y-10z+5y+x.$
17. $3p^2-3pq+4q^2-p^2+7pq-6p^2-2pq+5q^2.$
18. $5ab-4bc+ca-2ab+6ca+7bc-3ca+bc.$
19. $6x^2yz+xy^2z-4x^2yz-5xyz^2+3xy^2z+7xyz^2.$
20. $4ax^2y-5a^2xy+8ax^2y-7axy^2+2a^2xy-4axy^2.$

5. Addition of miscellaneous expressions.

Example 6. Add together $5a^3-4a^2b+6ab^2-2b^3$, $a^3-2ab^2-3a^2b$, and $b^3-3a^3+ab^2$.

Sol. Here, by arranging the like terms in columns and adding them up, we have

$$\begin{array}{r} 5a^3-4a^2b+6ab^2-2b^3 \\ a^3-3a^2b-2ab^2 \\ \underline{-3a^3\qquad\quad +ab^2\ +b^3.} \\ 3a^3-7a^2b+5ab^2-b^3 \end{array}$$

EXERCISE 13.

Add together :

1. $x^2+3xy, -2xy+y^2.$
2. $x^2+xy, -xy+y^2.$
3. $a^2+ab, -ab-b^2.$
4. $p^2-3pq+q^2, -5p^2+6pq+2q^2.$
5. $a^2+a+1,\ 3a^2-a+2,\ 5a^2-4a-7.$
6. $-5x^2+3xy-y^2, -8x^2+xy-y^2,\ 7x^2-3y^2.$
7. $2x+3y-4z,\ -5x-y-z,\ -3x-4y.$
8. $b^2-b^2c+c^2b,\ 2b^2c-3c^2b,\ -4b^2-2c^2b.$
9. $3a^2x^2-4ax-10,\ 2a^2x^2+ax-6,\ -4a^2x^2+ax+9.$
10. $a^2-5ab-4b^2,\ -3a^2+2ab-6b^2,\ 5a^2+3ab+2b^2.$
11. $5p^2-2pq+7q^2,\ -9p^2+3pq-6q^2,\ +2p^2-5pq+10q^2.$

12. $-1-3x-2x^3$, $5x-3+4x^3$, x^2+2x^3-2x+2.

13. $3a-4b+5c-d+6e$, $-2a+4b-2c-7d+3f$, $-6a+4b-3c-7d+9f$, $-5a-2b+5c-4d-3e$, $9a-2b-c+2e-6f$.

14. $\frac{1}{2}a^2-\frac{1}{3}ab+b^2$, $2ab-\frac{1}{2}b^2+\frac{1}{3}a^2$, $\frac{1}{2}a^2+\frac{2}{3}ab-\frac{1}{3}b^2$.

15. If $A=x^2-5x+9$, $B=4x-3$, $C=6-x^2$, find the value of $A+B+C$.

16. If $x=4a^2-3a+5$, $y=-2a+7a^2+4$ and $z=-9+4a-11a^2$, find the value of $x+y+z$.

6. Subtraction.

The subtraction of directed numbers involves two things (1) *the quantity* and (2) *the sign*. Suppose it is required to subtract $+4$ from $+7$. It is the same as asking, "What must be added to $+4$ to make $+7$?" The answer is, "$+3$".

Similarly,
$$\begin{aligned}(+7)-(-4) &= +11\\ (-7)-(+4) &= -11\\ (-7)-(-4) &= -3\end{aligned}$$

These can be illustrated with the help of a graduated scale as before.

As usual, upward direction is $+$ and downward is $-$.

(i) Required: $(+7)-(+4)$.

Start from $+4$. Count the number of steps to reach $+7$. It is 3.

In what direction do you move to read $+7$? Upward direction i.e. $+$.

$\therefore\ (+7)-(+4) = +3$

(ii) Required: $(+7)-(-4)$.

Start from -4 and find the number of steps to reach $+7$. It is 11.

In what direction do you move to reach $+7$? Upward direction i.e. $+$.

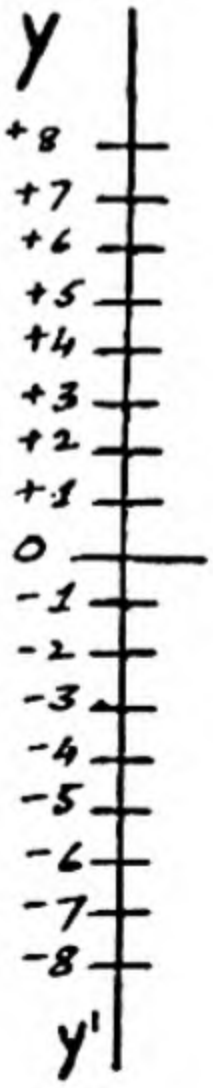
$\therefore\ (+7)-(-4) = +11$

(iii) Required: $(-7)-(+4)$.

The number of steps from $+4$ to -7 is 11 and the movement is downward i.e. $-$.

$\therefore\ (-7)-(+4) = -11$

(iv) Required: $(-7)-(-4) = -3$.



The number of steps from -4 to -7 is 3 and the movement is downward.

$\therefore\ (-7)-(-4) = -3.$

Let us now consider the following algebraic additions and subtractions :

Addition

$+7$	$+7$	-7	-7	$+4$	$+4$	-4	-4
$+4$	-4	$+4$	-4	$+7$	-7	$+7$	-7
$+11$	$+3$	-3	-11	$+11$	-3	$+3$	-11

Subtraction

$+7$	$+7$	-7	-7	$+4$	$+4$	-4	-4
$+4$	-4	$+4$	-4	$+7$	-7	$+7$	-7
$+3$	$+11$	-11	-3	-3	$+11$	-11	$+3$

Comparing the answers, we find that
subtracting $+4$ gives the same result as adding -4
subtracting -4 ,, ,, ,, ,, adding $+4$
subtracting $+7$,, ,, ,, ,, adding -7
subtracting -7 ,, ,, ,, ,, adding $+7$.

Hence the following rule may be adopted.

Rule : *In order to subtract a given quantity, change its sign and add.*

Example 7 : Subtract $3a-5b-c$ from $5c-8b+4a$.

Sol. It means : Arrange the like terms in columns, change the signs of all the terms in the subtrahend, and then add.

subtraction

$$\begin{array}{r} 4a-8b+5c \\ 3a-5b-\ c \\ \hline a-3b+6c \\ \hline \end{array}$$

corresponding addition

$$\begin{array}{r} 4a-8b+5c \\ -3a+5b+\ c \\ \hline a-3b+6c \\ \hline \end{array}$$

NOTE : With a little practice, the subtraction can and should be done mentally straight-away without writing the corresponding addition form.

Example 8: Subtract
$12xy-5x^2+2c^2-1$ from $32x^2+20xy-3y^2$.

Sol. Arranging the terms properly, changing the signs in the subtrahend, and adding the columns, we have

$$\begin{array}{r} 32x^2+20xy-3y^2 \\ +5x^2-12xy \qquad -2c^2+1 \\ \hline 37x^2+\ 8xy-3y^2-2c^2+1 \end{array}$$

EXERCISE 14.

With the help of a graduated scale, subtract:

1. + 7 from +12.
2. − 7 from +12.
3. + 7 from −12.
4. − 7 from −12.
5. +12 from + 7.
6. −12 from + 7.
7. +12 from − 7.
8. −12 from − 7.

9. Subtract:

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
$+7a$	$+7a$	$-7a$	$-7a$	$+12a$	$+12a$	$-12a$	$-12a$
$+12a$	$-12a$	$+12a$	$-12a$	$+7a$	$-7a$	$+7a$	$-7a$

10. Subtract:

(i) + $5x$ from $-5x$.
(ii) − $7m$ from $+7m$.
(iii) + $9n$ from $+9n$.
(iv) $-13q$ from $-13q$.

11. Subtract:

(i) 0 from m
(ii) 0 from $-m$
(iii) + m from 0.
(iv) − m from 0.

12. Subtract:

(i) $\begin{array}{r} x+y \\ x-y \\ \hline \end{array}$ (ii) $\begin{array}{r} x+y \\ x+y \\ \hline \end{array}$ (iii) $\begin{array}{r} x+y \\ y+x \\ \hline \end{array}$ (iv) $\begin{array}{r} x-y \\ y-x \\ \hline \end{array}$

13. Do the following questions on subtraction:

(i) $\begin{array}{r} 5x+2y+3z \\ 3x+2y+\ z \\ \hline \end{array}$

(ii) $\begin{array}{r} -ab+4bc+6ca \\ 2ab-3bc+7ca \\ \hline \end{array}$

(iii) $\begin{array}{r} -x^2-2y^2+3xy \\ -x^2+4y^2-\ xy \\ \hline \end{array}$

(iv) $\begin{array}{r} -7+3y^2 \\ 11-y^2+3x^2 \\ \hline \end{array}$

Subtract by arranging like terms in columns:

14. $1-m^2+m^3$ from $3-2m^2-m^4$.

15. $x^2+2y^2+z^2-6$ from $4x^2-7+z^2-2y^2$.

16. $a^2-2ab+3b^2$ from $-6a^2+ab-9b^2$.

17. $-8x^2y^2-3y^2z^2-z^2x^2$ from $-12x^2y^2-4y^2z^2+7z^2x^2$.

18. $3p^3+4p^2q-pq^2-2q^3+3$ from $4p^3-p^2q-pq^2+2q^3-3$.

19. Subtract a^2+b^2 from the sum of $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

20. Subtract the sum of x^2+xy and $xy+y^2$ from the sum of x^2+xy+y^2 and x^2-xy+y^2.

If $x = 3a^2-1$, $y = 2b^2+4$, $z = a^2-b^2+7$, find:

21. $x-y$. **22.** $y-z$. **23.** $z-x$.

24. $x+y-z$. **25.** $x-y+z$. **26.** $y+z-x$.

7. Removal of Brackets.

(1) Because $(5+4) = 9$, $8+(5+4) = 8+9 = 17$

We know also that $8+5+4$ is equal to 17

$\therefore\ 8+(5+4) = 8+5+4$

(2) Because $(5-4) = 1$, $8+(5-4) = 8+1 = 9$

Also, $8+5-4$ is equal to 9.

$\therefore\ 8+(5-4) = 8+5-4$

(3) Because $(-5+4) = -1$, $8+(-5+4) = 8-1 = 7$

and also $8-5+4$ is equal to $12-5 = 7$

$\therefore\ 8+(-5+4) = 8-5+4$

(4) Because $(-5-4) = -9$, $8+(-5-4) = 8+(-9) = -1$

and also $8-5-4$ is equal to -1.

$\therefore\ 8+(-5-4) = 8-5-4$.

Applying the same principle, we have the following relationship in Algebra:

	Brackets with a + sign before.		Brackets removed.
(i)	$a+(+b+c)$	$=$	$a+b+c$
(ii)	$a+(+b-c)$	$=$	$a+b-c$
(iii)	$a+(-b+c)$	$=$	$a-b+c$
(iv)	$a+(-b-c)$	$=$	$a-b-c$

	Brackets with a—sign before.		Brackets removed.
(i)	$a-(+b+c)$	$=$	$a-b-c$
(ii)	$a-(+b-c)$	$=$	$a-b+c$
(iii)	$a-(-b+c)$	$=$	$a+b-c$
(iv)	$a-(-b-c)$	$=$	$a+b+c$

From these two sets of examples, we can conclude that

(1) when there is a + sign before the brackets, the brackets can be removed *without* making any change in the signs of the enclosed terms and

(2) when there is $a-$ sign before the brackets, the brackets can be removed on *changing* the sign of every term within, from + to − and from − to +.

Example 9: Simplify by removing brackets:

$$(1+3x)-(2-5x)+(2-3x)-(1+5x)$$

Sol. $(1+3x)-(2-5x)+(2-3x)-(1+5x)$

$= 1+3x-2+5x+2-3x-1-5x$

$= 1-2+2-1+3x+5x-3x-5x$

$= 0.$

Example 10: Simplify:

$$(a+b+c+d)-(a-b+c-d)+(b-c+d-a)$$

Sol. The given expression

$= (a+b+c+d)-(a-b+c-d)+(b-c+d-a)$

$= a+b+c+d-a+b-c+d+b-c+d-a$

$= a-a-a+b+b+b+c-c-c+d+d+d$

$= a-2a+3b+c-2c+3d$

$= -a+3b-c+3d.$

When there are 2 or more pairs of brackets, one within another, they can be removed by proceeding from the *innermost* brackets first.

Example 11: Simplify $7x+\{-3x+(5-2x)\}$.

Sol. The expression

$= 7x+\{-3x+(5-2x)\}$

$= 7x+\{-3x+5-2x\}$

$= 7x+\{-5x+5\}$

$= 7x-5x+5$

$= 2x+5.$

Example 12: Simplify $7x-\{-3x-(5-2x)\}$

Sol. The expression $= 7x-\{-3x-(5-2x)\}$
$= 7x-\{-3x-5+2x\}$
$= 7x-\{-x-5\}$
$= 7x+x+5$
$= 8x+5.$

EXERCISE 15.

Write down the value of the following :

1. (i) $+(+5)$, (ii) $-(+3)$, (iii) $+(-7)$, (iv) $-(-6)$.

2. (i) $-(-3m)$, (ii) $+(-5q)$, (iii) $-2-(2-2)$, (iv) $-(2-4)-2$.

3. (i) $-x^2-(-a^2)+(-a^2)+(+x^2)$, (ii) $(+p-p)-(p-p)$.

4. (i) $+(-1)+(-1)+(-1)$, (ii) $-(+3)-(+3)-(+3)$, (iii) $-(-3)-(-3)-(-3)$.

Simplify :

5. $4p+(-3q-5p)$.

6. $(4a-5b)+(-2a+4b)$.

7. $(6p-2q)+(3q-4p)$.

8. $4p-(-3q+5p)$.

9. $7k-(-4k-2l)$.

10. $(3a-4b)-(-2a+5b)$.

11. $(l-5m+2n)-(-3n+4m-5l)$.

12. $-3a-\overline{3b+c}+\overline{3a+3b}-(+c)$.

13. $p+\{2q+\overline{3-5p}-(-q+7)+5\}$.

14. $2p-(-3+\overline{p-2q}-5)-7$.

15. $2a+(+5b-\overline{7b-2a})$.

16. $2-\{a+3+(a-\overline{4a-5})\}$

17. $5y-[3x+\{2y-(3y-z)+5z\}]$.

18. $1-[1-\{1-(\overline{1-1})\}]$.

19. If $a=-1$, $b=-2$, $c=-3$, find the value of
$1-b+a-[c-\{a+(b-c-\overline{a-b-c})\}]$.

Find the value of:

20. $(a+b+c)-(a-b-c)+(-a-b-c)+(a+b-c)$.

21. $(x+y-z)+(y+z-x)-(z+x-y)-(x+y+z)$.

22. $(a-b+c)-(3a-3b+3c)+(2a-2b+2c)$.

23. $(a^3+b^3-c^3+abc)-(a^3-b^3+c^3-abc)+(a^3-b^3+c^3-abc)-(a^3+b^3-c^3+abc)$.

24. $[a-\{b+(c+d)\}]-[a+\{b-(c-d)\}]+[a-\{b-(c+d)\}]-[a-\{b-(c-d)\}]$.

8. Insertion of Brackets.

This is just the *reverse* of 'removal' of brackets and the following rules will apply:

(i) *Any number of terms in an expression may be put within a pair of brackets and the sign + prefixed.*

(ii) *Any number of terms in an expression may be put within a pair of brackets and the sign — prefixed, provided the signs of all the terms placed within the brackets be changed, i.e., from + to — and from — to +.*

For example,

$a+b-c = a+(b-c)$ or $a-(-b+c)$
$p-q+r = p+(-q+r)$ or $p-(+q-r)$
$l+m+n = l+(m+n)$ or $l-(-m-n)$
$x-y-z = x+(-y-z)$ or $x-(y+z)$

EXERCISE 16.

Enclose all the terms within a pair of brackets, preceded by (i) + sign (ii) — sign.

1. $2a+3b-4c$. **2.** $2a-3b+4c$.
3. $3x+4y+5z$. **4.** $3x-4y-2z$.

Enclose the second and the third terms within a pair of brackets, preceded by (i) + sign (ii) — sign.

5. $ax+by-c$. **6.** $x-2y+z$.
7. $3p+4q-5r$. **8.** $5l-4m-n$.

Enclose all the terms after the first, within a pair of brackets preceded by (i) + sign (ii) — sign.

9. $-a^2+m^2+b^2-n^2$. **10.** $x^2-a^2+bc-b^2-ca$.

11. Fill in the blanks in the following by bringing together the like powers of x:

$2x^4+3x^2-4x^3+x^2-5x^4+x^3$
$= (2x^4\ldots\ldots)+(x^3\ldots\ldots)+(3x^2\ldots\ldots)$
$= -(-2x^4\ldots\ldots)-(-x^3\ldots\ldots)-(-3x^2\ldots\ldots)$

12. Put the proper sign (+ or —) in the place of each star.

$a^2-b^2+c^2-ab+bc-ca$
$= (a^2* \ ab)+(-b^2* \ bc)+(c^2* \ ca)$
$= (* \ a^2* \ b^2* \ c^2)-(* \ bc* \ ca* \ ab)$.

9. Multiplication.

A labour camp consists of workers and children. Each worker pays Rs. 8 to the camp-fund. Each child receives Rs. 8 from the fund. Let us consider the changes in the camp-fund by the arrival and the departure of these people.

(i) If 5 workers *join*, the fund *increases* by Rs. 40.
(ii) If 5 workers *leave*, the fund *decreases* by Rs. 40.
(iii) If 5 children *join*, the fund *decreases* by Rs. 40.
(iv) If 5 children *leave*, the fund *increases* by Rs. 40.

"Joining", "income", "increase" are represented by + sign.
"Leaving", "expenditure", "decrease" are represented by − sign.

The above four cases can now be written in symbols thus :—

(i) $(+5)\times(+8) = +40$ (ii) $(-5)\times(+8) = -40.$
(iii) $(+5)\times(-8) = -40.$ (iv) $(-5)\times(-8) = +40.$

By examining similar suitable examples, we can generalise thus :

$(+a)\times(+b) = +ab$ ... ... (i)
$(-a)\times(+b) = -ab$ ... ... (ii)
$(+a)\times(-b) = -ab$ ... ... (iii)
$(-a)\times(-b) = +ab$ ... ... (iv)

Hence the Law :

The product of two quantities with like signs is positive.
The product of two quantities with unlike signs is negative.

If one of the factors is zero, the product is zero.

$(+a)\times 0 = +0 = 0$; $(-a)\times 0 = -0 = 0.$

Zero × any quantity is zero.

If there are three or more quantities to be multiplied, the result of multiplying them all is called their continued product.

Example 13 : Find the continued product :

(i) $(+a)\times(+b)\times(+c)$; (ii) $(2a)\times(-3b)\times(+3c)$;
(iii) $(+3a)\times(-4b)\times(-2c)$;
(iv) $(-a)\times(-2b)\times(-5c).$

Sol. (i) $(+a)\times(+b)\times(+c) = (+ab)\times(+c)$
$= +abc.$

(ii) $2a \times (-3b) \times (+3c) = (2) \times (-3) \times a \times b \times (+3c)$
$= -6ab \times (+3c)$
$= (-6) \times (+3) \times (ab) \times (c)$
$= -18abc.$

(iii) $(+3a) \times (-4b) \times (-2c) = (+3 \times -4) \times a \times b \times (-2c)$
$= (-12) + (ab) \times (-2) \times (c)$
$= (-12) \times (-2) \times (+ab) \times (c)$
$= (+24)(+abc)$
$= +24abc.$

(iv) $(-a) \times (-2b) \times (-5c) = (+2ab) \times (-5c)$
$= (+2) \times (-5) \times (ab) \times c$
$= -10abc.$

EXERCISE 17.

1. Multiply :

(i) $(+5)$ by $(+7)$, (ii) $(+7)$ by (-2).
(iii) (-9) by $(+5)$. (iv) (-8) by (-4).
(v) 12 by -5. (vi) -15 by 8.
(vii) $+5a$ by $+8$. (viii) $+7m$ by -4.
(ix) $-8m$ by $+4$. (x) $-7m$ by -8.
(xi) $6a$ by -12. (xii) $+12$ by $-2m$.

2. Multiply :

(i) $4 \times 0 \times (-3)$. (ii) $(+4m) \times 0 \times (-3m)$.
(iii) $(+2a) \times (-3b) \times 0$. (iv) $0 \times (-7p) \times (-2q)$.
(v) $(+5) \times (+5)$. (vi) $(-5) \times (-5)$.
(vii) $(-a) \times (-a)$. (viii) $(-b) \times (-b) \times (-b)$.
(ix) $(-c) \times (-c) \times (-c) \times (-c)$.
(x) $(-p) \times (-p) \times (-p) \times (-p) \times (-p)$.
(xi) $(-1) \times (-1)$. (xii) $(-1) \times (-1) \times (-1)$.

3. Show that :

(i) $(+a)^2 = +a^2$ and $(-a)^2 = +a^2$.
(ii) $(+x)^3 = +x^3$ and $(-x)^3 = -x^3$.
(iii) $(+p)^4 = +p^4$ and $(-p)^4 = +p^4$.
(iv) $(+m)^5 = +m^5$ and $(-m)^5 = -m^5$.
(v) $(+1)^2 = +1$ and $(-1)^2 = +1$.
(vi) $(+1)^3 = +1$ and $(-1)^3 = -1$.
(vii) $(+1)^4 = +1$ and $(-1)^4 = +1$.
(viii) $(+1)^5 = +1$ and $(-1)^5 = -1$.

4. (i) What is the value of $(-1)^n$, when n is an even number ?

(ii) What is the value of $(-1)^n$, when n is an odd number ?

(iii) What is the sign of each of the following :

$(-a)^{83}$, $(-a)^{94}$, $(+p)^{45}$, $(+m)^{28}$?

5. (i) $8x = 0$; what is the value of x ?

(ii) $4(x-3) = 0$; what is the value of $x-3$?

6. Find the value of $3x^3-2x^2y-5xy^2+y^3$ by substituting for x and y the values given in the table :

x	-3	-2	$+1$	-1	-1	0
y	$+2$	-3	-2	-1	0	-1

7. Find the value of :

(i) $5x^3-3x^2+4x+1$.

(ii) $-3x^4+x^3+2x^2-x+4$.

(iii) $-4x^3-2x^2+6x-9$.

when x has values -2, $+3$, -1, 0.

8. Find the continued product of :

(i) $(-2y)^3$, $(3y)^4$, $(-y)^5$.

(ii) $(-a)^6$, $(-2ab^2)^2$, $(-a^2b)^3$.

10. **Product of a compound expression and a simple expression.**

Just as $(a+b)\times 2 = a\times 2+b\times 2$

$= 2a+2b$,

and $(a-b)\times 2 = a\times 2-b\times 2$

$= 2a-2b$

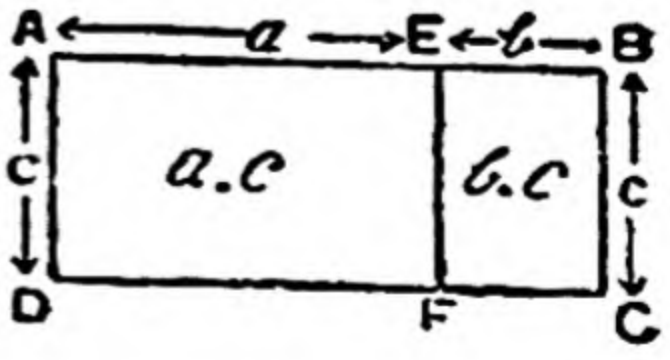
so $(a+b)\times c = ac+bc$,

and $(a-b)\times c = ac-bc$.

Illustrations.

(i) Take a rectangle $ABCD$, having $AD = c$ units, $AE = a$ units, and EB $AB = (a+b)$ units, $= b$ units.

Draw EF parallel to AD, dividing the rectangle $ABCD$ into 2 smaller rectangles.



The whole rect. AC = rect. AF + rect. EC.

Now, area of rect. $AC = (a+b)c$ sq. units,
the area of ,, $AF = ac$ sq. units, and
,, ,, ,, $EC = bc$ sq. units.

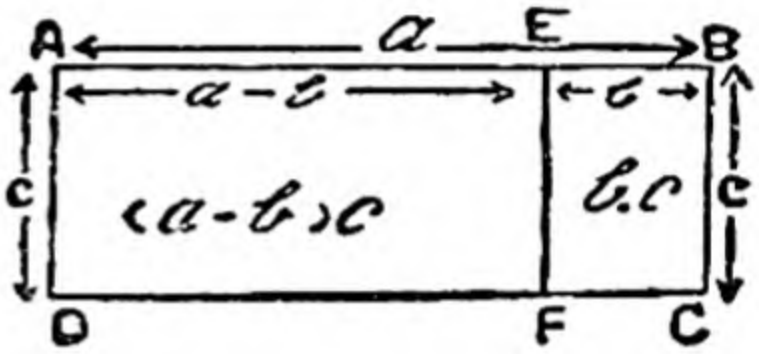
Substituting, we have: $(a+b)c = ac+bc$.

(ii) Take another rectangle $ABCD$ in which $AB = a$ units, $AD = c$ units.

On AB, mark a point E making $EB = b$ units.

$\therefore$ $AE = (a-b)$ units.

Draw EF parallel to AD.



The rect. AF = rect. AC—rect. EC.

But area of rect. $AF = (a-b)c$ sq. units
,, ,, $AC = ac$ sq. units
and ,, ,, $EC = bc$ sq. units

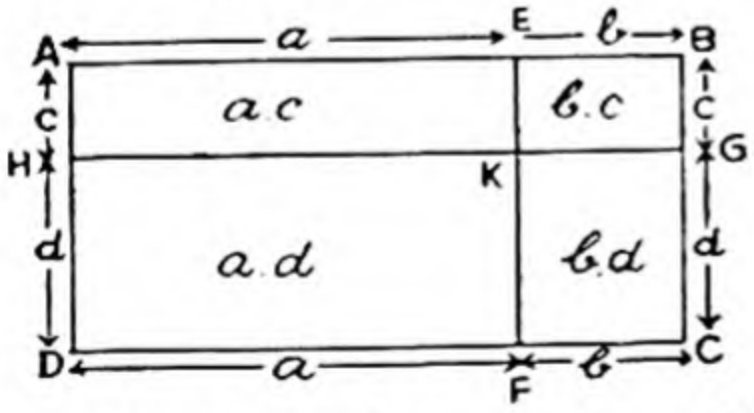
Substituting, we have $(a-b)c = ac-bc$.

NOTE: $(a+b) \times c$ is the same as $c \times (a+b)$

Rule: The product of a compound expression and a single term is found by multiplying in succession each term of the expression by that term. Since $(a+b)\ m = am+bm$,
if we substitute $c+d$ for m, we have

$$(a+b)\ (c+d) = a\ (c+d)+b\ (c+d)$$
$$= ac+ad+bc+bd.$$

Graphic Illustration. Take a rectangle $ABCD$ in which $AE = a$ units, $EB = b$ units, $AH = c$ units, and $HD = d$ units.



Through E and H draw straight lines EF and HG, respectively parallel to BC and AB, and intersecting at K.

Rect. AC = rect. AK+rect. HF+rect. EG+rect. KC.

The area of rect. $AC = (a+b)\ (c+d)$ sq. units, the area of rect. $AK = ac$ sq. units, the area of rect. $HF = ad$ sq. units, the area of rect. $EG = bc$ sq. units, and that of rect. $KC = bd$ sq. units.

$\therefore$ $(a+b)\ (c+d) = ac+ad+bc+bd.$

Rule : Multiply each term of the first expression by each term of the second and add all the partial products thus obtained.

Example 14 : Multiply (i) $3a^2+2b^2$ by $+ab$,
(ii) $3a^2-2b^2$ by $+ab$,
(iii) $3a^2+2b^2$ by $-ab$,
(iv) $3a^2-2b^2$ by $-ab$.

Sol. (i) $(3a^2+2b^2)\times(+ab) = (3a^2)\times(+ab)+(2b^2)\times(+ab)$
$= 3a^3b+2ab^3.$

(ii) $(3a^2-2b^2)\times(+ab) = \{(+3a^2)+(-2b^2)\}\times(+ab)$
$= (+3a^2)\times(+ab)+(-2b^2)\times(+ab)$
$= 3a^3b-2ab^3.$

(iii) $(3a^2+2b^2)\times(-ab) = (3a^2)\times(-ab)+(+2b^2)\times(-ab)$
$= -3a^3b-2ab^3.$

(iv) $(3a^2-2b^2)\times(-ab) = \{(+3a^2)+(-2b^2)\}\times(-ab)$
$= (+3a^2)(-ab)+(-2b^2)(-ab)$
$= -3a^3b+2ab^3.$

Example 15 : Multiply (i) $(3x+2y)$ by $(3x+4y)$,
(ii) $(3x+2y)$ by $(3x-4y)$,
(iii) $(3x-2y)$ by $(3x-4y)$.

Sol. (i) $(3x+2y)\times(3x+4y) = (3x+2y)(+3x)+(3x+2y)(+4y)$
$= 9x^2+6xy+12xy+8y^2$
$= 9x^2+18xy+8y^2.$

The process is usually arranged thus :

$(3x+2y)\times(+3x) =$ ... $3x+2y$
$(3x+2y)\times(+4y) =$ ... $3x+4y$
Product = ... $9x^2+6xy$
$\quad +12xy+8y^2$
$9x^2+18xy+8y^2$

After some practice, each partial product is written with like terms in columns and the left-hand side of the process is omitted, as illustrated below :

(ii)
$3x+2y$
$3x-4y$
$9x^2+6xy$
$-12xy-8y^2$
$9x^2-6xy-8y^2$

(iii)
$3x-2y$
$3x-4y$
$9x^2-6xy$
$-12xy+8y^2$
$9x^2-18xy+8y^2$

EXERCISE 18.

1. Multiply :

(i) $(4a+5b)\times(+3)$, (ii) $(5x+7y)\times(-3)$,
(iii) $(2m-3n)\times(+7)$, (iv) $(3m-5n)\times(-7)$,
(v) $(3p-2q)\times(+4p)$, (vi) $(3p-2q)\times(-5q)$,
(vii) $(2a+b-3c)\times(+4c)$, (viii) $(2a-b-2c)\times(-5c)$.

2. $(x+2)\times(x+3)$. **3.** $(x+5)\times(x+2)$.
4. $(a+b)\times(b+c)$. **5.** $(a+b)\times(a+b)$.
6. $(2a+b)\times(3a+2b)$. **7.** $(2x+3y)\times(x+y)$.
8. $(a^2-ab)\times(ab-b^2)$. **9.** $(x^3-10)\times(-x^3+10)$.
10. $(-x^2+2xy)\times(3xy-y^2)$. **11.** $(x^2+xy)\times(-x-y)$.
12. $(2x^2+5ab)\times(2x^2-5ab)$. **13.** $(3m^2-2mn)\times(n^2-3mn)$.
14. $(-\frac{1}{2}+5x)\times(\frac{1}{4}x-3)$. **15.** $(-2a^2-\frac{1}{4})\times(5a-\frac{1}{3})$.

11. More Examples of Multiplication.

Example 16: Multiply x^2+xy-y^2 by x^2-xy+y^2.

$$\begin{array}{r} x^2+xy-y^2 \\ x^2-xy+y^2 \\ \hline x^4+x^3y-x^2y^2 \\ -x^3y-x^2y^2+xy^3 \\ +x^2y^2+xy^3-y^4 \\ \hline x^4 \quad -x^2y^2+2xy^3-y^4 \end{array}$$

Before the actual process of multiplication, it is useful to arrange the multiplicand and the multiplier according to descending or ascending powers of some common letter.

Example 17: Multiply $1+a^2+a$ by $-a+1+a^2$.

(i) Arranging according to descending powers of a,

$$\begin{array}{r} a^2+a+1 \\ a^2-a+1 \\ \hline a^4+a^3+a^2 \\ -a^3-a^2-a \\ +a^2+a+1 \\ \hline a^4 \quad +a^2 \quad +1 \end{array}$$

(ii) Arranging according to ascending powers of a,

$$\begin{array}{r} 1+a+a^2 \\ 1-a+a^2 \\ \hline 1+a+a^2 \\ -a-a^2-a^3 \\ +a^2+a^3+a^4 \\ \hline 1 \quad +a^2 \quad +a^4 \end{array}$$

Example 18: Multiply $3a^2b^2-ab^3+2a^4-a^3b$ by $-2ab+b^2-3a^2$.

Arranging according to descending powers of a,

$$\begin{array}{r} 2a^4-a^3b+3a^2b^2-ab^3 \\ -3a^2-2ab+b^2 \\ \hline -6a^6+3a^5b-9a^4b^2+3a^3b^3 \\ \quad -4a^5b+2a^4b^2-6a^3b^3+2a^2b^4 \\ \quad\quad +2a^4b^2-a^3b^3+3a^2b^4-ab^5 \\ \hline -6a^6-a^5b-5a^4b^2-4a^3b^3+5a^2b^4-ab^5 \end{array}$$

Product is :

EXERCISE 19.

Multiply :

1. $1+a+a^2$ by $1-a$. **2.** $1-a+a^2$ by $1+a$.

3. a^2+ab+b^2 by $a+b$. **4.** a^2-ab+b^2 by $a-b$.

5. $a^4+a^2b^2+b^4$ by a^2-b^2. **6.** $a^4-a^2b^2+b^4$ by a^2+b^2.

7. $7a^2-12ab+9b^2$ by $3a^2-4b^2$.

8. $6x^2-3xy+4y^2$ by $2x^2+5xy-3y^2$.

9. $7x^3-2x^2y+xy^2-4y^3$ by $3x^2-4xy+y^2$.

10. x^2+xy+y^2 by x^2-xy+y^2.

11. $x^3+y^3+z^3-3xyz$ by $x+y+z$.

12. $x^3+y^3-z^3+3xyz$ by $x+y-z$.

13. $1+a+a^3+a^2+a^4$ and $1-a$.

14. $a^2+b^2+c^2-ab-ac-bc$ and $b+a+c$.

15. $y^2+z^2+xy+x^2-yz+xz$ and $x-z-y$.

16. $\frac{1}{2}ab+\frac{1}{3}b^2-\frac{1}{4}a^2$ and $\frac{1}{2}ab+\frac{1}{3}a^2-\frac{1}{4}b^2$.

17. $3a^2+a^6+1-5a-a^3$ and $2a^5-1+a^2-a^3$.

Find the product of:

18. a^3-3a^2+3a-1 and $1-2a+a^2$.
Check the result by putting $a=2$.

19. $3a+2a^2+1$ and $1+2a^2-3a$.
Check the result by putting $a=-3$.

20. $1+x^2+y^2-xy+x+y$ and $y+x-1$.
Check the result by putting $x=2$ and $y=-1$.

12. Continued Product.

Example 19 : Find the continued product of
$a-b$, $a+b$, a^2+b^2 and a^4+b^4.

Sol. Here we multiply $a-b$ by $a+b$, the product by a^2+b^2, and that product by a^4+b^4.

(i) $a-b$
$a+b$
a^2-ab
$+ab-b^2$
$a^2 \quad -b^2$

(ii) a^2-b^2
a^2+b^2
$a^4-a^2b^2$
$+a^2b^2-b^4$
$a^4 \quad -b^4$

(iii) a^4-b^4
a^4+b^4
$a^8-a^4b^4$
$+a^4b^4-b^8$
$a^8 \quad -b^8$

EXERCISE 20.

Find the continued product of:

1. $1+a$, $1-a$, $1+a^2$.
2. $a+1$, $a+2$, $a+3$.
3. $a-2$, $a+3$, $a+5$.
4. $x+a$, $x+b$, $x+c$.
5. $(x+y)$, $(x+2y)$, $(x+3y)$.
6. $(a+b)$, $(a+b)$, $(a+b)$.
7. $(a-b)$, $(a-b)$, $(a-b)$.
8. $(x+y)$, $(x-y)$, $(x^4+x^2y^2+y^4)$.
9. (a^2+ab+b^2), (a^2-ab+b^2), $(a^4-a^2b^2+b^4)$.
10. $(a-b)$, (a^2+ab+b^2), (a^3+b^3), (a^6+b^6).
11. $(a+b+c)$, $(a-b+c)$, $(a+b-c)$, $(-a+b+c)$.

Find the square of:

12. $a+b+c$.
13. $a-b+c$.
14. $2a-3b+4c$.

Find the cube of:

15. $a+b+c$.
16. $a-b+c$.

13. Coefficients of particular terms.

Example 20: Find by inspection, the co-efficient of x^2 in the product of $(3x^2-5x+1)$, $(2x-5)$.

Sol. Here 6 partial products are possible. From them, we have to collect the terms containing x^2.

$2x \times 3x^2$	gives us $6x^3$; hence ignored.
$2x \times (-5x)$	gives us $-10x^2$.
$2x \times 1$	is ignored.
$-5 \times 3x^2$	gives us $-15x^2$.
$-5 \times -5x$	gives us $25x$: ignored.
$-5 \times +1$	gives us a number; ignored.

Thus the terms containing x^2 are
$(-10x^2)+(-15x^2)$ or $-25x^2$.
$\therefore$ The coefficient of x^2 is -25.

NOTE: Usually, only the terms containing the required power of x are taken down.

Example 21: If $A = x+y$ and $B = x-y$, find the value of $(3A+B)(2A-B)$.

Sol. $3A+B = 3(x+y)+(x-y)$
$= 3x+3y+x-y = 4x+2y.$
$2A-B = 2(x+y)-(x-y)$
$= 2x+2y-x+y = x+3y.$
$\therefore (3A+B)(2A-B) = (4x+2y)(x+3y)$
$= 4x^2+2xy+12xy+6y^2$
$= 4x^2+14xy+6y^2.$

EXERCISE 21.

Find by inspection the coefficient of:

1. x^2 in the product of (x^2-5x+2), $(x+4)$.
2. x^2 ,, ,, $(3x^2+2x-5)$, $(7x-4)$.
3. x ,, ,, $(2x^2+x-1)$, $(x-3)$.
4. x ,, ,, $(6x^2+xy-y^2)$. $(2x-y)$.
5. a^2 ,, ,, $(4a^2-7a+3)$, $(3a+5)$.
6. a ,, ,, $(a-4)$, $(2a^2-9a+3)$.
7. x^2 ,, ,, $(1-3x)$, $(4-5x^2+6x)$.

Find the term containing:

8. x^2 in the continued product of $(x-1)$, $(x+2)$, $(x-3)$.
9. x ,, ,, ,, $(2x+5)$, $(3x-4)$, $(2x-1)$.
10. x ,, ,, ,, $(4x-5)$, $(x+1)$, $(x-2)$.
11. y ,, ,, ,, $(1-3y+3y^2)$, $(2+3y)$.
12. If $A = x+y$ and $B = x-y$, find the value of $3A^2+4AB+5B^2$.
13. Find the value of $(A+B)(A-B)$ when $A = (2x-y)$ and $B=(x-2y)$.
14. Evaluate $P^2+2PQ+Q^2$ when $P = (x+a)^2$, $Q = (x-a)^2$.
15. If $x = a^2-bc$, $y = b^2-ca$, $z = c^2-ab$, find the values of (i) $ax+by+cz$, (ii) $bx+cy+az$, (iii) $cx+ay+bz$.
16. Multiply $2x^2+3x+1$ by $3x^2+x+2$, and hence find the product of 231 and 312. [*Hint*: *put* $x=10$]

17. Multiply x^2+2x+3 by $3x^2+2x+1$; hence write down the product 123×321.

18. Multiply $2x^3+x^2+3$ by x^3+3x+4; hence write down the product of 2103 and 1034.

14. Division.

If 37 is divided by 8, the quotient is 4 and the remainder is 5. Arranging in the form of a "Long Division", we have

$$8\overline{)37}\,(\,4 \qquad \text{or} \qquad \text{Divisor})\overline{\text{Dividend}}\,(\,\text{Quotient}$$
$$\underline{\quad\quad}\quad\quad\quad

Similarly, dividing by q is the *same* as multiplying by $\frac{1}{q}$.

Dividing by $\frac{x}{y}$ is the *same* as multiplying by $\frac{y}{x}$.

Hence, *to divide by a quantity is the same thing as to multiply by its reciprocal.*

Thus division is the *inverse* of multiplication.

In a chain of multiplication and division, the operations can be performed in any order.

$$a \times b \div c = a \div c \times b$$

$$a \div b \div c = a \div c \div b \quad \left[= a \times \frac{1}{b} \times \frac{1}{c} \right]$$

15. **Division of a Multinomial Expression by a Simple Expression.**

Just as $(15+12+9) \div 3 = (15 \div 3) + (12 \div 3) + (9 \div 3)$ so, in Algebra, $(a+b+c) \div d = (a \div d) + (b \div d) + (c \div d)$

When a multinomial expression is to be divided by a simple expression—*divide each term of the multinomial by the divisor and take the sum of the partial quotients.*

16. **Rule of Signs.**

$\because (+5) \times (+3) = +15, \quad \therefore (+15) \div (+5) = +3 \ldots$ (i)
$\because (-5) \times (-3) = +15, \quad \therefore (+15) \div (-5) = -3 \ldots$ (ii)
$\because (-5) \times (+3) = -15, \quad \therefore (-15) \div (-5) = +3 \ldots$ (iii)
$\because (+5) \times (-3) = -15, \quad \therefore (-15) \div (+5) = -3 \ldots$ (iv)

In (i) and (iii), the dividend and the divisor have the *same* sign, and the quotient is *positive.*

In (ii) and (iv), the dividend and the divisor have *opposite* signs, and the quotient is *negative.*

Similarly in Algebra,

$(+ab) \div (+a) = +b, \quad (-ab) \div (-a) = +b$
$(+ab) \div (-a) = -b, \quad (-ab) \div (+a) = -b$

In division, like signs produce $+$ *; unlike signs produce* $-$.

Example 22: Divide $-15a^6b^4$ by $5a^3b^2$.

Sol. $(-15a^6b^4) \div (5a^3b^2)$
$= (-15 \div 5) \times (a^6 \div a^3) \times (b^4 \div b^2)$
$= -3a^{6-3}b^{4-2} = -3a^3b^2.$

Example 23: Divide $x^6-5ax^5+3a^2x^4$ by $-2x^4$.

Sol. The quotient $= \dfrac{x^6-5ax^5+3a^2x^4}{-2x^4}$

$$= \frac{x^6}{-2x^4} - \frac{5ax^5}{-2x^4} + \frac{3a^2x^4}{-2x^4}$$

$$= -\tfrac{1}{2}x^2 + \tfrac{5}{2}ax - \tfrac{3}{2}a^2.$$

EXERCISE 22.

Divide :

1. $9x^5$ by $3x^3$.
2. $18a^2b^4$ by $-6ab^2$.
3. $-12a^4b^5$ by $-4a^3b^3$.
4. $-15a^3b^2$ by $10a^2b^2$.
5. $-18a^3x^3b^3y^4$ by $-6a^3x^3b^2y^3$.
6. $-m^4n^3p^4q^3$ by $m^2n^2p^2q^2$.
7. $-14a^3b^5c^4$ by $+2a^2b^2c^2$.
8. $-\frac{3}{5}p^3q^4$ by $-\frac{9}{20}p^2q^3$.
9. $-abcd+axcm$ by $-ac$.
10. $6abxy-4acxz+2abcxyz$ by $12ax$.
11. $-a^2b^2c-ab^2c^2-a^2bc^2$ by $-abc$.
12. $25a^4x^5-20a^3x^4-35a^2x^3$ by $5a^2x^2$.
13. $18a^4b^5x^3-12a^2b^3x^4+6a^3b^4x^5$ by $-6a^2b^3x^3$.
14. $12a^4b^2c-18a^2b^4c-36ab^2c^4$ by $-6ab^2c$.

17. Division by a Compound Expression.

Division of one compound expression by another is analogous to 'Long Division' in Arithmetic, as illustrated below in the division of 483 by 23;

Compact process.

$$23\overline{)483}\,(21$$
$$\underline{46}$$
$$23$$
$$\underline{23}$$

Expanded process.

$$2.10+3\overline{)4.10^2+8.10+3}\,(2.10+1$$
$$\underline{4.10^2+6.10}$$
$$2.10+3$$
$$\underline{2.10+3}$$

If we replace 10 by x, 10^2 by x^2, etc., $483\div23$ becomes $(4x^2+8x+3)\div(2x+3.)$ The full working is given below :

$$2x+3\overline{)4x^2+8x+3}\,(2x+1$$

Line 1. $\underline{4x^2+6x}$

Line 2. $2x+3$

Line 3. $\underline{2x+3}$

Line 4. $\times\quad -$

Explanation: Proceed from left to right. Take the first terms of the dividend and the divisor for trial. $4x^2 \div 2x = 2x$. Put this as the first term of the quotient. Multiply the divisor by $2x$ and write the product below line 1 —as in line 2. Subtract and bring down the next term $+3$—as in line 3. $2x \div 2x = 1$ and this is the second term of the quotient. Multiply the divisor by 1 and put down the product as in line 4 and subtract. There is no remainder.

Example 24: Divide $4x^3+7x^2-3x-15$ by $4x-5$.

$$
\begin{array}{l|l}
4x-5 & 4x^3+7x^2-3x-15 \\
 & \underline{4x^3-5x^2} \\
 & \quad 12x^2-3x \\
 & \underline{12x^2-15x} \\
 & \quad\quad 12x-15 \\
 & \underline{12x-15} \\
 & \quad\quad\quad 0
\end{array}
\qquad
\begin{array}{l}
x^2+3x+3 \\
\\
\\
\

Divide :

5. $2x^2-3x-2$ by $2x+1$.
6. $10x^2-14x-12$ by $2x-4$.
7. $3m^2+m-2$ by $m+1$.
8. $3a^2+10a+3$ by $3a+1$.
9. $6a^2-7a-3$ by $3a+1$.
10. $4p^2-4p-3$ by $2p-3$.
11. $5m^2-17m+6$ by $5m-2$.
12. $-21a^2+a+10$ by $-7a+5$.
13. $1-x+x^2-x^3$ by $1+x^2$.
14. $3a-3a^2+a^3-1$ by $a-1$.
15. $3x^3-2y^3+9xy^2-11x^2y$ by $3x-2y$.
16. $3a^3-a^2b-2b^3-5ab^2$ by $2b+3a$.
17. $2a^4+a^2-3a^3-9+9a$ by $2a^2-3+a$.
18. $4x-1-8x^2-6x^4+11x^3$ by $1-x+3x^2$.
19. $1+q+a^2+a^3+a^4+a^5$ by $1+a^2+a^4$.
20. $3x^5+3x^4+2x^3+1$ by $3x^3-x+1$.

18. **Division with certain terms missing.**

Sometimes, the dividend may not contain some terms. It means that the *coefficients* of such terms are *zero*. In the expression $1-x^3$, x and x^2 are missing. So, $1-x^3$ may be written as $1+0x+0x^2-x^3$. Similarly $x^4+x^2y^2+y^4$ may be written as $x^4+0x^3y+x^2y^2+x0y^3+y^4$. Before commencing the division, it is useful to supply the omissions.

Example 26: Divide $1-a^4$ by $1-a$.

Sol. $1-a^4 = 1+0a+0a^2+0a^3-a^4$

$$
\begin{array}{r|l}
1-a & 1+0+0+0-a^4 \\
 & 1-a \\
\hline
 & +a+0 \\
 & +a-a^2 \\
\hline
 & +a^2+0 \\
 & +a^2-a^3 \\
\hline
 & +a^3-a^4 \\
 & +a^3-a^4 \\
\hline
 & \times \\
\hline
\end{array}
\qquad 1+a+a^2+a^3
$$

The required quotient is $1+a+a^2+a^3$.

Example 27: Divide $a^4+a^2b^2+b^4$ by a^2-ab+b^2

Sol. Dividend $= a^4+0a^3b+a^2b^2+0ab^3+b^4$

$$
\begin{array}{l}
a^2-ab+b^2\big)\,a^4+0+a^2b^2+0+b^4\,\big(\,a^2+ab+b^2 \\
\quad\underline{a^4-a^3b+a^2b^2} \\
\qquad +a^3b+0\quad+0 \\
\qquad\underline{+a^3b-a^2b^2+ab^3} \\
\qquad\qquad +a^2b^2-ab^3+b^4 \\
\qquad\qquad\underline{+a^2b^2-ab^3+b^4} \\
\qquad\qquad\quad \times \\
\end{array}
$$

The quotient is a^2+ab+b^2

16458.

EXERCISE 24.

Divide :

1. x^3+y^3 by $x+y$.
2. x^5+y^5 by $x+y$.
3. x^5-y^5 by $x-y$.
4. x^8-y^8 by $x+y$.
5. x^6-1 by x^3+2x^2+2x+1.
6. x^6-2x^3+1 by x^2-2x+1.
7. x^9+y^9 by x^3+y^3.
8. $x^3+3xy+y^3-1$ by $x+y-1$.

Divide and check the result by putting $x=2$, $y=1$; $a=-1$, $b=-2$:

9. x^6-y^6 by $x^3+2x^2y+2xy^2+y^3$.
10. $x^{12}-2x^6y^6-3y^{12}$ by $x^4-x^2y^2+y^4$.
11. $a^4-\frac{1}{81}b^4$ by $a-\frac{1}{3}b$.
12. $\frac{1}{4}x^2-\frac{1}{9}y^2-\frac{1}{36}z^2+\frac{1}{9}yz$ by $\frac{1}{2}x-\frac{1}{3}y+\frac{1}{6}z$.
13. $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$.
14. $a^3-b^3-c^3-3abc$ by $a^2+b^2+c^2+ab+ac-bc$.
15. $x^3+8y^3-27z^3+18xyz$ by $x-3z+2y$
16. $a^3+8b^3+27c^3-18abc$ by $a^2+4b^2+9c^2-2ab-6bc-3ca$.

19. Division with a Remainder.

As already pointed out, if we denote the dividend by D, the divisor by d, the quotient by Q and remainder by R, we have $\mathbf{D=d\times Q+R}$.

But the complete quotient is $\mathbf{Q+\frac{R}{d}}$

Example 28. Divide $x^4-3x^3-13x^2+12x+4$ by x^2-x+2.

Sol. Arrange the divisor and the dividend according to *descending powers of* x.

$$
\begin{array}{r|l}
x^2-x+2 & x^4-3x^3-13x^2+12x+4 \;\big(\; x^2-2x-17 \\
 & \underline{x^4-\ x^3+\ 2x^2} \\
 & \quad -2x^3-15x^2+12x \\
 & \quad \underline{-2x^3+\ 2x^2-\ 4x} \\
 & \qquad -17x^2+16x+\ 4 \\
 & \qquad \underline{-17x^2+17x-34} \\
 & \qquad\quad -\ x+38
\end{array}
$$

The quotient $= x^2-2x-17$ and the remainder $= -x+38$.

The complete quotient is $x^2-2x-17+\dfrac{-x+38}{x^2-x+2}$.

When the divisor and the dividend are arranged according to ascending powers of x, the division may be *endless* and the answer *different* from the one obtained above.

Arrange the divisor and the dividend according to *ascending powers of* x.

$$
\begin{array}{r|l}
2-x+x^2 & 4+12x-13x^2-3x^3+x^4 \;\big(\; 2+7x-4x^2-7x^3 \\
 & \underline{4-\ 2x+\ 2x^2} \\
 & \quad 14x-15x^2-\ 3x^3 \\
 & \quad \underline{14x-\ 7x^2+\ 7x^3} \\
 & \qquad -\ 8x^2-10x^3+\ x^4 \\
 & \qquad -\ 8x^2+\ 4x^3-4x^4 \\
 & \qquad\quad \underline{\qquad\qquad\qquad} \\
 & \qquad\quad -14x^3+5x^4 \\
 & \qquad\quad -14x^3+7x^4-7x^5 \\
 & \qquad\quad\quad \underline{\qquad\qquad\qquad} \\
 & \qquad\quad\quad -2x^4+7x^5
\end{array}
$$

The quotient $= 2+7x-4x^2-7x^3$ and the remainder is $-2x^4+7x^5$.

The complete quotient is $2+7x-4x^2-7x^3+\dfrac{-2x^4+7x^5}{2-x+x^2}$.

Example 29: Divide $1+x$ by $1-x$ up to 4 terms.

Sol.
$$
\begin{array}{r|l}
1-x & 1+x+0+0+0+\ldots \\
 & 1+2x+2x^2+2x^3+\ldots \\
\hline
 & +2x+0 \\
 & +2x-2x^2 \\
\hline
 & +2x^2+0 \\
 & +2x^2-2x^3 \\
\hline
 & +2x^3+0 \\
 & +2x^3-2x^4 \\
\hline
 & +2x^4+\ldots
\end{array}
$$

The complete quotient is $1+2x+2x^2+2x^3+\dfrac{2x^4}{1-x}$

Example 30: Find a and b in order that $x^4-2x^3+8x^2+ax+b$ may be exactly divisible by x^2+x+5.

Sol.
$$
\begin{array}{r|l}
x^2+x+5 & x^4-2x^3+8x^2+ax+b \\
 & x^4+x^3+5x^2 \quad\quad\quad\

Prove that:

9. $\dfrac{a^2+11a+35}{a+5} = a+6+\dfrac{5}{a+5}$.

10. $\dfrac{x^2-9x+25}{x-4} = x-5+\dfrac{5}{x-4}$.

11. $\dfrac{a^2-13a+45}{a-7} = a-6+\dfrac{3}{a-7}$.

12. $\dfrac{x^2-16x+60}{x-9} = x-7-\dfrac{3}{x-9}$.

13. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$ is the product of two expressions of which one is x^2-2x+1. Find the other.

14. Find the divisor when the dividend is $x^3-6x^2+12x-5$, the quotient $x-3$ and the remainder $x+1$.

15. What must be subtracted from $12a^2+8a-9$, so that it may be exactly divisible by $2a-1$?

16. What must be added to $3a^3-8a^2+7a$, so that it may be exactly divisible by $3a-2$?

17. What must be added to $3a^3+7a^2-ba-5$, so that it may be exactly divisible by a^2+2a-1?

18. Find the condition that x^4-3x^2+m may be exactly divisible by x^2+1?

19. For what value of k is $9a^3-6a^2+3a-k$ exactly divisible by a^2-2a+3?

20. For what value of x will $2x^3+5x^2-mx+4$ be exactly divisible by x^2+2x-1?

CHAPTER III

SIMPLE EQUATIONS

1. **An algebraic statement that two expressions are equal is called an equation.**

Thus (i) $2a-3b = -3b+2a$, (ii) $(x+1)^2 = x(x+2)+1$, (iii) $x+3 = 5$ are all equations.

The part of an equation which lies on the left-hand side of the $=$ sign is called the "*Left-Hand Side*" or briefly L.H.S.; and the part of an equation on the right hand side of the $=$ sign is called the "*Right-Hand Side*" or briefly R.H.S.

In (i) put $a = 3$ and $b = 1$, and simplify.

L.H.S. $= 2\times3-3\times1 = 6-3 = +3$ and

R.H.S. $= -3\times1+2\times3 = -3+6 = +3$.

$\therefore$ L.H.S. $=$ R.H.S.

In (i) put $a = 2$ and $b = 3$ and simplify.

Even then L.H.S. $=$ R.H.S.

This equation is true for *all* values of a and b.

Similarly equation (ii) *i.e.* $(x+1)^2 = x(x+2)+1$ is true for *all* values of x *i.e.* when $x = 0, 1, 2, 3$, etc.

Such equations which are true for *all* values of the letters involved in them are called *Identities*.

On the other hand, equation (iii) $x+3=5$ is true only when $x = 2$ and for no other value of x.

Equations which are true for *particular* values of the unknown quantities involved in them are called "*Conditional Equations*" or merely "*Equations*."

An equation $x-5=0$ is true only when $x=5$. The value 5 is said to satisfy the equation. The process of finding that value of x (or the unknown quantity) which will satisfy the equation is called **"Solving the equation"**, and the value itself is called a root of the equation. Sometimes it is also called a **"solution"** of the equation.

An equation which, when simplified, involves *only one* letter in the first degree only is called a **simple equation.**

2. Methods of Solving Simple Equations.

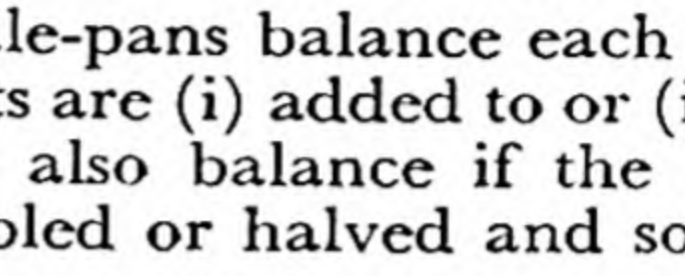
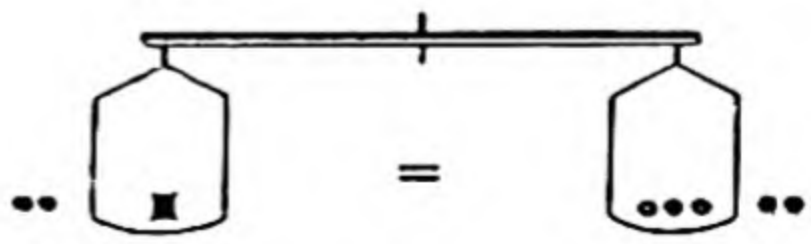
An equation may be compared with a balance. The two scale-pans are the two sides of the equation. The sides of the equation are connected by the sign of equality. The two scale-pans are connected by a beam, and with equal weights in them, the beam is horizontal.

If the contents of the 2 scale-pans balance each other, they will still do so if equal amounts are (i) added to or (ii) removed from both sides. They will also balance if the contents of *both* sides are doubled or trebled or halved and so on.

Let us see how the principle of the balance works in the solution of an equation.

Suppose there are a few packets of equal weights containing rupee-coins. Let one scale-pan contain three full packets and two loose coins ; and let the other pan contain two full packets and 5 rupee-coins and let them balance. Also let x units be the weight of one packet which has to be determined.

We have the equation $3x+2=2x+5$.

Symbolic representation	*Diagrammatic representation*
3 packets and 2 coins balance 2 packets and 5 coins. or $3x+2=2x+5$ Remove 2 packets *i.e.* $2x$ from each side. Now 1 packet and 2 coins balance 5 coins OR $x+2=5$ Again remove 2 coins from each side. Then we have one packet balancing 3 coins. OR $x=3$. The weight of a packet is 3 units.	 

In the above illustration, we have only been removing equal quantities from both sides. On the other hand, if we add equal quantities, or multiply or divide both sides by equal quantities, still they would balance.

Thus, from the analogy of the balance, it can be seen that the solution of equations is based on the following four axioms, namely :

If equals be added to equals, the sums are equal. [Axiom I]

If equals be subtracted from equals, the remainders are equal. [Axiom II]

If equals be multiplied by equals, the products are equal. [Axiom III]

If equals be divided by equals, the quotients are equal. [Axiom IV]

Example 1: Solve the equation $x-7 = 10$.

Sol. Given $x-7 = 10$.

Adding 7 to both sides, we have

$$x-7+7 = 10+7 \qquad \text{[Axiom I]}$$

$$\therefore \quad x = 17.$$

Example 2: Solve the equation $x+9 = 13$.

Sol. Given $x+9 = 13$.

Subtracting 9 from both sides, we have

$$x+9-9 = 13-9 \qquad \text{[Axiom II]}$$

$$\therefore \quad x = 4.$$

Example 3: Solve the equation $\frac{x}{5} = 4$.

Sol. Given $\frac{x}{5} = 4$

Multiplying both sides by 5, we have

$$\frac{x}{5} \times 5 = 4 \times 5 \qquad \text{[Axiom III]}$$

$$\therefore \quad x = 20.$$

Example 4: Solve the equation $3x = 12$.

Sol. Given $3x = 12$.

Dividing both sides by 3, we have

$$\frac{3x}{3} = \frac{12}{3} \qquad \text{[Axiom IV]}$$

$$\therefore \quad x = 4.$$

Example 5: Solve the equation $3y+2 = 17$.

Sol. Given $3y+2 = 17$.

Subtracting 2 from both sides, we have

$3y+2-2 = 17-2$ [Axiom II]

or $3y = 15$.

Dividing both sides by 3, we have

$\frac{3y}{3} = \frac{15}{3}$ [Axiom IV]

$\therefore \quad y = 5$

Example 6: Solve the equation $\frac{z}{4}-6 = 10$.

Sol. Given $\frac{z}{4}-6 = 10$.

Adding 6 to both sides, we have

$\frac{z}{4}-6+6 = 10+6$ [Axiom I]

or $\frac{z}{4} = 16$.

Multiplying both sides by 4, we have

$\frac{z}{4}\times 4 = 16\times 4$ [Axiom III]

$\therefore \quad z = 64$.

EXERCISE 26.

Solve the following equations showing the *full process*:

1. $3x = 18$. **2.** $5x = 35$. **3.** $7x = 21$.

4. $11x = 44$. **5.** $\frac{y}{4} = 7$. **6.** $\frac{y}{6} = 3$.

7. $\frac{y}{8} = 5$. **8.** $\frac{y}{12} = 4$. **9.** $\frac{1}{3}z = 6$.

10. $\frac{1}{5}z = 2$. **11.** $x-5 = 13$. **12.** $x-3 = 12$.

13. $x-9 = 4$. **14.** $x-14 = 3$. **15.** $x-7 = 6$.

16. $x-2 = 9$. **17.** $x+4 = 11$. **18.** $x+6 = 15$.

19. $x+8 = 17$. **20.** $x+10 = 21$. **21.** $2y+3 = 15$.

22. $3z+4 = 22$. **23.** $5x+2 = 27$. **24.** $4x+5 = 21$.

25. $3y-1 = 8$. **26.** $7y-6 = 15$.

27. $6z-3 = 33$. **28.** $5z-7 = 38$.

29. $3x-5-x = 13$ **30.** $5x-9-2x = 12.$

31. $2x+7+x = 10.$ **32.** $4x+3+x = 15.$

33. $6x-2 = 3x+7.$ **34.** $4z+6 = 2z+11.$

35. $5y-1 = 2y+9.$ **36.** $4y+7 = y+17.$

37. $\frac{x}{2}-1 = 2.$ **38.** $\frac{z}{3}+2 = 4.$

39. $\frac{z}{6}+\frac{1}{6} = \frac{5}{6}.$ **40.** $\frac{z}{5}-\frac{1}{5} = \frac{2}{5}.$

41. $\frac{x}{12}+\frac{5}{4} = 2.$ **42.** $\frac{5x}{6}-\frac{1}{4} = \frac{3}{8}.$

43. $5x-8+3x = 12-4x.$ **44.** $17x-3+2x = 7x+18+5x.$

45. $2x-(x+4) = 9.$ **46.** $5y-(3y-2) = 22.$

47. $17x+(8x-11) = 39.$ **48.** $z-(6-4z)+5 = 14.$

49. $7y+(5y-9)-6 = 21.$ **50.** $4x-(x+5)+3 = 0.$

3. **Transposition.**

Consider the equation $2x-7 = 5$ ... ... (i)

Adding 7 to both sides, we have

$$2x-7+7 = 5+7 \quad \ldots \quad \ldots$$

or $$2x = 5+7 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Comparing (i) and (ii), we notice that -7 *disappears* from the L.H.S. and appears on the R.H.S. with its sign *changed* from $-$ to $+$.

Again, consider another equation

$$5x = 15+2x \quad \ldots \quad \ldots \quad \text{(iii)}$$

Subtracting $2x$ from both sides, we have

$$5x-2x = 15+2x-2x$$

or $$5x-2x = 15 \quad \ldots \quad \ldots \quad \text{(iv)}$$

Comparing (iii) and (iv), it looks as if $+2x$ has been *transferred* from the R.H.S. to the L.H.S. and its sign *changed* from $+$ to $-$.

In actual practice, this is what is usually done, and this is allowable also *i.e. a term may transferred from one side of an eqation to the other, but its sign must be changed.*

This process is called the process of **transposition.**

Example 7: Solve $11x-5 = 4x+9$.

Sol. Given $11x-5 = 4x+9$.

Transposing $4x$ and 5, we have

$$11x-4x = 9+5$$

$$\therefore \quad 7x = 14 \quad \text{and} \quad x = 2.$$

[Verification.

Put 2 for x in the above equation.

L.H.S. $= 11\times 2-5 = 17.$

R.H.S. $= 4\times 2+9 = 17.$

$\therefore$ L.H.S. = R.H.S., and the solution is correct.]

Example 8: Solve $5x-18-2x = x+2-3x$.

Sol. Given $5x-18-2x = x+2-3x$

Transposing all the terms containing x to the left-hand side and all other terms to the right-hand side, we have

$$5x-2x-x+3x = +2+18$$

or $$5x+3x-3x = 20$$

or $$5x = 20$$

$\therefore$ $$x = 4.$$

[Verification.

Put 4 for x in the above equation.

L.H.S. $= 5\times 4-18-2\times 4 = -6.$

R.H.S. $= 4+2-3\times 4 = -6.$

$\therefore$ L.H.S. = R.H.S., and the solution is correct.]

NOTE: In practice it is *usual* to bring all the terms containing x to the left-hand side and all other terms to the right-hand side.

Example 9: Solve $4(2x+5)-2(x+1) = 0$.

Sol. Given $4(2x+5)-2(x+1) = 0.$

Removing the brackets, we have

$$8x+20-2x-2 = 0$$

or $$6x+18 = 0.$$

Transposing $+18$, we have

$$6x = -18$$

$\therefore$ $$x = -3.$$

[Verification.

Put -3 for x in the above equation.

L.H.S. $= 4\{2\times(-3)+5\}-2(-3+1)$

$= 4\times(-1)-2\times(-2) = -4+4 = 0.$

R.H.S. $= 0$

$\therefore$ L.H.S. = R.H.S., and the solution is correct.]

Example 10: Solve $(x-2)(4x-1) = 5+(2x-3)^2$.

Sol. Given $(x-2)(4x-1) = 5+(2x-3)^2$

Removing brackets, we have

$$4x^2-x-8x+2 = 5+4x^2-12x+9$$

$$[\because (2x-3)^2 = (2x-3)(2x-3)]$$

Transposing,

$$4x^2-x-8x-4x^2+12x = 5+9-2$$

$$+3x = 12$$

$$x = 4.$$

[Verification.

L.H.S. $= (4-2)(16-1) = 30.$
R.H.S. $= 5+(8-3)^2 = 30.$
L.H.S. $=$ R.H.S. and the solution is correct.]

NOTE: It will be found useful to remember the following hints:

(i) We may change the signs of *all* the terms on both sides of an equation without altering the equality. Thus, in $-x+5 = -8$, if we divide both sides by -1, we get $x-5 = 8$.

(ii) We may interchange the position of the two sides of an equation without changing the signs.

Thus, $11 = x-3$ may be written as $x-3 = 11$.

(iii) We may cancel the same term from both sides of an equation. Thus $2x-5 = 6+x-5$ is the same thing as $2x = 6+x$, and $17x-4+2x = 5x+6+2x$ is the same thing as $17x-4 = 5x+6$.

EXERCISE 27.

In solving the following equations apply the *method of transposition*, and *verify* each solution:

1. $3x-5 = 2x-3$.
2. $4x-1 = 6x+5$.
3. $2x+3 = x-8$.
4. $5x+7 = 2x+3$.
5. $11-2x = 15-x$.
6. $7+3x = 12-2x$.
7. $15+3x = 8+2x$.
8. $9-5x = 13+3x$.
9. $-4+2x = 6-3x$.
10. $12-4x = -5+2x$.
11. $-3y-5 = -7y+1$.
12. $-5+7x = -11-2x$.
13. $3(5x+6) = 4(2-x)$.
14. $9x+20-4x = 45$.
15. $3+3x-4 = 5x-1$.
16. $7x-6+x = 16+3x$.
17. $-5x+2 = 8x-20-4x$.

18. $5x-9+3x-7=0.$
19. $3x-6+5x-12=0.$
20. $4x+20-8(13-x)=0.$
21. $3(x-5)-7(2x+3)=0.$
22. $4(2x-3)+5(x-2)=0.$
23. $2(3x-1)-6(2x-5)=0.$
24. $-7(12+3x)+5(x-2)=0.$
25. $-3(5-4x)+2(3x-1)=0.$
26. $2x-(3-5x)+2=4x-7.$
27. $(4x-3)(2x-1)=(2x+5)(4x-1).$
28. $(4x+5)(3x-2)=(6x-1)(2x-5).$
29. $(x-6)(12x+1)=(3x+2)(4x-1).$
30. $(3x+2)(8x+5)=(6x-1)(4x+7).$
31. $-(x-8)(x+12)=(x+1)(6-x).$
32. $-(7-6x)(3-2x)=-(4x-3)(3x-2).$
33. $(x+3)^2+(4-x)^2=2x^2+x.$
34. $(x+5)^2-(2-x)^2=7.$
35. $(2x+1)^2-8=(2x-1)^2.$
36. $14-5(x-4)(x+8)=188-5(x+3)^2.$
37. $9(x-2)^2+3(x-4)^2=(3x-7)(4x-19)+42.$

38. Find the value of x which will make $5(x-4)+2$ equal to 52.

39. For what value of m will $(2m-3)(m+4)+5-2m^2$ be equal to zero ?

40. For what values of x are the following statements true :

(i) $(12x+5)(x-3)=(3x+4)(4x+3)+29$?
(ii) $(2x-1)(6x+5)=4(3x-2)(x+1)+3$?
(iii) $(2x-1)(3x-4)=(6x-5)(x-1)$?

4. Equations involving Fractions.

If fractions occur in an equation, the denominators can be got rid of, by multiplying both sides of the equation by the *L.C.M.* of the denominators.

Example 11 : Solve $\frac{3x}{4}+\frac{5x}{6}=38.$

Sol. Given $\frac{3x}{4}+\frac{5x}{6}=38.$

The L.C.M. of the denominators 4 and 6 is 12.

Multiplying both sides of the equation by 12, we have

$$12\times\frac{3x}{4}+12\times\frac{5x}{6}=12\times38$$

$$3\times3x+2\times5x=12\times38$$

or $$9x+10x=12\times38$$

or $$19x=12\times38$$

$\therefore$ $$x=\frac{12\times38}{19}$$

i.e. $$x=24.$$

[Verification.

Put 24 for x.

$$\text{L.H.S.}=\frac{3\times24}{4}+\frac{5\times24}{6}=18+20$$

$$=38\quad=\text{R.H.S.}$$

$\therefore$ the answer is correct]

Example 12: Solve $\frac{3x}{5}+\frac{5}{6}=\frac{x}{2}+\frac{3}{4}$.

Sol. Given $\frac{3x}{5}+\frac{5}{6}=\frac{x}{2}+\frac{3}{4}$.

The L.C.M. of 5, 6, 2 and 4 is 60.

Multiplying both sides by 60, we have

$$36x+50=30x+45$$

$$36x-30x=45-50$$

$$6x=-5$$

$$x=-\frac{5}{6}.$$

[Verification.

$\text{L.H.S}=\frac{3}{5}\times(-\frac{5}{6})+\frac{5}{6}=-\frac{1}{2}+\frac{5}{6}=\frac{1}{3}$.

$\text{R.H.S.}=\frac{1}{2}\times(-\frac{5}{6})+\frac{3}{4}=-\frac{5}{12}+\frac{3}{4}=\frac{1}{3}$.

$\therefore$ L.H.S. = R.H.S., and the solution is correct.]

Example 13: Solve $\frac{2x+7}{5}+6=\frac{3x+4}{10}+8$.

Sol. Given $\frac{2x+7}{5}+6=\frac{3x+4}{10}+8$.

The denominators are 5, 10 ; their L.C.M. is 10.

Multiplying both sides by 10, we have

$$10\times\frac{(2x+7)}{5}+10\times6=10\times\frac{(3x\times4)}{10}+10\times8$$

or $$2(2x+7)+60=3x+4+80$$

or $$4x+14+60=3x+4+80$$

Transposing, we have

$$4x-3x=4+80-14-60$$

or $$x=10.$$

[Verification.

Put 10 for x in the equation.

$$\text{L.H.S.}=\frac{27}{5}+6=\frac{57}{5}$$

$$\text{R.H.S.}=\frac{34}{10}+8=\frac{57}{5}$$

L.H.S. = R.H.S.

$\therefore$ Solution is correct.]

Example 14: Solve $x-\dfrac{2x-\cdot3}{\cdot7}=\dfrac{\cdot5-x}{\cdot35}$

Sol. Given $x-\dfrac{2x-\cdot3}{\cdot7}=\dfrac{\cdot5-x}{\cdot35}$

The L.C.M. of the denominators is 3·5.

Multiplying both sides by 3·5, we have

$$3\cdot5x-3\cdot5\times\frac{(2x-\cdot3)}{\cdot7}=3\cdot5\times\frac{(\cdot5-x)}{\cdot35}$$

or $$3\cdot5x-5(2x-\cdot3)=10(\cdot5-x).$$

Removing brackets,

$$3\cdot5x-10x+1\cdot5=5-10x$$

Multiply throughout by 10, to get rid of the decimals.

Then, $$35x-100x+15=50-100x$$

Transposing, $$35x-100x+100x=50-15.$$

or $$35x=35$$

$\therefore$ $$x=1.$$

EXERCISE 28.

Solve the following equations and *verify* each solution :

1. $\frac{x}{2}+\frac{x}{3}=5.$

2. $\frac{x}{4}-\frac{x}{6}=3.$

3. $\frac{x}{2}+\frac{2x}{3}=\frac{7}{2}.$

4. $\frac{5x}{6}-\frac{x}{4}=14.$

5. $\frac{x}{2}-5=7-\frac{5x}{6}.$

6. $\frac{2x}{5}-\frac{3}{4}=\frac{x}{2}-\frac{5}{6}.$

7. $\frac{3}{x}=2.$ [*Hint* : Multiply both sides by x.]

8. $\frac{1}{x}-\frac{1}{6}=\frac{1}{8}.$

9. $\frac{3x+1}{5}=\frac{2x-1}{11}.$

10. $\frac{y-3}{5}+\frac{y-5}{3}=0.$

11. $\frac{3y-2}{7}-\frac{5y-1}{14}=0.$

12. $\frac{x}{2}-3=\frac{3x}{4}-\frac{5x}{6}+7.$

13. $\frac{6x}{5}+\frac{3x}{4}-11=\frac{3x}{20}-18+\frac{5x}{2}.$

14. $\frac{4x-1}{5}=\frac{3x+1}{4}-1.$

15. $\frac{x-1}{14}+\frac{x-2}{21}=\frac{x-3}{7}.$

16. $\frac{2x+3}{11}+2=\frac{2x-1}{22}+\frac{x+3}{6}.$

17. $3+\frac{5x-8}{16}=\frac{x+4}{8}+6\frac{1}{2}.$

18. $12+\frac{x+8}{6}=4+\frac{3x-16}{8}.$

19. $\frac{2x-1}{3}-\frac{1}{4}=x-\frac{x+1}{2}.$

20. $\cdot5x+1\cdot2x=3\cdot4.$

21. $1\cdot32x+\cdot02x=1\cdot19+x.$

22. $\cdot2y+\cdot05=\cdot12+\cdot15y.$

23. $\frac{\cdot24-\cdot5x}{6}=\frac{\cdot3x+1}{4}.$

24. $\frac{\cdot3x-1}{5}=\frac{\cdot6-\cdot2x}{3}.$

25. $1\cdot2x-\frac{\cdot16x-\cdot05}{\cdot5}=\cdot4x.$

26. $\frac{1}{2}(x-2)+\frac{1}{3}(x-3)=\frac{1}{4}(x-4).$

27. $\frac{7x-3}{2}+1=\frac{3x+5}{4}+\frac{7x-2}{3}.$

28. $\frac{2x-1}{2}+\frac{3x-4}{6}=\frac{6+x}{4}-\frac{x+1}{9}$.

29. $\frac{5+3x}{5}-\frac{3x-1}{2}=\frac{3+5x}{3}+\frac{2x+1}{4}$.

30. What value of x will make $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-2$ equal to x ?

5. Formulae.

A formula is a *collection of symbols* expressing in a compact form the relation between certain quantities.

$I=\frac{P\times R\times T}{100}$ is a formula. It shows the relation between the interest I, the principal P, rate % R and the time T.

Here I is expressed in terms of P, R, and T. I is the *subject of the formula.* This formula can be put in *different* forms.

One form is: $P\times R\times T=100\times I$

Making P the subject, it becomes $P=\frac{100\times I}{R\times T}$.

If R is the subject, $R=\frac{100\times I}{P\times T}$.

If T is the subject, $T=\frac{100\times I}{P\times R}$.

Each quantity can be expressed in terms of the others. This is called *changing the subject of a formula.*

Example 15: A rectangular room is l metres long, b metres broad and h metres high. If the area of the four walls is A sq. metres, we know that $A=2h(l+b)$.

Here A is the subject of the formula ; change the formula so that l may become the subject.

Sol. $A=2h\ (l+b)$

or $2h\ (l+b)=A$.

Divide each side by $2h$:

We get $l+b=\frac{A}{2h}$.

Subtract b from both sides:

$$l=\frac{A}{2h}-b.$$

EXERCISE 29.

1. If A sq. cm is the area of a triangle whose base is b cm, and height h cm, $A = \frac{1}{2}bh$.

Make (i) b the subject, (ii) h the subject.

2. If c cm is the circumference of a circle of radius r cm, $c = 2\pi r$. Make r the subject.

3. In a trapezoid the parallel sides are a cm and b cm and the height is h cm. If its area is A sq. cm, $A = \frac{1}{2}(a+b)h$.

Make (i) h the subject; (ii) $(a+b)$ the subject.

4. If the radius of a circle is r metres and its area A sq. metres, $A = r^2\pi$. Make r the subject.

5. If x be increased by $r\%$, the result $y = x\left(1+\frac{r}{100}\right)$

Make r the subject. Find r, if $x = 20$ and $y = 21$.

6. If r cm is the radius of the base of a cylinder, h cm its height and V cubic cm its volume, $V = r^2\pi h$. Make (i) r the subject; (ii) h the subject.

7. Given $v = u+ft$, find (i) f in terms of u, v, t; (ii) t in terms of u, v, f.

8. If a cm is one side of a right-angled triangle, b cm the other side and c cm its hypotenuse, $c^2 = a^2+b^2$. Make a the subject.

9. If F degrees Fahrenheit is the same temperature as C degrees Centigrade, $F = 32+\frac{9C}{5}$; make C the subject, and find C when $F = 50$.

10. If the radius of the base of a circular cone is r cm, its height h cm and volume V cubic cm, $V = \frac{1}{3}\pi r^2 h$.

Make (i) h the subject; (ii) r the subject.

11. If $s = ut+\frac{1}{2}ft^2$, make f the subject.

12. The temperature t, the pressure p and the volume v of a quantity of gas are connected by the formula $\frac{pv}{t+273} = 10$.

Make t the subject, and find t if $p = 3$, $v = 1050$.

6. Symbolical Expression.

The framing of an equation depends upon the ability to express given statements in symbols. Sufficient practice in symbolical expression is therefore necessary to solve problems.

EXERCISE 30.

1. What is the sum of 7 and a ? ... $[7+a]$
2. What is the sum of m and n ?
3. What is the difference between q and 15 ?
4. What is the difference between p and q ?
5. One part of x is 7 ; what is the other ? ... $[x-7]$
6. One part of $2m$ is n ; what is the other ?
7. One part of $2m+n$ is $m+n$; what is the other ?
8. By how much is m greater than 17 ? ... $[m-17]$
9. By how much is m less than 20 ?
10. The sum of two numbers is m and one of them is n ; what is the other ?
11. The sum of two numbers is $2x+19$, one of them is $x+5$; what is the other ?
12. The difference between two numbers is d, the greater is 25 ; what is the other ?
13. What is the selling price if the cost price is Rs. 160 and the profit is Rs. x ?
14. What is the excess of 53 over n ? ... $[53-n]$
15. What is the excess of m over 34 ?
16. What is the excess of p over q ?
17. What is the defect of 19 from 23 ?
18. What is the defect of m from 40 ?
19. What is the defect of x from y ?
20. What number is less than 60 by m ?
21. What number is less than $2m$ by n ?
22. By how much does 21 exceed n ?
23. By how much does $2p$ exceed q ?
24. If n is a factor of 28, what is the other factor ? $\left[\frac{28}{n}\right]$
25. If m is a factor of n, what is the other factor ?
26. If m be divided into 5 equal parts, what is the value of each part ?

27. If the product of two numbers is $2p^2$ and one of them is p, what is the other ?

28. How many times is 8 contained in 48 ?

29. How many times is a contained in 48 ?

30. How many times is m contained in n ?

31. There are three numbers, each equal to a ; what is their product ?

32. What is five times x increased by 15 ? ... [$5x+15$]

33. What is seven times n diminished by 12 ?

34. What is three times p increased by q ?

35. What is six times m diminished by n ?

36. x is multiplied by 4 and 13 is added to the product ; what is the result ?

37. y is doubled and then 5 is added to it ; what is the result ?

(*Fractions*)

38. One-fifth of m is increased by 7 ; what is the result ?

39. One-third of x is diminished by 9 ; what is the result ?

40. Three-fourths of y is increased by 11 ; what is the result ?

41. Five-sixths of m is diminished by 13 ; what is the result ?

(*Consecutive Numbers*)

42. Write down three consecutive numbers of which (i) 15 is the middle one ; (ii) x is the middle one.... [14, 15, 16]

43. Write down three consecutive numbers of which 11 is the least one ; of which k is the least one.

44. Write down three consecutive numbers of which 27 is the greatest one ; of which p is the greatest one.

45. Write down three consecutive numbers of which $2m+3$ is the middle one.

46. Write down three consecutive even numbers of which $2n+4$ is the middle one.

47. Write down three consecutive even numbers of which $2x-6$ is the least one.

48. Write down three consecutive even numbers of which $2x+2$ is the greatest one.

49. Write down three consecutive odd numbers of which $2n-3$ is the least one.

50. Write down three consecutive odd numbers of which $2m+1$ is the middle one.

51. Write down three consecutive odd numbers of which $2n+5$ is the greatest one.

(Digits)

If the digits of a number beginning from the left-hand side be 3 and 7, the number is 37, and its value is $10\times3+7$; when the digits of 37 are reversed, it becomes 73, and its value is $10\times7+3$.

Similarly, if the digits of a number beginning from the left-hand side be x and y, the value of the number $= 10\times x+y = 10x+y$; when these digits are reversed, the value of the number thus formed $= 10\times y+x = 10y+x$.

52. Represent the numbers whose digit in the tens place is 8 and the digit in the units place is n.

53. Represent the number whose digit in the tens place is k and the digit in the units place is 4.

54. If the digits of a number beginning from the left be m and n, what is its value ?

55. In the preceding question, if the digits be reversed, how would you represent that number ?

56. Represent the number whose digit in the tens place is $3p$ and the digit in the units place is $2q$.

(Ages)

57. A man is m years old now ; how old was he 16 years back ? n years back ? ... [$(m-16)$ years]

58. A man is 35 years old now ; how old will he be 12 years hence ? x years hence ?

59. How old will a man be k years hence, if he is now y years old ?

60. How old is a man now who was 17 years of age 13 years ago ? x years ago ?

61. How old is a man now who was p years of age 18 years ago ? n years ago ?

62. A man will be 60 years old in k years ; how old is he now ?

63. A man will be n years old in m years ; how old is he now ?

64. The age of a boy is $(m-5)$ years ; what was his age k years ago ? What will be his age p years hence ?

65. A father is 37 years older than his son. The age of the son is $(m-3)$ years ; what is the age of the father ?

(*Speed*)

66. A man walks at the rate of x km an hour ; (i) how far does he walk in 3 hours ? (ii) in h hours ?

67. A man walks at the rate of x km an hour ; (i) how far does he walk in 4 hours ? (ii) in t hours ?

68. What is the speed per hour of a man who travels (i) 32 km in 8 hours ? (ii) 32 km in x hours ? (iii) m km in t hours ?

69. A train goes at the rate of 28 km per hour ; (i) in what time will it go 280 km ? (ii) m km ?

70. A train goes at the rate of s km per hour ; (i) in what time will it go 250 km ? (ii) m km ?

(*Trade*)

71. The cost price of an article is $3m$ rupees, and the gain is Rs. 5 : find its selling price.

72. The cost price of an article is Rs. $7x$, and the loss is Rs. 9 ; find its selling price.

73. The selling price of an article is Rs. $2n$, and the loss is Rs. 6 ; find its cost price.

74. The price of 7 cows is Rs. 350 ; what is the price of one cow ? of x cows ?

75. If one cow costs k rupees, how many can be bought for r rupees ?

76. What is $12\frac{1}{2}$ per cent of 40 ? of x ?

77. What is m per cent of Rs. 250 ? of Rs. k ?

78. A man buys goods for Rs. x, and sells them at a gain of 7 per cent ; what is the selling price ?

79. A man buys goods for Rs. $3m$, and sells them at a loss of 12 per cent ; what is the selling price ?

80. A man buys goods for Rs. c, and sells them at a gain of p per cent ; what is the selling price ?

7. Solution of Problems.

Problems are solved with the help of equations. So the first step in the solution of problems is the framing of equations. The following procedure may be followed in solving problems.

(i) Read the problem carefully and understand what is given and what is required.

(ii) Represent the *required quantity* by the symbol x or y or some other letter.

(iii) State the conditions of the problem in *symbolic* language.

(iv) Obtain two expressions which represent the *same* thing and put one equal to the other.

(v) Solve the equation thus formed.

(vi) Verify the solution.

Example 16: The excess of twice a certain number over 9 is 15 ; find the number.

Sol. Let the required number be x.

Twice that number $= 2x$

The excess of twice the number over $9 = 2x - 9$.

As this excess is equal to 15, we have

$$2x - 9 = 15$$

or $$2x = 24$$

and $$x = 12.$$

The required number is 12.

Verification

[$2 \times 12 - 9 = 15$

Solution is correct.]

Example 17: One number is greater than another by 11 and their sum is 49. Find the numbers.

Sol.

I	II
Let the smaller number be x.	Let the greater number be y.
Then the other number is $x+11$.	Then the other number is $y-11$.
As their sum is equal to 49, we have	As their sum is equal to 49, we have
$x+(x+11)=49$	$y+(y-11)=49$
or $2x+11=49$	or $2y-11=49$
or $2x=38$	or $2y=60$
$\therefore$ $x=19$	$\therefore$ $y=30$
The smaller number is 19 and the greater number is $19+11$ *i.e.* 30	The greater number is 30 and the smaller number is $30-11$ *i.e.* 19.

Verification

[$30-19=11$, the numbers differ by 11.
$30+19=49$, the sum of the numbers $=49$.
The solution is correct.]

Example 18: Divide Rs. 64 between A and B, so that three times A's share may be greater than four times B's share by Rs. 10.

Sol. Let A's share be Rs. x.
Then B's share is Rs. $(64-x)$.
$\therefore$ Three times A's share $=$ Rs. $3x$
and four times B's share $=$ Rs. $4(64-x)$

According to the condition, $3x$ is greater than $4(64-x)$ by 10.

$$3x=4(64-x)+10$$
or $$3x=256-4x+10$$
or $$3x+4x=256+10$$
or $$7x=266$$
or $$x=38.$$

$\therefore \quad 64-x = 64-38 = 26.$

$\therefore$ A's share = Rs. 38 and B's share = Rs. 26.

Verification

[Since 3×38 is greater than 4×26 by 10 and $38+26=64$, the answer is correct.]

Example 19: I thought of a number, doubled it, then added 7, the result was 31 ; what was the number thought of ?

Sol. Let x be the number thought of ; then the successive steps would be :

$x,\ 2x,\ 2x+7,$

Then $2x+7 = 31$

$\therefore \quad 2x = 24$

$\therefore \quad x = 12.$

The number thought of is 12.

Verification

[$12\times 2+7 = 31$, the answer is correct.]

EXERCISE 31.

1. The sum of two numbers is 83, and one of them exceeds the other by 17 ; find them.

2. If 28 be added to five times a number, the result is 78 ; find the number.

3. If 7 be taken away from four times a number, the result is 69 ; find the number.

4. If 9 be added to one-third of a number, the result is 19 ; find the number.

5. If 11 be taken away from one-fifth of a number, the result is 23 ; find the number.

6. Think of a number, double it, and add 15. If the result is 71, what was the number thought of ?

7. Think of a number, divide it by 4, and add 9 to it ; the result is 15. What was the number thought of ?

8. Think of a number, take away 5 from it, multiply the result by 4, and add 10 to it ; it becomes equal to 90. Find the original number.

9. A is twice as old as B, and the sum of their ages is 72 years ; find their ages.

10. *A* is three times as old as *B*, and the difference of their ages is 30 years ; find their ages.

11. One angle is greater than another by 32°, and their sum is 106° ; find them.

12. In an isosceles triangle, the vertical angle is one-half of each of the base angles. Find each angle.

13. One angle is one-fifth of another, and their difference is 20° ; find them.

14. Divide Rs. 96 between *A* and *B*, so that four times *A*'s share may be greater than three times *B*'s share by Rs. 13.

15. Divide Rs. 150 between *A* and *B*, so that five times *A*'s share may be less than six times *B*'s share by 9.

16. Divide Rs. 81 among *A*, *B* and *C*, so that *B* may get Rs. 7 more than *A*, and *C* Rs. 6 less than twice *A*'s share.

17. Divide Rs. 340 among *A*, *B* and *C*, so that *B* may get Rs. 30 more than *A* and *C* Rs. 50 less than twice *A*'s share.

18. Divide Rs. 500 among *A*, *B* and *C*, so that *B* may get Rs. 60 more than one-half of *A*'s share and *C* may get Rs. 90 more than one-fourth of *A*'s share.

8. Problems on Fractional parts.

Example 20: Divide 180 into two parts, so that one-third of one part may be greater than one-fourth of the other by 18.

Sol. Let x be one of the parts ; then $180-x$ would be the other part.

One-third of the first part $= \frac{x}{3}$.

One-fourth of the second part $= \frac{1}{4}(180-x)$.

According to the condition given in the problem, we have

$$\frac{x}{3} = \frac{1}{4}(180-x)+18.$$

Multiplying both sides by 12, we have

$$4x = 3(180-x)+(12\times 18)$$

or $$4x = 540-3x+216$$

or $$7x = 756$$

$\therefore$ $$x = 108,$$

and $$(180-x) = 180-108 = 72 ;$$

The two parts are 108 and 72.

Verification

[Since $\frac{1}{3} \times 108 = 36$, and $\frac{1}{4}$ of $72 = 18$, also 36 is greater than 18 by 18, the solution is correct.]

EXERCISE 32.

1. What is the number whose one-half, one-third, and one-fourth added together give us 65 ?

2. What is the number whose sixth part exceeds its seventh part by 10 ?

3. What is the number whose fourth part falls short of its third part by 5 ?

4. Divide 81 into two parts, so that five-sixths of the smaller may exceed seven-fifteenths of the larger by 9.

5. A post is one-sixth of its length in mud, one-fourth of its length in water, and 3·5 metres above the water ; find its length.

6. A man goes one-half of his journey by a railway train, one-fourth of it by a motor-car, one-sixth of it on a horse, and the remaining 10 km on foot ; find the total length of the journey.

7. Out of a cask of wine one-seventh had leaked away, 6 litres were sold, and then the cask was three-fourths full : find the capacity of the cask.

8. A man leaves one-fourth of his property to his wife, one-sixth of it to each of his two sons, one-eighth of it to each of his two daughters, and the rest amounting to Rs. 1,200 for a school ; find his total property.

9. The difference between two numbers is 16, and one-sixth of one is less than one-fourth of the other by 5 ; find the numbers.

10. The sum of two numbers is 678. When the greater is divided by the smaller, the quotient is 5 and the remainder is 6. Find the numbers.

11. The sum of two numbers is 1064. When the greater is divided by the other, the quotient is 4 and the remainder is 14. Find the numbers.

12. The difference between two numbers is 431. When the greater is divided by the smaller, the quotient is 3 and the remainder is also 3. Find the numbers.

9. Problems on Age.

Example 21 : *A* is 24 years older than *B* ; 10 years back *A*'s age was five times the age of *B* ; find their ages.

Sol. Let *A*'s age be x years ; then *B*'s age is $(x-24)$ years. 10 years ago, *A*'s age was $(x-10)$ years, and *B*'s age $(x-24-10)$ years *i.e.* $(x-34)$ years.

According to the condition given in the problem, we have

$$x-10 = 5(x-34)$$

or $$x-10 = 5x-170$$

or $$x-5x = -170+10$$

or $$-4x = -160$$

$\therefore$ $$x = 40 \text{ and } x-24 = 40-24 = 16$$

i.e. *A*'s age = 40 years, *B*'s age = 16 years.

Verification

[(i) $40-16 = 24$
(ii) $(40-10)$ is five times $(16-10)$
The conditions are satisfied.
$\therefore$ The answer is correct.]

EXERCISE 33.

1. *A*'s age is three-eighths of *B*'s age ; the difference of their ages is 15 ; find their ages.

2. A father is 20 years older than his son ; 12 years back the age of the father was six times than that of his son ; find their ages.

3. A father is 24 years older than his son ; 15 years hence the father will be $\frac{5}{3}$ times as old as his son ; find their ages.

4. What is *A*'s present age, if he is now three times as old as *B*, and was 5 times as old as *B* 8 years back ?

5. A father is twice as old as his son ; 14 years ago he was three times as old as his son ; find their ages.

6. *A* is 10 years older than *B* ; 12 years ago, five-sevenths of *A*'s age exceeded five-eighths of *B*'s age by 10 ; find their ages.

7. A father is 26 years older than his son ; 16 years hence three-fifths of the father's age will exceed two-thirds of his son's age by 14 ; find their ages.

10. Problems on Consecutive Numbers.

Remember that consecutive odd numbers differ by 2 and also consecutive even numbers differ by 2.

Example 22. Find three consecutive even numbers whose sum is 96.

Sol. Let $2n$ be the middle one of 3 consecutive even numbers,
then $2n-2$ = the least one of the 3 consecutive even numbers,
and $2n+2$ = the greatest one of the 3 consecutive even numbers,
Since their sum is given to be equal to 96, we have

$$(2n-2)+2n+(2n+2) = 96$$
or $$6n = 96$$
or $$n = 16.$$
$\therefore$ $2n-2 = 30$, $2n = 32$, and $2n+2 = 34$
The numbers are 30, 32, and 34.

Example 23. Find two consecutive odd numbers so that one-third of the greater may exceed one-fifth of the smaller by 8.

Sol. Let $2n+1$ be the greater of the consecutive odd numbers,
then $2n-1$ = ,, smaller ,, ,, ,, ,,
According to the condition given in the problem, we have

$$\tfrac{1}{3}(2n+1) = \tfrac{1}{5}(2n-1)+8$$
Multiplying by 15, $$5(2n+1) = 3(2n-1)+15\times 8$$
or $$10n+5 = 6n-3+120$$
or $$4n = 112$$
or $$n = 28.$$
$\therefore$ $2n+1 = 57$, and $2n-1 = 55$
The two consecutive numbers are 55 and 57.

EXERCISE 34.

1. Find three consecutive numbers whose sum is 87.
2. Find three consecutive even numbers whose sum is 48.
3. Find three consecutive odd numbers whose sum is 81.

4. Find two consecutive numbers such that one-third of the greater exceeds one-fifth of the smaller by 7.

5. Find two consecutive even numbers such that one-eighth of the greater may exceed one-tenth of the other by 3.

6. Find two consecutive odd numbers such that one seventh of the greater may fall short of one-fifth of the smaller by 4.

7. Find two consecutive numbers such that seven-eighths of the greater may exceed five-sixths of the smaller by 6.

8. Find two consecutive odd numbers such that two thirds of the greater may exceed three-fifths of the smaller by 9.

11. Problems on Digits of Numbers.

If x, y are the digits of a number, and if x is in the tens place, the *actual number* is $10x+y$, and the *sum* of the digits is $x+y$.

If the digits are *reversed*, the number becomes $10y+x$.

Example 24. There are two digits in a number, the digit in the units place being 8. If 9 be added to the number, the order of the digits is reversed ; find the number.

Sol. Let the digit in the tens place be x

The digit in the units place is 8.

$\therefore$ The number $= 10 \times x+8$.

When the order of the digits is reversed, the value of the new number $= 10 \times 8+x$.

According to the condition given in the problem, we have

$$(10x+8)+9 = 10 \times 8+x$$

or $$10x+17 = 80+x$$

or $$9x = 63$$

$$x = 7.$$

Thus the required number $= 78$.

Verification

[Since $78+9 = 87$, and we get 87 by reversing the digits of 78, the answer is correct.]

EXERCISE 35.

1. A number consists of two digits ; the sum of the digits is 12. If 36 be added to the number, the digits are reversed. Find the number.

2. A number consists of two digits ; the digit in the units place is 5. If 27 be added to it, the digits are reversed. Find the number.

3. A number consists of two digits whose sum is 8. If 54 be subtracted from it, the digits are reversed. Find the number.

4. A number consists of two digits, the digit in the tens place being 5. If 18 be subtracted from it, the digits are reversed. Find the number.

5. A number consists of two digits ; the digit in the units place is one-third of the digit in the tens place. If the digits be reversed, the new number falls short of the original by 36. Find the number.

6. A number consists of two digits, the digit in the units place is one-half of the digit in the tens place. If 9 be subtracted from the number, the digits are reversed. Find the number.

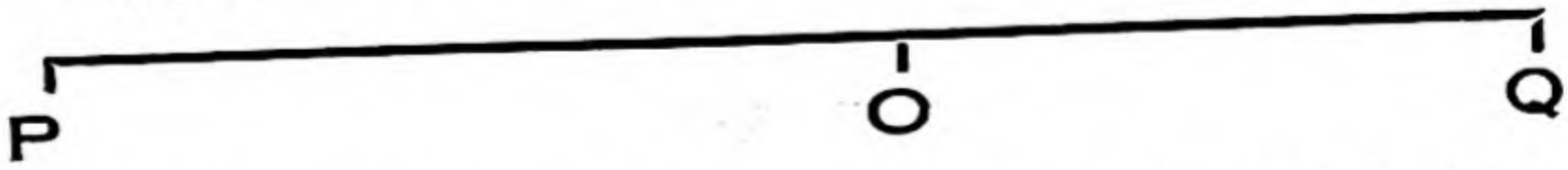
7. Reverse the digits of a number, it will be three-eighths of what it was before ; also the difference between the digits is 5. Find the number.

8. There is a number of two digits whose difference is 2. If it be increased by three times the sum of its digits, the digits are reversed. Find the number.

12. Problems on Time and Distance.

Example 25 : Two trains with speeds 32 km and 24 km per hour respectively, starting at the same time, run in opposite directions between two stations P and Q, 336 km apart, one going from P to Q and the other from Q to P. Find when and where they will meet.

Sol. Suppose that the two trains meet at O after x hours.



Then the distance travelled by the first train in x hours $= 32x$ kilometres and the distance travelled by the second train in x hours $= 24x$ kilometres.

The sum of these distances is equal to PQ or 336 km.

$$\therefore \quad 32x+24x = 336$$
$$\text{or} \quad 56x = 336$$
$$\text{or} \quad x = 6.$$

Hence the two trains meet after 6 hours and at a distance of 6×32, or 192 km from P.

Example 26: A and B are travelling on the same road in the same direction, A with the speed of 5 km an hour and B in a motor-car with the speed of 60 km an hour. A is 20 km. ahead of B. Find when and where B will overtake A.

Sol. Suppose B overtakes A after x hours.
Then the distance travelled by A in x hours $= 5x$ kilometres and the distance travelled by B in x hours $= 60x$ kilometres.
According to the condition given in the problem, we have

$$60x-20 = 5x$$
$$\text{or} \quad 55x = 20$$
$$\text{or} \quad x = \tfrac{4}{11}.$$

Hence B overtakes A after $\frac{4}{11}$ hour or $21\frac{9}{11}$ minutes, and at a distance of $\frac{4}{11} \times 60$, or $21\frac{9}{11}$ km from his own starting-point.

EXERCISE 36.

1. A train running at the rate of 24 km an hour leaves Amritsar for Jullundur at 7 p.m., and another train leaves Jullundur for Amritsar at 8 p.m. at the rate of 32 km an hour. Find the time when the two trains meet, the distance between Amritsar and Jullundur being 50 km.

2. There are two ports P and Q, the distance between them being 5625 km. A steamer sails from P to Q with the speed of 240 km a day ; two days after, another steamer starts from Q and sails towards P with the speed of 250 km a day. Find when and where they meet.

3. A sets out for a walk at the rate of 3·5 km an hour. 3 hours afterwards, B cycles after him at the rate of 9 km an hour. When will B overtake A ?

4. A and B travel in opposite directions from two towns, 135 km apart, and meet in 15 hours ; if A goes twice as fast as B, what will be the speed of B ?

5. A train with the speed of 32 km an hour starts from a station $\frac{1}{2}$ hour after a goods train, and overtakes it in 50 minutes. Find the speed of the goods train.

6. A man walked to the top of a hill at the rate of 2·5 km an hour and down the hill at the rate of 3·5 km an hour. The journey took him 4 hours in all ; find the length of his journey.

7. A man has $7\frac{1}{3}$ hours at his disposal ; how far can he go in a motor-car, running with the speed of 36 km an hour so as to return home in time in a tonga running with the speed of 8 km an hour ?

8. A and B start at the same time from two towns, 39 km apart, and meet in 4 hours and 20 minutes. If A walks 1 km an hour faster than B, find where he meets B.

9. A train takes 8 hours to travel from P to Q. Had it gone 2 km an hour faster, it would have taken $\frac{1}{2}$ hour less to reach Q. Find its speed and the distance between P and Q.

10. A and B travel along the same road. A on foot with the speed of 5 km an hour, and B in a motor-car with the speed of 60 km an hour. A is 10 km ahead of B ; find when and where B will overtake A.

13. Problems on Trade and Investment.

Example 27 : A person invested Rs. 880 in buying sheep and goats at the rate of Rs. 12 and Rs. 16 each respectively. If the total number of sheep and goats be 65 find the number of sheep and the number of goats.

Sol. Let the number of sheep purchased be x,
then the number of goats purchased $= 65-x$,
$\therefore$ the price of the sheep $=$ Rs. $12x$,
and the price of the goats $=$ Rs. $16(65-x)$.

According to the problem, we have

$$12x+16(65-x) = 880$$
or $$12x+1040-16x = 880$$
or $$-4x = -160$$
$\therefore$ $$x = 40$$

The number of sheep $= 40$
and number of goats $= 65-x = 65-40 = 25$.

Verification

[The price of the sheep = Rs. 40×12 = Rs. 480 ;
The price of the goats = Rs. 25×16 = Rs. 400
The total cost = Rs. 480 + Rs. 400 = Rs. 880
$\therefore$ The answer is correct.]

Example 28: The population of a town is 15,000 ; the annual death-rate of males is $4\frac{1}{2}\%$ and that of females is $5\frac{1}{2}\%$. The total number of deaths during the year is 745 ; find the number of males and females in the town.

Sol. Let the number of males in the town be x,
then the number of females in the town $= 15000 - x$.
Annual number of males deaths

$$= x \times \frac{4\frac{1}{2}}{100} = \frac{9x}{200}.$$

Annual number of female deaths

$$= (15000 - x) \times \frac{5\frac{1}{2}}{100} = \frac{11(15000 - x)}{200}.$$

According to the condition given in the problem, we have

$$\frac{9x}{200} + \frac{11(15000 - x)}{200} = 745.$$

Multiplying both sides by 200, we have

$$9x + 11(15000 - x) = 149000$$
or $$9x + 165000 - 11x = 149000$$
or $$2x = 16000$$
or $$x = 8000,$$ (the number of males),
$$15000 - x = 15000 - 8000 = 7000,$$ (the number of females).

The students should verify the answer.

EXERCISE 37.

1. Two pieces of cloth, measuring 80 metres cost together Rs. 144. One piece was bought at Rs. 1·50 per metre and the other at Rs. 2 per metre. Find the length of each.

2. If 8 kg. of tea and 12 kg. of coffee together cost Rs. 154 and a kilogram of tea cost 50 paise more than a kilogram of coffee, find the cost of each per kg.

3. A person bought sugar of two different kinds, and paid in all Rs. 74, the better one at the rate of Rs. 1·60 per kg and the other at Rs. 1·40 per kg. ; the total amount of both kinds of sugar was 50 kg. Find the number of kilograms of each kind.

4. A person had Rs. 9,900, a part of which he lent out at 8% and the rest at 6% ; the interest amounted to Rs. 702. How much was lent out at 8% ?

5. A sum of Rs. 3,200 was lent out at simple interest, partly at 5% and partly at $6\frac{1}{2}$% per annum. The total annual interest amounted to Rs. 190. How much was lent out at each rate ?

6. A person had a capital of Rs. 4,600, a part of which he invested in a business yielding 10% annual profit, and the rest he lent out at 8% per annum ; the whole annual income amounted to Rs. 416. Find how much he invested in the business.

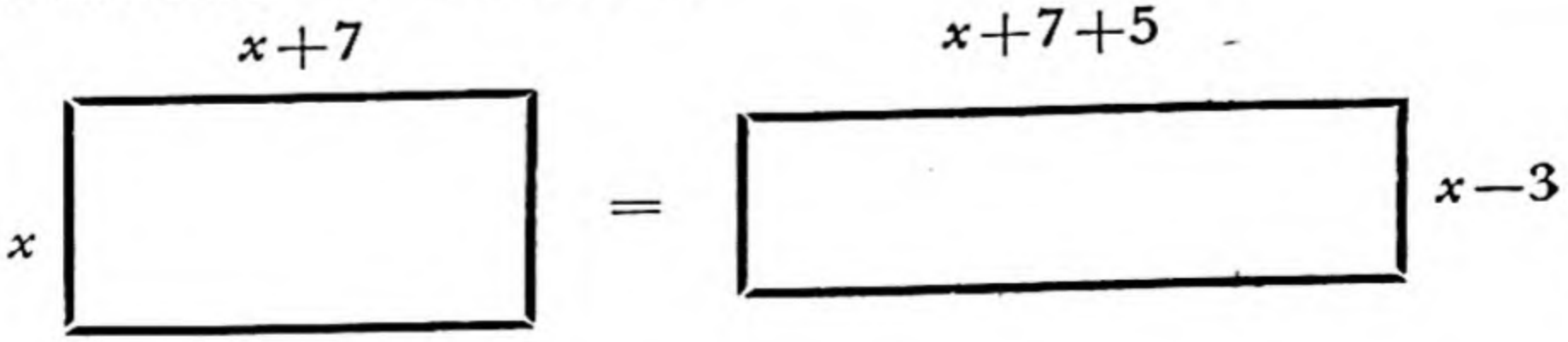
7. The population of a town is 6,600 ; the annual death-rate of males is $3\frac{1}{2}$% and that of females is $3\frac{3}{4}$% ; the total number of deaths during the year is 239. Find the number of males and females in the town.

8. A tradesman bought an equal number of goats of two kinds, one at Rs. 12 each and the other at Rs. 15 each. If he had spent his money equally on the two kinds, he would have had one goat more. How many did he buy of each kind ?

14. Problems on Rectangles, Coins, etc.

Example 29 : The length of a room exceeds its breadth by 7 m. If the length be increased by 5 m and the breadth be diminished by 3 m, the area remains unaltered. Find the dimensions of the room.

Sol.



In the first case, let the breadth of the rectangle be x metres.
Then the length $= (x+7)$ metres

The number of square metres in the area $= x(x+7)$

In the second case, the length $= (x+7)+5$ metres
$= (x+12)$ metres
and the breadth $= (x-3)$ metres
Then the number of square metres in the area $= (x+12)(x-3)$

According to the condition given in the problem, we have

$$x(x+7) = (x+12)(x-3)$$
$$\text{or } x^2+7x = x^2+12x-3x-36$$
$$\text{or } x^2-x^2+7x-12x+3x = -36$$
$$\text{or } -2x = -36$$
$$\therefore x = 18 \text{ and } x+7 = 18+7 = 25$$
$$\therefore \text{the length} = 25 \text{ m and breadth} = 18 \text{ m.}$$

Example 30: Divide 36 into 4 parts such that if the first be increased by 2, the second be diminished by 2, the third be multiplied by 2, and the fourth divided by 2, the result in each case is the same.

Sol. Let x be the result we get in each case.

(First part $+2$) $= x$, $\therefore$ first part $= x-2$.
(Second part -2) $= x$, $\therefore$ second part $= x+2$.
(Third part $\times 2$) $= x$, $\therefore$ third part $= \frac{x}{2}$.
(Fourth part $\div 2$) $= x$, $\therefore$ fourth part $= 2x$.

Since the sum of the parts is equal to the whole,

$$(x-2)+(x+2)+\frac{x}{2}+2x = 36$$
$$\text{or } \frac{9x}{2} = 36$$
$$\text{or } x = 8.$$

Hence the first part $= x-2 = 8-2 = 6$.
the second part $= x+2 = 8+2 = 10$.
the third part $= \frac{x}{2} = \frac{8}{2} = 4$.
and the fourth part $= 2x = 2\times 8 = 16$.

Example 31 : A bag contains Rs. 450 in rupees and 25-paise coins. The amount of the latter is double that of the former. Find the total number of coins.

Sol. Let the amount of the rupee coins be Rs. x.

Then the amount of the 25-paise coins = Rs. $2x$.

By the question, $x+2x = 450$

or $3x = 450$

$\therefore$ $x = 150$

The value of the rupees = Rs. 150 and the value of the 25-paise coins = Rs. 300

$\therefore$ number of rupee coins = 150 ;

number of 25-paise coins $= 300 \div \frac{1}{4} = 1200$

$\therefore$ Total number of coins $= 150+1200 = 1350$.

EXERCISE 38.

1. Find the length of the side of a square whose area is exactly equal to the area of a rectangle whose sides are 225 m and 289 m.

2. The length of a rectangular room exceeds its breadth by 9 m. If the length be increased by 7 m and the breadth diminished by 5 m, the area remains the same. Find the dimensions of the room.

3. The length of a rectangular room exceeds its breadth by 6 m. If the length be increased by 9 m and the breadth diminished by 3 m the area is increased by 15 sq. m. Find its dimensions.

4. The length of a room exceeds its breadth by 12 m. If the length be diminished by 6 m and the breadth increased by 2 m, the area is diminished by 48 sq. m. Find its dimensions.

5. Divide 180 into four parts such that if the first part be increased by 5, the second diminished by 5, the third multiplied by 5 and the fourth divided by 5, the result in each case is the same.

6. Divide 155 into four parts such that if the first part be diminished by 4, the second be increased by 5, the third be divided by 2, and the fourth be multiplied by 3, the result is the same.

7. Divide 81 into four parts such that if the first part be diminished by 5, the second be increased by 1, the third be multiplied by 2, and the fourth be divided by 3, the result is the same.

8. A bag contains Rs. 325 in rupees and 50-paise coins. If the amount of the latter be less than that of the former by Rs. 15, how many coins are there of each kind ?

9. A purse contains three times as many 25-paise coins as 10-paise coins. The value of all the coins is Rs. 17. How many coins are there of each kind ?

10. A certain sum of money is to be distributed amongst a number of boys ; on calculation it is found that if Rs. 4 be given to each, Rs. 13 would be left over, and if Rs. 5 be given to each, Rs. 15 would be wanting. Find the number of boys and the sum.

SECTIONAL REVISION I

PAPER 1

1. Simplify $3x-5y-x-3y+2x$.

2. Find the value of $2x^3-3x^2+x-5$ when $x = -2, -1, 0$.

3. Divide $x^4-x^2y^2-x^2z^2+y^2z^2$ by $x^2-xy+xz-yz$.

4. Prove by different methods that $6a-2(8a-3a) = -4a$.

5. Solve the equation $(3x+20)-(x-3x+2) = 21$.

6. A number consists of two digits whose difference is 4. If it be increased by three times the sum of its digits, the digits are reversed ; find the number.

PAPER 2

1. Subtract $b\{a-(b+c)\}$ from the sum of $a\{a-(c-b)\}$ and $c\{a-(b-c)\}$.

2. What must be added to $8x^3-12x^2-16x+21$ to make it exactly divisible by $2x-3$?

3. Simplify $\frac{a}{6}-\left[\frac{3b-4a}{6}-\left\{b-\frac{a-3b}{6}-\left(3b-\frac{4a+b}{3}\right)\right\}\right]$.

4. Find the continued product of

$$(a-b),\ (a^2+ab+b^2),\ (a^3+b^3).$$

5. Solve the equation $\frac{5}{3}(x-4)-\frac{7}{3}(2x-6) = 20$.

6. A father is 21 years older than his son ; 12 years hence three-fourths of the father's age will exceed two-thirds of the son's age by 18 years ; find their ages.

PAPER 3

1. A person walks $3a-2b+c$ km east ; he then walks $2a-b-3c$ km west, again walks $5a-b-4c$ km east. How far is he then from the starting-point ?

2. Remove the brackets and simplify $7(x-2y)-5(x-3y)$.

3. Divide $1+x$ by $1-2x+2x^2$ up to four terms.

4. If $x = 2a-3b$ and $y = 3a-2b$, find the value of $(2x-3y)(x-2y)$.

5. Solve $\frac{5}{8}(x-2)-\frac{3}{8}(2x+1)+2 = 0$.

6. A and B are travelling on the same road in the same direction, A with the speed of 4 km an hour and B with the

speed of 4·5 km an hour. *B* gives *A* a start of 40 minutes; find when and where *B* will overtake *A*.

PAPER 4

1. Simplify $3a-[a-b-2\{a-b+c-(a-b-\overline{c-a})\}+a]$.

2. The product of two expressions is $x^4+4x^3+3x^2+2x-1$, and one of them is x^2+x+1; find the other.

3. Multiply x^4+x^2+1 by x^2-1.

4. Find the value of $\frac{2a-b}{a+b}+\frac{2b-c}{b+c}+\frac{2c-a}{c+a}$ when $a=2$, $b=a+2$ and $c=a-2$.

5. Prove diagrammatically:
(i) $(a+b)\ c=ac+bc$, (ii) $(a-b)\ c=ac-bc$,
(iii) $(a+b)\ (c+d)=ac+ad+bc+bd$.

6. Solve the equation $\frac{5(3-2x)}{4}=3(x+1)-\frac{3x+8}{6}$.

PAPER 5

1. Distinguish between (i) an index and a power,
(ii) an identical equation and a conditional equation.

2. Find the value of
(i) $(+1)^{2n}$, (ii) $(+1)^{2n-1}$, (iii) $(-1)^{2n}$, (iv) $(-1)^{2n+1}$.

3. Divide 1 by $1+a+a^2$ up to four terms.

4. Find the co-efficient of x^3 in the product of $3x^2-2x-1$ and $2x^2+3x+1$.

5. Find the continued product of (a^2+a+1), (a^2-a+1) and (a^4-a^2+1).

6. *A* cycles at the rate of 12 km an hour from *P* to *Q* and returns at the rate of 8 km an hour. *B* cycles both ways at the rate of 10 km an hour, and takes 15 minutes less than *A*. Find the distance between *P* and *Q*.

PAPER 6

1. Find the co-efficient of x in the continued product of $(x-a)$, $(x-2b)$, $(x-2c)$.

2. Multiply x^3-12x^2+5x-7 by $x^4-3x^3+6x^2-4x+8$, and arrange the product in ascending powers of x.

3. If $x = am^2+5bm+5c$ and $y = am^2-6bm-6c$, find the value of $6x+5y$.

4. Simplify $(x-a)(x-b)(x-c)$
$$-[bc(x-a)-\{(a+b+c)x-a(b+c)\}x].$$

5. Solve the equation $(x-3)(x-5)+1 = (x+1)(x-2)+4$.

6. A travels from P to Q, a distance of 60 km, and back again, at the rate of 12 km an hour. On his way back he meets B who travels at the rate of 8 km an hour and who started at the same time from P. Find the distance from P of the place where they meet.

PAPER 7

1. (i) From the formula $D = d \times Q+R$, find the value of R when $D = 895$, $d = 25$, $Q = 35$.

(ii) One factor of $27a^4+11a-10$ is $3a^2-a+2$, find the other.

2. Add $a-3b+4c$, $3a+2b-5c$, $4a-7b+3c$, and multiply the result by the difference between $9a+6c$ and $10a-b-5c$.

3. Simplify $(5x+4y+1)(5x+4y-1)+(3x-5y)(3x+5y)$
$$-(4x-3y)(4x+3y)+1.$$

4. What must be added to $a^3-5a(a-3)-1$ to make it equal to $a^3+5a(a+3)+1$?

5. Solve the equation $(x+6)^2-(5-x)^2 = 18x+7$.

6. A person invested Rs. 5,800 at simple interest, partly at 4% and partly at 5% per annum. The total annual interest was Rs. 267. How much was lent at 4%?

PAPER 8

1. Prove that $x(x-1)(x-2)(x-3)+1 = (x^2-3x+1)^2$.

2. If $x = m(p+q)$ and $y = n(p-q)$, find the value of
$$\frac{x}{m}+\frac{y}{n} \text{ and } \frac{x}{m}-\frac{y}{n}.$$

3. Find the dividend when the quotient is a^2+a+2, the divisor is a^2-3a+4, and the remainder is $7a+3$.

4. A man walks 6 km North, then 10 km South, then again 8 km North. How far is he from the starting-point? Illustrate graphically.

5. Solve the equation $\frac{\cdot 3x-\cdot 2}{5}+1=\frac{\cdot 6-\cdot 5x}{3}-1$.

6. A farmer bought an equal number of sheep of two kinds, one at Rs. 30 each and the other at Rs. 40 each. If he had spent his money equally on the two kinds, he would have had one sheep more. How many of each kind did he buy ?

PAPER 9

1. Simplify $3x+5-7x^2+4x-3x^2-4x^5--x^4+x^3+3x^4+6$, and arrange the result in ascending powers of x.

2. Find the value of $x(x+1)(x+2)(x+3)+1$ when $x=1$, 2, 3, 4...&c. Show that in each case the result is a perfect square.

3. In the formula $V=\frac{1}{3}\pi r^2h$, V is the subject. Make (i) r, (ii) h the subject of the formula.

4. Find without actual multiplication the term containing x^2 in the product of $(2x^3-x^2+x+4)$, $(3x^2+2x+1)$.

5. Solve the equation $\frac{\cdot 5x}{2}-\frac{\cdot 2x}{3}=\frac{\cdot 02x-\cdot 9}{6}$.

6. The length of a room exceeds its breadth by 11 m. If its length be increased by 5 m and the breadth diminished by 4 m, the area is diminished by 44 sq. m ; find the dimensions of the room.

OBJECTIVE TYPE ITEMS—1

1. *Complete each of the following statements by supplying the proper words/letters.*

(i) There are three.....in the expression $3-7x+5x^2$.

(ii) The co-efficient of m^3 in the expression $27m-7m^3$ is.....

(iii) The figure 6 in the expression $3x^6$ is called.....

(iv) The value of the *unknown* that satisfies an equation is called a.....of the equation.

(v) An equation that is true for only one value of the variable is called a.....equation.

(vi) The expression $7-3x^2+5x^5-7x^7$ has been arranged in.....powers of x.

(vii) $2x-5$, $2x-3$, $2x-1$ are three consecutive..... numbers.

2. *For each part below, choose the right answer and indicate it by the Capital letter printed alongside.*

(i) In the equality $8x-3y=17$, the letters x and y are known as
A. variables B. constants
C. exponents D. integers.

(ii) The reciprocal of $1-\frac{m}{2}$ is

A. $1+\frac{m}{2}$ B. $1-\frac{2}{m}$ C. $\frac{2}{2-m}$ D. $\frac{2}{2+m}$.

(iii) In p^4, p^3, p^2, p, the numerical co-efficients are
A. Zero in all cases B. Unity in all cases
C. 4, 3, 2, 1 D. 4, 3, 2, 0.

3. *Which of the following are three consecutive* numbers, if the middle number is a ?

A. $a+1$, a, $a+2$. B. $a-1$, a, $a+1$.

C. $a-2$, a, $a+2$. D. $\frac{a}{2}$, a, $2a$.

CHAPTER IV

FORMULAE

1. A few important multiplication formulae are given in the following pages and they should be thoroughly learnt by heart.

Formula I: $(a+b)^2=a^2+2ab+b^2$

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \\ \hline \end{array}$$

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

i.e. The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

$$(\bigcirc+\bigtriangledown)^2=\bigcirc^2+2\times\bigcirc\times\bigtriangledown+\bigtriangledown^2$$

Example 1: Find the square of $3x+4b$.

Sol. $(3x+4y)^2$ is of the form $(a+b)^2$ having $3x$ in place of a and $4y$ in place of b.

$$\begin{aligned}(3x+4y)^2 &= (3x)^2+2(3x)(4y)+(4y)^2\\ &= 9x^2+24xy+16y^2\end{aligned}$$

Example 2: Find the missing term in the following perfect square:

$$49a^2+70ab+(\ldots\ldots)$$

Sol. $49a^2=(7a)^2$, $70ab=2(7a)(5b)$

$\therefore$ the second term of the binomial $=5b$

$\therefore$ the missing term $=25b^2$.

EXERCISE 39.

Write down the square of:

1. $2x+5y$. **2.** $3a+7b$. **3.** $5x+3$. **4.** $3+4p$.

5. $3ab+4b^2$. **6.** $8a^2+3ab$. **7.** $9ax+4y^2$. **8.** m^2+n^2.

9. $px+qy$. **10.** p^3+pq^2. **11.** $2a^2b+3ab^2$. **12.** $a+(-b)$.

Find the missing terms in the following perfect squares:

13. $16x^2+(\ldots\ldots)+49y^2$. **14.** $36m^2+60mn+(\ldots\ldots)$.

15. $(\ldots\ldots)+112pq+64p^2$. **16.** $121a^2+(\ldots\ldots)+81$.

17. $144m^2+72m+(\ldots\ldots)$. **18.** $(\ldots\ldots)+104p+169$.

Formula II: $(\mathbf{a-b})^2=\mathbf{a^2-2ab+b^2}$

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \\ \hline \end{array}$$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b)-b(a-b)\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

Statement of the formula in words:

The square of the difference of two quantities is equal to the sum of their squares minus twice their product.

Example 3: Find the square of $(3x-4y)^2$.

Sol. $(3x-4y)^2$ is of the form $(a-b)^2$ having $3x$ in place of a and $4y$ in place of b.

$$\begin{aligned}(3x-4y)^2 &= (3x)^2-2(3x)(4y)+(4y)^2\\ &= 9x^2-24xy+16y^2.\end{aligned}$$

Example 4: Find the missing term in the perfect square $(\ldots\ldots)-60mn+36n^2$.

Sol. $36n^2=(6n)^2$, $-60mn=-2(5m)(6n)$

The first term of the binomial $=5m$

$\therefore$ the missing term $=(5m)^2=25m^2$.

EXERCISE 40

Write down the square of:

1. $3a-5b$. **2.** $5m-3$. **3.** $6p-5$.

4. $7x-1$. **5.** $4-7p$. **6.** $a-(-b)$.

7. $4m^2-5n^2$. **8.** $ax-by$. **9.** $2a^2x-3b^2y$.

10. $7a^2-4b^2$. **11.** $2a^3b-b^3$. **12.** $7p^3q^2-8p^2q^3$.

Find the missing term in the following perfect squares:

13. $16a^2-(\quad)+9b^2$. **14.** $81x^2-72xy+(\quad)$.

15. $(\quad)-48pq+36p^2$. **16.** $49m^2-(\quad)+121$.

17. $64p^2-96pq+(\quad)$. **18.** $(\quad)-120xy+144y^2$.

2. Application of formulae I and II.

Example 5: Find by formula the value of $(309)^2$.

Sol. $(309)^2 = (300+9)^2$
$= (300)^2+2\times300\times9+(9)^2$
$= 90000+5400+81$
$= 95481.$

Example 6: Find the square of 395 by formula.

Sol. $(395)^2 = (400-5)^2$
$= (400)^2-2\times400\times5+(5)^2$
$= 160000-4000+25$
$= 156025.$

EXERCISE 41

Find algebraically the value of:

1. $(302)^2$. **2.** $(405)^2$. **3.** $(703)^2$.
4. $(806)^2$. **5.** $(611)^2$. **6.** $(1001)^2$.

Find algebraically the value of:

7. $(198)^2$. **8.** $(245)^2$. **9.** $(396)^2$.
10. $(138)^2$. **11.** $(497)^2$. **12.** $(599)^2$.

Formula III. $\mathbf{(a+b)(a-b) = a^2-b^2.}$

The product of the sum and the difference of two quantities is equal to the difference of their squares.

Proof. $(a+b)(a-b) = a(a-b)+b(a-b)$
$= a^2-ab+ba-b^2 = a^2-b^2.$

Example 7: Multiply $4a+3b$ by $4a-3b$.

Sol. $(4a+3b)(4a-3b) = (4a)^2-(3b)^2 = 16a^2-9b^2.$

Example 8: Multiply $a+b+c$ by $a-b-c$.

Sol. $(a+b+c)(a-b-c) = \{a+(b+c)\}\{a-(b+c)\}$
$= a^2-(b+c)^2$
$= a^2-(b^2+2bc+c^2)$
$= a^2-b^2-2bc-c^2$
$= a^2-b^2-c^2-2bc.$

Example 9: Simplify $(x-2y+3z)^2-(x+2y-3z)^2$.

Sol. The given expression $= \{(x-2y+3z)+(x+2y-3z)\}$
$\times\{(x-2y+3z)-(x+2y-3z)\}$
$= 2x\times(-4y+6z)$
$= -8xy+12xz.$

EXERCISE 42.

Write down the product of:

1. $3x+5$ and $3x-5$.
2. $am+bn$ and $am-bn$.
3. $ax+b^2$ and $ax-b^2$.
4. $px+qy$ and $px-qy$.
5. $4a+5b$ and $4a-5b$.
6. $3xy-7xz$ and $3xy+7xz$.
7. $a+b$, $a-b$ and a^2+b^2.
8. $a+1$, $a-1$ and a^2+1.
9. a^2+b^2, a^2-b^2 and a^4+b^4.
10. $x+y-z$ and $x-y+z$.
11. $x-3y+4z$ and $x-3y-4z$.
12. x^2-xy+y^2 and x^2+xy+y^2.
13. x^2+x+1 and x^2-x+1.
14. x^4+x^2+1 and x^4-x^2+1.

Simplify :

15. $(2a-b+c)^2-(a+b-2c)^2$.
16. $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.
17. $(a+b-c+d)^2-(a-b+c-d)^2$.
18. $(2a+3b-4c+5d)^2-(2a-3b+4c-5d)^2$.

3. Corollaries.

$\because \quad (a+b)^2 = a^2+2ab+b^2,\ a^2+b^2 = (a+b)^2-2ab$
$\because \quad (a-b)^2 = a^2-2ab+b^2,\ a^2+b^2 = (a-b)^2+2ab$
Also, we know that $a^2-b^2 = (a+b)(a-b)$.

Example 10 : Find the value of a^2+b^2, if $a+b = 9$ and $ab = 20$.

Sol. We know that $a^2+b^2 = (a+b)^2-2ab$.
Substituting the values of $a+b$ and ab, we have
$a^2+b^2 = (9)^2-2\times20$
$= 81-40 \qquad = 41.$

Example 11 : Find the value of a^2+b^2 when $a-b = 7$ and $ab = 60$.

Sol. $a^2+b^2 = (a-b)^2+2ab$ (By formula)
$= (7)^2+2\times60$ $\left[\because \quad a-b = 7,\ ab = 60\right.$
$= 49+120$
$= 169$

Example 12 : Find the value of $(236)^2-(232)^2$.

Sol. $(236)^2-(232)^2 = (236+232)(236-232)$
$= 468\times4$
$= 1872.$

EXERCISE 43.

Find the value of a^2+b^2 when

1. $a+b=27$ and $ab=180$. **2.** $a+b=19$ and $ab=88$.
3. $a-b=7$ and $ab=120$. **4.** $a-b=9$ and $ab=112$.

Find the value of:

5. $(547)^2-(542)^2$. **6.** $(784)^2-(779)^2$.
7. $328\times328-318\times318$. **8.** $614\times614-607\times607$.

Simplify:

9. $(a+b)^2-(a-b)^2$. **10.** $(2a-3b)^2+(3a-2b)^2$.
11. $(px+qy)^2-(px-qy)^2$. **12.** $(2mx-ny)^2-(mx+3ny)^2$
13. $(3ab-2b^2)^2-4(a^2-b^2)^2$.

Formula IV: $(\mathbf{x+a})(\mathbf{x+b})=\mathbf{x^2+(a+b)x+ab}$.

$$\begin{aligned}(x+a)(x+b) &= x(x+b)+a(x+b)\\ &= x^2+xb+ax+ab\\ &= x^2+(a+b)x+ab\end{aligned}$$

$\therefore\ (x+a)(x+b)=x^2+$(sum of the second terms) x +(product of the 2nd terms).

Example 13: Write down the product of:

(i) $x+4$ and $x+5$ (ii) $x+6$ and $x-2$.
(iii) $x-3$ and $x+7$. (iv) $x-5$ and $x-6$.

Sol. (i) $(x+4)(x+5)=x^2+(4+5)x+4\times5$.
$=x^2+9x+20$.

(ii) $(x+6)(x-2)=x^2+(6-2)x+6\times(-2)$
$=x^2+4x-12$.

(iii) $(x-3)(x+7)=x^2+(-3+7)x+(-3)\times7$
$=x^2+4x-21$.

(iv) $(x-5)(x-6)=x^2+(-5-6)x+(-5)\times(-6)$
$=x^2-11x+30$.

EXERCISE 44.

Find out the product of:

1. $x+3$ and $x+5$. **2.** $x+4$ and $x+6$.
3. $x+5$ and $x+7$. **4.** $x+7$ and $x+9$.
5. $m-8$ and $m+5$. **6.** $m-11$ and $m+7$.
7. $p-14$ and $p+9$. **8.** $a-20$ and $a+11$.
9. $x-7$ and $x+15$. **10.** $x-8$ and $x+12$.
11. $x-6$ and $x+17$. **12.** $x-12$ and $x+20$.

13. $x+10$ and $x-11$. **14.** $x+13$ and $x-8$.
15. $x+16$ and $x-9$. **16.** $x+12$ and $x-7$.
17. $x+7$ and $x-17$. **18.** $x+3$ and $x-15$.
19. $x-5$ and $x-18$. **20.** $x+7$ and $x-19$.
21. $x-7$ and $x-9$. **22.** $x-12$ and $x-15$.
23. $x-13$ and $x-10$. **24.** $x-9$ and $x-14$.

Multiply :

25. $2x+3$ and $2x+5$. **26.** $3m+5$ and $3m-2$.
27. $1+2x$ and $1-3x$. **28.** $3+4x$ and $3-5x$.

Formula V : $(\mathbf{x+a})(\mathbf{x+b})(\mathbf{x+c})$
$= \mathbf{x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.}$

By actual multiplication,

$$(x+a)(x+b)(x+c) = \{x^2+(a+b)x+ab\}(x+c)$$
$$= x^3+(a+b)x^2+abx+cx^2+(a+b)cx+abc$$
$$= x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$$

From this formula, the following results can be easily deduced, by putting $-a$, $-b$, $-c$ for a, b, c respectively wherever required.

$$(x+a)(x+b)(x-c) = x^3+\{a+b-(c)\}x^2+$$
$$\{ab+a(-c)+b(-c)\}x+ab(-c).$$
$$= x^3+(a+b-c)x^2+(ab-ac-bc)x-abc \quad \text{(i)}$$

Similarly, $(x+a)(x-b)(x-c)$
$$= x^3+(a-b-c)x^2+(-ab-ac+bc)x+abc \quad \text{(ii)}$$

and $(x-a)(x-b)(x-c)$
$$= x^3+(-a-b-c)x^2+(ab+ac+bc)x-abc \quad \text{(iii)}$$

Example 14: Write down the product of $x-2$, $x+3$ and $x-6$.

Sol. $(x-2)(x+3)(x-6)$
$= x^2+(-2+3-6)x^2+$
$\{(-2)(3)+(3)(-6)+(-6)(-2)\}x+(-2)(3)(-6)$
$= x^3-5x^2-12x+36.$

EXERCISE 45.

Write down the product of:

1. $(x+1)(x+2)(x+3)$. **2.** $(x+2)(x+3)(x+4)$.
3. $(x+4)(x+5)(x+6)$. **4.** $(x+3(x+4)(x-5)$.
5. $(x+4)(x+5)(x-7)$. **6.** $(x+5)(x+2)(x-4)$.

7. $(x+3)(x-4)(x-5)$. **8.** $(x+4)(x-6)(x-3)$.
9. $(x-3)(x-4)(x-5)$. **10.** $(x-4)(x-1)(x-7)$.
11. $(x-3)(x-5)(x-2)$. **12.** $(x-4)(x-6)(x-8)$.
13. $(2x+1)(2x+3)(2x+5)$. **14.** $(3x-1)(3x-2)(3x-4)$.
15. $(ax+2)(ax+3)(ax+6)$. **16.** $(mx+1)(mx-4)(mx-5)$.

Complete the following:

17. $(x+p)(\quad+q)(x+\quad) = x^3+(\quad)x^2+(\quad)x+pqr$.
18. $(\quad+a)(p+2b)(p-3c) = (p)^3+(\quad)p^2+(\quad)p+(\quad)$.
19. $(m+3)(\quad-4)(m+\quad) = m^3+(\quad)m^2+(\quad)m-60$.

Formula VI: $\mathbf{(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b)}$

The cube of the sum of two quantities is equal to the sum of their cubes plus three times their product multiplied by their sum.

The formula can be easily established by actual multiplication or can be deduced from formula V, as shown below:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= a^3+(b+b+b)a^2+(b\times b+b\times b+b\times b)a+b\times b\times b\\ &= a^3+3ba^2+3b^2a+b^3\\ &= a^3+3a^2b+3ab^2+b^3\\ &= a^3+b^3+3ab(a+b).\end{aligned}$$

$$\left(\bigcirc+\bigtriangledown\right)^3=\bigcirc^3+3\bigcirc^2\times\bigtriangledown+3\bigcirc\times\bigtriangledown^2+\bigtriangledown^3$$

Example 15: Find the cube of $2x+3y$.

Sol. Reducing the expression to the standard form, we have

$$\begin{aligned}(2x+3y)^3 &= \{(2x)+(3y)\}^3\\ &= (2x)^3+(3y)^3+3(2x\times 3y)(2x+3y)\\ &= 8x^3+27y^3+36x^2y+54xy^2,\end{aligned}$$

or $$\begin{aligned}(2x+3y)^3 &= (2x)^3+3(2x)^2(3y)+3(2x)(3y)^2+(3y)^3\\ &= 8x^3+36x^2y+54xy^2+27y^3.\end{aligned}$$

Example 16: Find the missing terms as well as the missing parts of terms in

$$(\quad+\quad)^3 = (2x)^3+3(2x)^2(\quad)+3(\quad)(5y)^2+(\quad)^3.$$

Sol. Since in the expansion, the 1st term is $(2x)^3$ and the 3rd term contains $(5y)^2$,

the left-hand side $= (2x+5y)^3$

and the right-hand side $= (2x)^3+3(2x)^2(5y)+3(2x)(5y)^2+(5y)^3$.

EXERCISE 46.

Find the cube of:

1. $2x+y$. **2.** $x+3y$. **3.** $2a+3b$. **4.** $ax+by$.
5. $1+3x$. **6.** $2+5x$. **7.** $3x^2+2$. **8.** a^2+b^2.
9. $3a^2+2b^2$.

Fill in the gaps in the following:

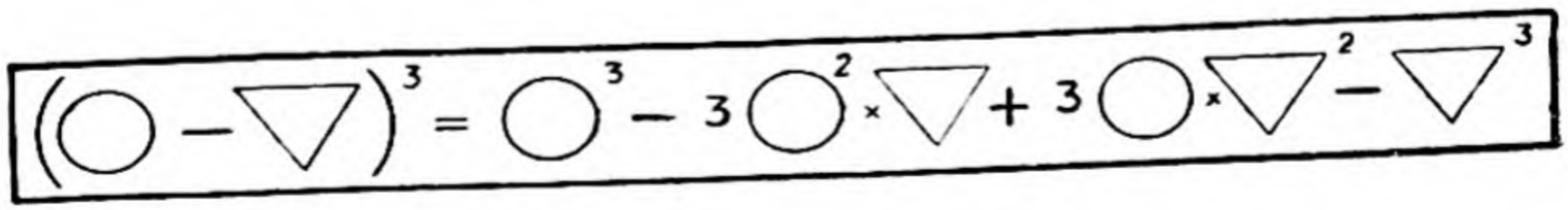
10. $(\quad+\quad)^3 = (4m)^3+3(\quad)^2(\quad)+3(\quad)(\quad)^2+(2n)^3$.
11. $(2p+\quad)^3 = (\quad)^3+3(\quad)(\quad)+3(2p)(\quad)+(3q)^3$.
12. $(\quad+\quad)^3 = (5p)^3+3(\quad)(5p)^2+3(\quad)(\ 2q)^2+(\quad)^3$.
13. $(2a+\quad)^3 = (\quad)^3+(3b)^3\ +3(\qquad)(2a+3b)$.

Formula VII: $\mathbf{(a-b)^3 = a^3-3a^2b+3ab^2-b^3}$
$\mathbf{= a^3-b^3-3ab(a-b)}$

The cube of the difference of two quantities is equal to the difference of their cubes minus three times their product multiplied by their difference.

This formula can be proved by actual multiplication.
or by putting $(-b)$ for $(+b)$ in formula VI, as shown below:

$$(a-b)^3 = \{a+(-b)\}^3 = a^3+3a^2(-b)+3a(-b)^2+(-b)^3$$
$$= a^3-3a^2b+3ab^2-b^3$$
$$= a^3-b^3-3ab(a-b).$$



Example 17: Find the cube of $3x-2y$.

Sol. $(3x-2y)^3$ is of the form $(a-b)^3$.

$$(3x-2y)^3 = (3x)^3-(2y)^3-3(3x)(2y)(3x-2y)$$
$$= 27x^3-8y^3-54x^2y+36xy^2$$
$$= 27x^3-54x^2y+36xy^2-8y^3.$$

EXERCISE 47.

Find the cube of:

1. $2x-1$. **2.** $3-2x$. **3.** $3a-4b$. **4.** $4x-3y$.
5. $5x-4y$. **6.** $4m^2-3n$. **7.** $2ax-3by$. **8.** $1-a^2$.

Fill in the gaps in the following:

9. $(\ldots-3y)^3 = (2x)^3-3(\ldots)^2(\ldots\ldots)+3(\ldots\ldots)(\ldots)^2-(3y)^3$.
10. $(4m-\ldots)^3 = (\ldots)^3-3(\ldots)^2(\ldots\ldots)+3(4m)(\ldots)^2-(5x)^3$.
11. $(\ldots-6b)^3 = (\ldots)^3-3(4a)^2(\ldots\ldots)+3(\ldots\ldots)(6b)^2-(\ldots)^3$.
12. $(2p-\ldots)^3 = (\ldots)^3-(\ldots\ldots)^3-3(\ldots\ldots)(\ldots\ldots)(2p-3q)$.

Simplify:

13. $(6m-7n)^3-(3m+5n)^3-9(6m-7n)(3m+5n)(m-4n)$.
14. $(5p+3q)^3-(4p-2q)^3-3(5p+3q)(4p-2q)(p+5q)$.
15. $(x+3)^3-(x-1)^3$. **16.** $(2x+3)^3-(2x-5)^3$.

4. Corollaries to formulae VI and VII.

$\because\ (a+b)^3 = a^3+b^3+3ab(a+b),\ a^3+b^3 = (a+b)^3-3ab(a+b)$
$\because\ (a-b)^3 = a^3-b^3-3ab(a-b),\ a^3-b^3 = (a-b)^3+3ab(a-b)$

Example 18: If $a+b=6$ and $ab=8$, find the value of a^3+b^3.

Sol. We know that $a^3+b^3 = (a+b)^3-3ab(a+b)$
Substituting the values of $(a+b)$ and ab, we have
$a^3+b^3 = (6)^3-3\times8\times6$
$= 216-144 \qquad = 72.$

Example 19: If $a-b=4$ and $ab=12$, find the value of a^3-b^3.

Sol. We know that $a^3-b^3 = (a-b)^3+3ab(a-b)$
Substituting the values of $(a-b)$ and ab, we have
$a^3-b^3 = (4)^3+3\times12\times4$
$= 64+144 \qquad = 208.$

Example 20: Simplify:
$(x+2y)^3+(2x-y)^3+3(3x+y)(x+2y)(2x-y)$

Sol. The given expression
$= (x+2y)^3+(2x-y)^3+3(x+2y)(2x-y)(3x+y)$
Putting a for $(x+2y)$ and b for $(2x-y)$, we have
the expression $= a^3+b^3+3ab(a+b) \qquad [\because\ a+b=3x+y]$
$= (a+b)^3$
Substituting the values of a and b, we have,
the expression $= \{(x+2y)+(2x-y)\}^3$
$= (3x+y)^3$.

EXERCISE 48.

Find the value of a^3+b^3 when

1. $a+b=5$ and $ab=6$. **2.** $a+b=7$ and $ab=12$.
3. $a+b=9$ and $ab=18$.

Simplify :

4. $(3a+5b)^3+(4a-3b)^3+3(3a+5b)(4a-3b)(7a+2b)$.
5. $(2x-7y)^3+(3x+12y)^3+15(2x-7y)(3x+12y)(x+y)$.
6. $(a+2)^3+(a+1)^3$. **7.** $(2a+3)^3-(2a+1)^3$.

Find the value of a^3-b^3 when

8. $a-b=3$ and $ab=18$. **9.** $a-b=5$ and $ab=14$.
10. $a-b=4$ and $ab=32$.

Formula VIII: $(\mathbf{a+b})(\mathbf{a^2-ab+b^2})=\mathbf{a^3+b^3}$.

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a(a^2-ab+b^2)+b(a^2-ab+b^2)\\ &= a^3-a^2b+ab^2+ba^2-ab^2+b^3\\ &= a^3+b^3.\end{aligned}$$

Example 21: Find the product of $3a+2b$ and $9a^2-6ab+4b^2$.

Sol. $(3a+2b)(9a^2-6ab+4b^2) = \{(3a)+(2b)\}\{(3a)^2-(3a)(2b)+(2b)^2\}$
$= (3a)^3+(2b)^3 = 27a^3+8b^3$.

Example 22: Simplify
$(2x+3)(4x^2-6x+9)-(3x+2)(9x^2-6x+4)$.

Sol. $\because$ $(2x+3)(4x^2-6x+9) = \{(2x)+(3)\}\{(2x)^2-(2x)(3)+(3)^2\}$
$= (2x)^3+(3)^3 = 8x^3+27$,
and $(3x+2)(9x^2-6x+4) = \{(3x)+(2)\}\{(3x)^2-(3x)(2)+(2)^2\}$
$= (3x)^3+(2)^3 = 27x^3+8$,
the given expression $= (8x^3+27)-(27x^3+8)$
$= 8x^3+27-27x^3-8 = 19-19x^3$.

EXERCISE 49.

Find the product of:

1. $x+3$ and x^2-3x+9. **2.** $x+5$ and $x^2-5x+25$.
3. $x+7$ and $x^2-7x+49$. **4.** $x+11$ and $x^2-11x+121$.
5. $25a^2-15a+9$ and $5a+3$.
6. $9m^2-15mn+25n^2$ and $3m+5n$.
7. $x^2y^2-4xyz+16z^2$ and $xy+4z$.
8. $4x+9y$ and $16x^2-36xy+81y^2$.

9. a^2+bc and $a^4-a^2bc+b^2c^2$.
10. $a^4-4a^2b^2+16b^4$ and a^2+4b^2.
11. x^2+1 and x^4-x^2+1.

Simplify :

12. $(x+5)(x^2-5x+25)-(x+3)(x^2-3x+9)$.
13. $(x+2y)(x^2-2xy+4y^2)+(2x+y)(4x^2-2xy+y^2)$.
14. $(4x+3y)(16x^2-12xy+9y^2)-(3x+4y)(9x^2-12xy+16y^2)$.
15. $(x^2+4)(x^4-4x^2+16)-(x^2+2)(x^4-2x^2+4)$.

Formula IX: $(\mathbf{a-b})(\mathbf{a^2+ab+b^2}) = \mathbf{a^3-b^6}$.

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a(a^2+ab+b^2)-b(a^2+ab+b^2)\\ &= a^3+a^2b+ab^2-ba^2-ab^2-b^3\\ &= a^3-b^3.\end{aligned}$$

Example 23: Find the product of $3xy-z$ and $9x^2y^2-3xyz+z^2$.

Sol. $(3xy-z)(9x^2y^2+3xyz+z^2)$
$= \{(3xy)-z\}\{(3xy)^2+(3xy)z+z^2\}$
$= (3xy)^3-z^3 = 27x^3y^3-z^3$.

Example 24: Simplify $(x-3)(x^2+3x+9)-(x-2)(x^2+2x+4)$.

Sol. $(x-3)(x^2+3x+9)$
$= \{(x)-(3)\}\{(x)^2+(x)(3)+(3)^2\}$
$= x^3-3^3 = x^3-27$,

$$\begin{aligned}(x-2)(x^2+2x+4) &= \{(x)-(2)\}\{(x)^2+(x)(2)+(2)^2\}\\ &= x^3-2^3 = x^3-8.\end{aligned}$$

$\therefore$ the given expression $= (x^3-27)-(x^3-8)$
$= x^3-27-x^3+8 = -19$.

EXERCISE 50.

Write down the product of:

1. $(x-2)(x^2+2x+4)$.
2. $(x-6)(x^2+6x+36)$.
3. $(x-8)(x^2+8x+64)$.
4. $(x-12)(x^2+12x+144)$.
5. $(3x-2)(9x^2+6x+4)$.
6. $(5x-3y)(25x^2+15xy+9y^2)$.
7. $(pq-2r)(p^2q^2+2pqr+4r^2)$.
8. $(a^2-bc)(a^4+a^2bc+b^2c^2)$.

Simplify :

9. $(3m-4n)(9m^2+12mn+16n^2)-(2m-3n)(4m^2+6mn+9n^2)$
10. $(4p-3q)(16p^2+12pq+9q^2)-(3p-4q)(9p^2+12pq+16q^2)$.
11. $(m^2-6)(m^4+6m^2+36)-(m^2-3)(m^4+3m^2+9)$.

Fill in the gaps:

12. $(a)^3+(\ldots)^3=\{(\ldots)+(3b)\}\{(a)^2-(\ldots)(\ldots)+(\ldots)^2\}$.
13. $(\ldots)^3-(5y)^3=\{(2x)-(\ldots)\}\{(\ldots)^2+(\ldots)(\ldots)+(5y)^2\}$.
14. $(4p)^3+(\ldots)^3=\{(\ldots)+(\ldots)\}\{(\ldots)^2-(4p)(3q)+(\ldots)^2\}$.
15. $(\ldots)^3-(6b)^3=\{(\ldots)-(\ldots)\}\{49a^2+(\ldots)(\ldots)+36b^2\}$.

Formula X: Square of multinomial.

$$\begin{aligned}(a+b+c)^2&=\{(a+b)+c\}^2\\&=(a+b)^2+2(a+b)c+c^2\\&=a^2+2ab+b^2+2ac+2bc+c^2\\&=a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

Similarly, $(a+b+c+d)^2$
$=\{(a+b)+(c+d)\}^2$
$=(a+b)^2+2(a+b)(c+d)+(c+d)^2$
$=a^2+b^2+2ab+2ac+2ad+2bc+2bd+c^2+d^2+2cd$
$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$

From such examples, we establish the following rule:

The square of any multinomial is equal to the sum of the squares of its terms plus twice the sum of the products of the terms, taken two at a time.

Example 25: Find the square of $2a-3b+c$.

Sol. $$\begin{aligned}(2a-3b+c)^2&=(2a)^2+(-3b)^2+c^2+2(2a)(-3b)+2(2a)(c)\\&\qquad+2(-3b)(c)\\&=4a^2+9b^2+c^2-12ab+4ac-6bc.\end{aligned}$$

Example 26: Find the square of $a-3b+c-2d$.

Sol. $$\begin{aligned}(a-3b+c-2d)^2&=a^2+(-3b)^2+c^2+(-2d)^2+2a(-3b)\\&\qquad+2ac+2a(-2d)+2(-3b)c\\&\qquad+2(-3b)(-2d)+2c(-2d)\\&=a^2+9b^2+c^2+4d^2-6ab+2ac-4ad-6bc\\&\qquad+12bd-4cd.\end{aligned}$$

Example 27: Find the value of $a^2+9b^2-6ab-2a+6b+64$ when $a=33$ and $b=24$.

Sol. The expression $$\begin{aligned}&=(a)^2+(-3b)^2+2(a)(-3b)+2(a)(-1)\\&\qquad+2(-3b)(-1)+(-1)^2+63\\&=(a-3b-1)^2+63\end{aligned}$$

Substituting the values of a and b, we have

the expression $$\begin{aligned}&=(33-72-1)^2+63\\&=(-40)^2+63\\&=1600+63=1663.\end{aligned}$$

EXERCISE 51.

Write down the square of:

1. $a+b-c$. **2.** $a-b-c$. **3.** $2a-b-c$.
4. $2a+b-c$ **5.** $2a-b+3c$. **6.** $5x-2y+3z$.
7. $2x-3y+1$. **8.** x^2-xy+y^2. **9.** $px^2-qxy+ry^2$.
10. $2a+b+3c+d$. **11.** $a-2b-3c+4d$.
12. $2p-3q+r-2s+t$. **13.** $3x-4y+2z-w$.

Simplify:

14. $(2a-b+c)^2-(a+b-2c)^2$.
15. $(2a+b+c)^2-(a-b-c)^2-(a+b-c)^2$.
16. $(a+2b+c)^2-(a-b+2c)^2+(2a+b-c)^2$.
17. $(a+b+c+d)^2+(a-b+c-d)^2$.
18. $(a-b-c-d)^2+(a+b-c-d)^2$.
19. $(2a-b+c-d)^2-(a+b-c+d)^2-(a-b+c-d)^2$.

Find the value of:

20. $x^2+y^2+z^2-2xy+2xz-2yz$
when $x = 15$, $y = 20$, $z = 24$.

21. $p^2+4q^2+9z^2-4pq+6pz-12qz$
when $p = 8$, $q = 12$, $z = 15$.

22. $x^2+4y^2-4xy+10x-20y+36$ when $x = 16$, $y = 9$.
[Hint: *Split* 36 *into* $25+11$]

23. $9a^2+b^2-6ab+6a-2b-28$ when $a = 15$, $b = 37$.

24. $9x^2+y^2-6xy-6x+4y$ when $x = y = -2$.

Formula XI: $\mathbf{(a+b)^2-(a-b)^2 = 4ab}$.

We know that $(a+b)^2 = a^2+2ab+b^2$
$(a-b)^2 = a^2-2ab+b^2$

Subtracting, $(a+b)^2-(a-b)^2 = 4ab$
Adding, $(a+b)^2+(a-b)^2 = 2a^2+2b^2$.

Formula XII: $\mathbf{(a+b)^2 = (a-b)^2+4ab}$.

Formula XIII: $\mathbf{(a-b)^2 = (a+b)^2-4ab}$.

These two formulae can be obtained from formula XI by transposing the relevant term.

These formulae have to be learnt by heart.

Formula XIV: $\frac{1}{2}\{(\mathbf{a-b})^2+(\mathbf{b-c})^2+(\mathbf{c-a})^2\}$
$= \mathbf{a^2+b^2+c^2-ab-ac-bc.}$

$\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$
$= \frac{1}{2}\{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)\}$
$= \frac{1}{2}\{2a^2+2b^2+2c^2-2ab-2bc-2ca\}$
$= a^2+b^2+c^2-ab-ac-bc.$

Example 28: Evaluate $(a+b)^2$ when $a-b=3$ and $ab=10$

Sol. $(a+b)^2 = (a-b)^2+4ab$
$= 3^2+4\times 10 = 9+40 = 49.$

Example 29: Find the value of ab when $a+b=9$ and $a-b=3$.

Sol. Since $4ab = (a+b)^2-(a-b)^2$,
$\therefore \quad ab = \frac{1}{4}\{(a+b)^2-(a-b)^2\}$
$= \frac{1}{4}(9^2-3^2) = \frac{1}{4}(81-9) = 18.$

Example 30: Find the value of $a^2+b^2+c^2-ab-ac-bc$ when $a=32$, $b=30$ and $c=27$.

Sol. $a^2+b^2+c^2-ab-ac-bc = \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$
$= \frac{1}{2}\{2^2+3^2+(-5)^2\} = \frac{1}{2}(4+9+25) = 19.$

EXERCISE 52.

Find the value of:

1. $(a+b)^2$ when $a-b=5$, $ab=36$.
2. $(a+b)^2$ when $a-b=4$, $ab=77$.
3. $(a-b)^2$ when $a+b=12$, $ab=35$.
4. $(a-b)^2$ when $a+b=15$, $ab=36$.
5. ab when $a+b=11$, $a-b=3$.
6. ab when $a+b=22$, $a-b=8$.
7. $a^2+b^2+c^2-ab-ac-bc$ when $a=15$, $b=16$, $c=17$.
8. $a^2+b^2+c^2-ab-ac-bc$ when $a=1{\cdot}8$, $b=1{\cdot}3$, $c={\cdot}9$.
9. $a^2+b^2+c^2-ab-ac-bc$ when $a=p+1$, $b=p+2$, $c=p+3$.
10. Simplify $(x+1)^2+(x+2)^2+(x+3)^2-(x+1)(x+2)-(x+1)(x+3)-(x+2)(x+3)$.

5. Formulae involving $x+\frac{1}{x}$, $x-\frac{1}{x}$.

Example 31: If $x+\frac{1}{x}=6$, find the value of $x^2+\frac{1}{x^2}$.

Sol. $x^2+\frac{1}{x^2}$

$=\left(x^2+2+\frac{1}{x^2}\right)-2$

$=\left(x+\frac{1}{x}\right)^2-2$

$=(6)^2-2$

$=36-2 \quad =34$

Given $x+\frac{1}{x} \quad =6.$

Squaring both sides, we have

$\left(x+\frac{1}{x}\right)^2=(6)^2$

or $x^2+2\times x\times\frac{1}{x}+\frac{1}{x^2} \quad =36$

or $x^2+2+\frac{1}{x^2} \quad =36$

$\therefore \quad x^2+\frac{1}{x^2} \; = \; 36-2 \quad =34.$

Example 32: If $x-\frac{1}{x}=4$, find the value of $x^3-\frac{1}{x^3}$.

Sol. $x^3-\frac{1}{x^3}$

$=(x)^3-\left(\frac{1}{x}\right)^3$

$=\left(x-\frac{1}{x}\right)^3+3(x)\times\frac{1}{x}\times\left(x-\frac{1}{x}\right)$

$=(4)^3+3\times1\times4$

$=64+12$

$=76$

Given $x-\frac{1}{x} \quad =4$

Cubing both sides, we have

$\left(x-\frac{1}{x}\right)^3 \quad =(4)^3$

or $x^3-\frac{1}{x^3}-3\times x\times\frac{1}{x}\left(x-\frac{1}{x}\right)=64$

or $x^3-\frac{1}{x^3}-3\times1\times4 \quad =64$

or $x^3-\frac{1}{x^3}=64+12 \quad =76.$

EXERCISE 53.

Find the value of:

1. $x^2+\frac{1}{x^2}$ and $x^4+\frac{1}{x^4}$ when $x+\frac{1}{x}=5.$

2. $x^2+\frac{1}{x^2}$ and $x^4+\frac{1}{x^4}$ when $x-\frac{1}{x}=3$.

3. $x^4+\frac{1}{x^4}$ when $x-\frac{1}{x}=p$.

4. $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=5$.

5. $x^3-\frac{1}{x^3}$ when $x-\frac{1}{x}=2$.

6. $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=m$.

7. $x^3-\frac{1}{x^3}$ when $x-\frac{1}{x}=a$.

Formula XV: $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)=a^3+b^3+c^3-3abc.$

This formula can be proved by actual multiplication.

Example 33: Simplify $(2x+3y-z)\ (4x^2+9y^2+z^2-6xy+2xz+3yz)$.

Sol. The expression $=\{2x+3y+(-z)\}\{(2x)^2+(3y)^2+(-z)^2-(2x)(3y)-(2x)(-z)-(3y)(-z)\}$
$=(2x)^3+(3y)^3+(-z)^3-3(2x)(3y)(-z)$
$=8x^3+27y^3-z^3+18xyz.$

Example 34: Find the value of $a^3+b^3+c^3-3abc$ when $a+b+c=9$ and $ab+ac+bc=26$.

Sol. Now $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$
$=9^2-2\times26=29.$
$\therefore\ a^3+b^3+c^3-3abc=(a+b+c)\{(a^2+b^2+c^2)-(ab+ac+bc)\}$
$=9\ (29-26)=9\times3=27.$

EXERCISE 54.

Simplify :

1. $(x+y-1)(x^2+y^2+1-xy+x+y)$.
2. $(x-y-1)(x^2+y^2+1+xy+x-y)$.
3. $(2x+y+1)(4x^2+y^2+1-2xy-2x-y)$.
4. $(x+2y-z)(x^2+4y^2+z^2-2xy+xz+2yz)$.
5. $(x+2y-3z)(x^2+4y^2+9z^2-2xy+3xz+6yz)$.
6. $(2x-y-3z)(4x^2+y^2+9z^2+2xy+6xz-3yz)$.

Complete the following identities:

7. $(3x-y-z)(\quad) = 27x^3-y^3-z^3-9xyz.$

8. $(\quad)(25x^2+9y^2+z^2-15xy+5xz+3yz)$
$= 125x^3+27y^3-z^3+45xyz$

Find the value of $a^3+b^3+c^3-3abc$ when

9. $a+b+c = 13, \quad ab+ac+bc = 71.$

10. $a+b+c = 21, \quad a^2+b^2+c^2 = 149.$

11. $ab+bc+ca = 121, \quad a^2+b^2+c^2 = 158.$

12. If $x = a^2-bc$, $y = b^2-ca$, $z = c^2-ab$, shew that
$ax+by+cz = (a+b+c)(x+y+z).$

Example 35: Express $(x+5y+z)^2-4(x+3y)(2y+z)$ as a perfect square.

Sol. Let $x+3y = a$ and $2y+z = b$, then $x+5y+z = a+b.$
Then the given expression $= (a+b)^2-4ab=(a-b)^2$
$= \{(x+3y)-(2y+z)\}^2$
$= (x+y-z)^2.$

Example 36: If $x = a+m$, $y = b+m$ and $z = c+m$, prove that
$x^2+y^2+z^2-xy-xz-yz = a^2+b^2+c^2-ab-ac-bc.$

Sol. $x^2+y^2+z^2-xy-xz-yz = \frac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\}$
$= \frac{1}{2}\{(a+m-b-m)^2+(b+m-c-m)^2$
$+(c+m-a-m)^2\}$
$= \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$
$= a^2+b^2+c^2-ab-ac-bc.$

Example 37: If $x = a+b$, $y = b+c$, $z = c+a$, prove that
$x^3+y^3+z^3-3xyz = 2(a^3+b^3+c^3-3abc).$

Sol. $x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2$
$+(z-x)^2\}.$
Substituting the values of x, y, z, we get the expression
$= \frac{1}{2}\{2(a+b+c)\}\{(a-c)^2+(b-a)^2+(c-b)^2)\}$
$= 2[\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}]$
$= 2(a^3+b^3+c^3-3abc).$

EXERCISE 55.

Express as a perfect square:

1. $(3x-y+z)^2+4(3x+y)(2y-z).$

2. $(2x+y-z)^2+4(2x+3y)\ (2y+z).$

3. $(x+3y+3z)^2-4(x+y+2z)\ (2y+z).$

4. $(x-z)^2-4(x-y+z)\,(y-2z)$.

5. If $a=p+q$, $b=q+r$ and $c=r+p$, prove that $a^2+b^2+c^2-ab-ac-bc=p^2+q^2+r^2-pq-pr-qr$.

6. If $x=a+b-c$, $y=b+c-a$ and $z=c+a-b$, prove that $x^2+y^2+z^2-xy-xz-yz=4(a^2+b^2+c^2-ab-ac-bc)$.

7. Find the value of $x^2+y^2+z^2-xy-xz-yz$ when $x=a(a+7)$, $y=(a+2)(a+5)$, $z=(a+3)(a+4)$.

8. Find the value of $8xy(x^2+y^2)$ when $x+y=15$ and $x-y=11$.

9. If $a=y+z-x$, $b=z+x-y$ and $c=x+y-z$, prove that $a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz)$

10. If $a=y+z-3x$, $b=z+x-3y$ and $c=x+y-3z$, prove that $a^3+b^3+c^3-3abc=-16(x^3+y^3+z^3-3xyz)$.

11. If $2s=a+b+c$, prove that $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)=\frac{1}{2}(a^3+b^3+c^3-3abc)$.

A systematised list of multiplication formulae :

1. $(a+b)^2=a^2+2ab+b^2$.

2. $(a-b)^2=a^2-2ab+b^2$.

3. $a^2+b^2=(a+b)^2-2ab$.

4. $a^2+b^2=(a-b)^2+2ab$.

5. $(a+b)^2=(a-b)^2+4ab$.

6. $(a-b)^2=(a+b)^2-4ab$.

7. $(a+b)^2-(a-b)^2=4ab$.

8. $(a+b)(a-b)=a^2-b^2$.

9. $(a+b)(a^2-ab+b^2)=a^3+b^3$.

10. $(a-b)(a^2+ab+b^2)=a^3-b^3$.

11. $(x+a)(x+b)=x^2+(a+b)x+ab$.

12. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.

13. $(x+a)^3=x^3+3x^2a+3xa^2+a^3$, or $=x^3+a^3+3xa(x+a)$.

14. $(x-a)^3=x^3-3x^2a+3xa^2-a^3$, or $=x^3-a^3-3xa(x-a)$.

15. $x^3+a^3=(x+a)^3-3xa(x+a)$.

16. $x^3-a^3=(x-a)^3+3xa(x-a)$.

17. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$.

18. $(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.

19. $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}=a^2+b^2+c^2-ab-ac-bc$.

20. $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)=a^3+b^3+c^3-3abc$, or $\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}=a^3+b^3+c^3-3abc$.

CHAPTER V

FACTORS

1. **Type 1:** Factors of expressions of the form **ab+ac.**

Example 1: Resolve into factors $3a^2-6a$.

Sol. $3a^2-6a = (3a)(a)-(2)(3a)$
$= 3a(a-2)$.

[*Verify by actual multiplication*].

Example 2: Resolve into factors $2a^3b^2c-4ab^3c^2+6a^2bc^3$.

Sol. Here we notice that $2abc$ is a factor of every term,

$\therefore\ 2a^3b^2c-4ab^3c^2+6a^2bc^3$
$= 2abc(a^2b-2b^2c+3ac^2)$.

Example 3: Resolve into factors
$a^2(x-y)+3ab(x-y)+5b^2(x-y)$.

Sol. Here a compound expression $(x-y)$ is a factor of every term.

$\therefore\ a^2(x-y)+3ab(x-y)+5b^2(x-y)$
$= (x-y)(a^2+3ab+5b^2)$.

EXERCISE 56.

Resolve into factors: (*Verify by multiplication*).

1. $ma+mb$.
2. x^2+2x.
3. m^2n+mn^2.
4. $3x^2-6$.
5. $abc-bcd$.
6. $pq+qa-ra$.
7. ax^3+bx^2+cx.
8. $a^2x^3-2a^3x^2+a^2x^2$.
9. $4a^5b-6a^4b+8a^3b^3$.
10. $ax^3y^2+a^2xy^3+a^3x^2y$.

Factorise:

11. $c(a+b)+d(a+b)$.
12. $7m(x-1)+2(x-1)$.
13. $2a(x^2+x+1)+3b(x^2+x+1)$.
14. $3p(ax+b)+q(ax+b)+2r(ax+b)$.
15. $m^2(pq+r)+mn(pq+r)+n^2(pq+r)$.
16. $a(x-y)+b(y-x)+(c+d)(x-y)$.
17. $(x^2+7x)^2-25(x^2+7x)$.
18. $15(p^2-qr)^3-10(p^2-qr)^2$.
19. $(2p+3q)(x-y)+(p-2q)(x-y)+(3p+q)(x-y)$.
20. $(x+y)^3-3xy(x+y)$.

2. **Type II:** Factors of expressions of the form **ac+bc+ad+bd.**

Example 4: Resolve into factors $3ab+6bc+4ad+8cd$.

Sol. $3ab+6bc+4ad+8cd = (3ab+6bc)+(4ad+8cd)$
$= 3b(a+2c)+4d(a+2c)$
$= (a+2c)(3b+4d)$.

Example 5: Factorise $3x^2+4yz-4xy-3xz$.

Sol. Re-arranging the terms, the expression
$= 3x^2-3xz-4xy+4yz$
$= 3x(x-z)-4y(x-z)$
$= (x-z)(3x-4y)$.

Example 6: Factorise $a^3x-a^2b(x-y)-ab^2(y-z)-b^3z$.

Sol. Re-moving the brackets and re-arranging, we have the expression $= a^3x-a^2bx+a^2by-ab^2y+ab^2z-b^3z$
$= a^2x(a-b)+aby(a-b)+b^2z(a-b)$
$= (a-b)(a^2x+aby+b^2z)$.

EXERCISE 57.

Resolve into factors:

1. $5x+5y+ax+ay$.
2. $ax-ay-bx+by$.
3. $mx-3y-my+3x$.
4. $xy+3x+2y+6$.
5. $x^2-3ax+bx-3ab$.
6. $a^3+3a^2+9a+27$.
7. $3p^3-5p^2+6p-10$.
8. $6p^2-9pq+4pr-6qr$.
9. $pq+p+q+1$.
10. a^3+a^2-a-1.
11. $2a^3-5a^2+2a-5$.
12. $5a^3-2a^2+5a-2$.
13. $x^2-xy-3x+3y$.
14. $6a^2+3ab-2ac-bc$.
15. $2a^4-a^3+4a-2$.
16. $y^3-y^2-ay+y+a-1$.
17. $11x^3+7x+35+55x^2$.
18. ax^3+b+bx^2+ax.
19. $6xy+6-9y-4x$.
20. $x^2-2ax+2ab-bx$.
21. $ax^2+(a-1)x-1$.
22. $abx^2+(ay-b)x-y$.
23. $(a-2b)^2+3a-6b$.
24. $ab(c^2+1)+c(a^2+b^2)$.
25. $ab(x^2+y^2)-xy(a^2+b^2)$.
26. $a^3x+a^2(x-y)-a(y+z)-z$.
27. $a(x^2-1)+x^2(bx-c)+x(cx^3-b)$.
28. $x^3(kx+l)-x(kx+l)+m(x^2-1)$.

3. **Type III:** Factors of expressions of the form $\mathbf{a^2+2ab+b^2}$ or $\mathbf{a^2-2ab+b^2}$.

$$a^2+2ab+b^2 = a^2+ab+ab+b^2 = a(a+b)+b(a+b) = (a+b)(a+b) = (a+b)^2$$

$$a^2-2ab+b^2 = a^2-ab-ab+b^2 = a(a-b)-b(a-b) = (a-b)(a-b) = (a-b)^2$$

Example 7: Factorise $x^2+14x+49$.

Sol. $x^2+14x+49 = x^2+2.x.7+7^2 = (x+7)^2$.

Example 8: Factorise $36a^2-12a+1$.

Sol. $36a^2-12a+1 = (6a)^2-2.\ (6a).1+1^2 = (6a-1)^2$.

Example 9: Factorise $9+42xy+49x^2y^2$.

Sol. $9+42xy+49x^2y^2 = 3^2+2.3.(7xy)+(7xy)^2 = (3+7xy)^2$.

EXERCISE 58.

Resolve into factors:

1. $4a^2+28a+49$.
2. $25p^2-30p+9$.
3. $25a^2+40ab+16b^2$.
4. $49m^2-56mn+16n^2$.
5. $36a^2+60ab+25b^2$.
6. $36a^2-84ab+49b^2$.
7. $49a^2+42ab+9b^2$.
8. $25x^2-60xy+36y^2$.
9. $16x^2+56xy+49y^2$.
10. $81a^2-126ab+49b^2$.
11. $81x^2+90xy+25y^2$.
12. $9m^2-66mn+121n^2$.
13. $64x^2+16xy+y^2$.
14. $16p^2-48pq+36q^2$.
15. $9x^2+54xy+81y^2$.
16. $25m^2-70mn+49n^2$.
17. $x^2+2+\frac{1}{x^2}$.
18. $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$.
19. $a^2x^2+6abxy+9b^2y^2$.
20. $16a^2b^2-8abc^2+c^4$.
21. $(2a+3b)^2+2(2a+3b)(c+d)+(c+d)^2$.
22. $(3p-q)^2-2(3p-q)(r+s)+(r+s)^2$.

4. **Type IV:** Factors of expressions of the form $\mathbf{a^2-b^2}$.

$$a^2-b^2 = a^2+ab-ab-b^2 = a(a+b)-b(a+b) = (a+b)(a-b).$$

Example 10: Factorise (i) $16a^2-25b^2$ (ii) $1-49x^2$.

Sol. (i) $16a^2-25b^2$
$= (4a)^2-(5b)^2$
$= (4a+5b)(4a-5b)$.

(ii) $1-49x^2$
$= (1)^2-(7x)^2$
$= (1+7x)(1-7x)$.

NOTE: The reduction of the given expression to the standard form is essential.

Example 11: Resolve into factors $16x^5-x$.

Sol. $16x^5-x = x(16x^4-1)$
$= x\{(4x^2)^2-(1)^2\}$
$= x(4x^2+1)(4x^2-1)$
$= x(4x^2+1)\{(2x)^2-(1)^2\}$
$= x(4x^2+1)(2x+1)(2x-1)$.

Example 12: Resolve into factors $4(a-b)^2-9(c-d)^2$.

Sol. $4(a-b)^2-9(c-d)^2 = 2^2(a-b)^2-3^2(c-d)^2$
$= (2a-2b)^2-(3c-3d)^2$
$= (2a-2b+3c-3d)(2a-2b-3c+3d)$.

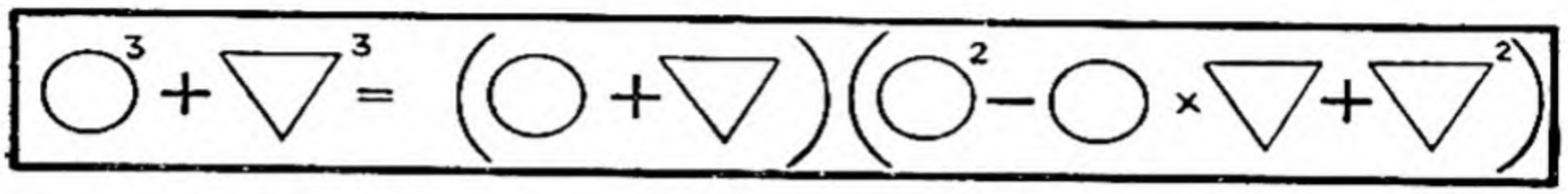
EXERCISE 59.

Resolve into factors:

1. $16x^2-9$. **2.** $81a^2-25b^2$. **3.** $1-36x^2$.
4. $64a^2-1$. **5.** $9x^2-100$. **6.** $49-25a^2$.
7. $121-c^2$. **8.** $1-49a^2c^2$. **9.** a^4-25.
10. $81x^4-1$. **11.** $16x^5-81x$.
12. $1-16x^4$. **13.** a^2-81a^6.
14. $36x^7-25x^3a^4$. **15.** $192x^9-243x^5y^4$.
16. $(a+5b)^2-49c^2$. **17.** $x^2-(2y-3z)^2$.
18. $(a+b)^2-(a-b)^2$. **19.** $(3x-2y)^2-(2x+y)^2$.
20. $16(x-y)^2-9(x+y)^2$. **21.** $49(a+b)^2-36(a-b)^2$.
22. $(a+b-c)^2-(a-b+c)^2$. **23.** $(x+a)^4-(x-a)^4$.
24. $4(3m-2n)^2-9(2m-3n)^2$.
25. $(8a+5)^2-(2a-7)^2$. **26.** $81(m+2n)^2-25(2m-n)^2$.
27. $16(a+3b-4c)^2-9(2a-b+3c)^2$.

5. **Type V**(a)**:** Factors of expressions of the form $\mathbf{a^3+b^3}$.

$$\begin{aligned} a^3+b^3 &= (a+b)^3-3ab(a+b) && \text{(By formula)}\\ &= (a+b)(a+b)^2-3ab(a+b)\\ &= (a+b)\{(a+b)^2-3ab\}\\ &= (a+b)(a^2+2ab+b^2-3ab)\\ &= (a+b)(a^2-ab+b^2). \end{aligned}$$



Example 13: Resolve into factors $8a^3+27b^3$.

Sol. $$\begin{aligned} 8a^3+27b^3 &= (2a)^3+(3b)^3\\ &= (2a+3b)\{(2a)^2-(2a)(3b)+(3b)^2\}\\ &= (2a+3b)(4a^2-6ab+9b^2). \end{aligned}$$

NOTE: The reduction of the given expression to the standard form is essential.

Example 14: Factorise $27(x+y)^3+8(2x-y)^3$.

Sol. $$\begin{aligned} 27(x+y)^3 &= 3^3(x+y)^3 = (3x+3y)^3\\ 8(2x-y)^3 &= 2^3(2x-y)^3 = (4x-2y)^3.\\ \therefore \text{ Expression} &= (3x+3y)^3+(4x-2y)^3\\ &= [(3x+3y)+(4x-2y)]\times\\ &\quad [(3x+3y)^2-(3x+3y)(4x-2y)+(4x-2y)^2]\\ &= (7x+y)[9x^2+18xy+9y^2-(12x^2+6xy-6y^2)\\ &\qquad +16x^2-16xy+4y^2)]. \end{aligned}$$

$$= (7x+y)(13x^2-4xy+19y^2).$$

EXERCISE 60.

Resolve into factors:

1. m^3+1. **2.** p^3+64. **3.** p^3q^3+216.

4. $64a^3+1000$. **5.** $125x^6+64y^3$. **6.** $64m^3n^3+1$.

7. $27m^6+125n^6$. **8.** $343a^4b+64ab^4$

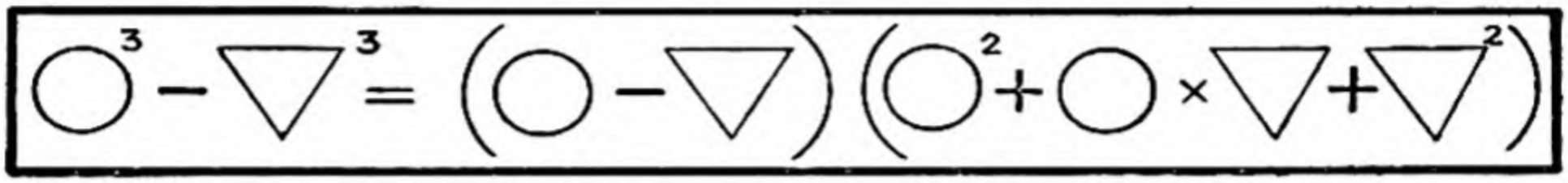
9. $(2a-3b)^3+27c^3$. **10.** $(a-2b)^3+1$.

11. $x^3-3x^2y+3xy^2-y^3-1$. **12.** $64(x-3y)^3+343(2x+y)^3$.

13. $8(x-y)^3+125(y-2x)^3$ **14.** $3(2p-3q)^3+81(q-p)^3$.

6. **Type V**(b)**:** Factors of expressions of the form $\mathbf{a^3-b^3}$.

$$\begin{aligned} a^3-b^3 &= (a-b)^3+3ab(a-b) && \text{(By formula)}\\ &= (a-b)(a-b)^2+3ab(a-b)\\ &= (a-b)\{(a-b)^2+3ab\}\\ &= (a-b)(a^2-2ab+b^2+3ab)\\ &= (a-b)(a^2+ab+b^2). \end{aligned}$$



Example 15: Resolve into factors $27x^3-1$.

Sol. $$\begin{aligned} 27x^3-1 &= (3x)^3-(1)^3\\ &= (3x-1)\{(3x)^2+(3x)(1)+(1)^2\}\\ &= (3x-1)(9x^2+3x+1). \end{aligned}$$

NOTE: The reduction of the given expression to the standard form is essential.

Example 16: Factorise a^7-27ab^6.

Sol. $$\begin{aligned} a^7-27ab^6 &= a(a^6-27b^6)\\ &= a[(a^2)^3-(3b^2)^3]\\ &= a(a^2-3b^2)[(a^2)^2+(a^2)(3b^2)+(3b^2)^2]\\ &= a(a^2-3b^2)(a^4+3a^2b^2+9b^4). \end{aligned}$$

Example 17: Resolve into factors x^6-64.

Sol. $$\begin{aligned} x^6-64 &= (x^3)^2-(8)^2\\ &= (x^3+8)(x^3-8)\\ &= (x+2)(x^2-2x+4)(x-2)(x^2+2x+4). \end{aligned}$$

EXERCISE 61.

Resolve into factors:

1. $1-a^3$.	**2.** $8-x^3$.	**3.** $125a^3-8$.
4. $729x^3-216$.	**5.** $8x^3-1000b^3$.	**6.** $729x^3y^3-512z^3$
7. a^6-8b^6.	**8.** a^9-b^9.	**9.** $a^{12}-8b^6c^6$.
10. $(3x+2y)^3-z^3$.	**11.** $(2a+b)^3-(a+2b)^3$.	
12. $125(a-2b)^3-27(3a-4b)^3$.		
13. a^6-b^6.	**14.** $a^{12}-b^{12}$.	**15.** $64a^{12}-1$.
16. $729a^6-b^6$.	**17.** a^7-ab^6.	**18.** $3x^{13}-192x^7y^6$

7. **Type VI:** Factors of trinomials like $\mathbf{x^2+px+q}$.

Analysis: The first term of the given expression is x^2. Therefore, it can have two binomial factors ; the first term of each factor will be x.

Let us suppose that the 2nd terms are m & n, so that $(x+m)\ (x+n)$ are the required factors.

Then $(x+m)(x+n)$ must be equal to x^2+px+q
or $x^2+(m+n)x+mn$ must be equal to x^2+px+q
or $m+n = +p$ and $mn = +q$
Thus, the second terms are such that
their product $= +q$ ... (i)
and *their algebraic sum* $= +p$. ... (ii)

So, to be able to factorise the expression, we have to *think out* two numbers satisfying conditions (i) and (ii) above.

The working is illustrated in the following examples.

Example 18: Resolve into factors $x^2+11x+24$.

Sol. (a) The expression can have 2 binomial factors of the form $(x+\ldots\ldots)(x+\ldots\ldots)$.
(b) According to the analysis mentioned above,
(i) the product of the two 2nd terms $= +24$ and
(ii) their algebraic sum $= +11$.

Factors of 24 with ± sign before each	1×24	2×12	$\mathbf{3\times8}$	4×6
Their algebraic sum with ± before each	25, 23	14, 10	**11**, 5	10, 2

The two factors of 24 which add up to $+11$ are $+3$ and $+8$
$\therefore$ The factors of $x^2+11x+24$ are $(x+3)$ and $(x+8)$.

NOTE: It is easy to guess the factors 8 and 3 at once. But it will be very helpful, if the factors are selected in the order 1, 2, 3, 4, etc.

Example 19: Factorise $x^2-11x+24$.

Sol. The factors must be of the form $(x+\ldots\ldots)(x+\ldots\ldots)$.

Find two numbers whose (i) product is $+24$ and (ii) algebraic sum is -11.

By referring to the table in the previous example, the two numbers can evidently be seen to be -3 and -8.
$\therefore$ $x-3$ and $x-8$ are the factors.

Example 20: Resolve into factors $x^2+17x-60$.

Sol. The factors are of the form $(x+\ldots\ldots)(x+\ldots\ldots)$.

We have to find two numbers whose product is -60 and whose sum is $+17$.

Factors are $+$ or $-$	1×60	2×30	$\mathbf{3\times20}$	4×15	5×12	6×10
Algebraic sum $+$ or $-$	61, 59	32, 28	23, **17**	19, 11	**17**, 7	16, 4

Here it is interesting to note that 17 occurs in two boxes.

17 is the sum of $(+12)$ and $(+5)$ or $(+20)$ and (-3). But the product of $(+12)$ and $(+5)$ is $+60$ and not -60. So they are rejected. $(+20)$ and (-3) satisfy our requirements.

$\therefore$ The factors are $x+20$ and $x-3$.

Example 21: Resolve into factors $x^2-17x-60$.

Sol. Here the factors of -60 whose algebraic sum is -17 are -20 and $+3$.

Hence the factors are $x-20$ and $x+3$.

Note: -12 and -5 will add up to -17 & the product will be $+60$. So, $x-12$ and $x-5$ are the factors of $x^2-17x+60$.

The detailed working of all these four examples are shown below:

$$x^2+11x+24$$
$$= x^2+3x+8x+24$$
$$= x(x+3)+8(x+3)$$
$$= (x+3)(x+8)$$

$$x^2-11x+24$$
$$= x^2-3x-8x+24$$
$$= x(x-3)-8(x-3)$$
$$= (x-3)(x-8)$$

Note: $x^2+11x-24$ and $x^2-11x-24$ cannot be factorised.

$x^2+17x-60$
$= x^2+20x-3x-60$
$= x(x+20)-3(x+20)$
$= (x+20)(x-3)$

$x^2-17x-60$
$= x^2-20x+3x-60$
$= x(x-20)+3(x-20)$
$= (x-20)(x+3).$

EXERCISE 62.

Resolve into factors:

1. $x^2+13x+42$.
2. $a^2-18a+65$.
3. $x^2+18x+65$.
4. $p^2-15p+54$.
5. $x^2+18x+72$.
6. $x^2+20x+91$.
7. $x^2-22x+117$.
8. $x^2+21x+104$.
9. $a^2+5a-104$.
10. $x^2+21x+108$.
11. $a^2+3a-108$.
12. $x^2+27x+180$.
13. $p^2+p-156$.
14. $a^2-3a-180$.
15. $a^2-2a-143$.
16. $a^2-a-240$.
17. $x^2+8x-105$.
18. $a^2-26a-120$.
19. $1-19x+60x^2$.
20. $1-49a-102a^2$.
21. $x^2+21x+98$.
22. $a^2-17a+72$.
23. $p^2-5p-176$.
24. $m^2+6m-91$.
25. $p^2-10p+24$.
26. $x^2+7x-30$.
27. $m^2+9m-36$.
28. $x^2-11x-80$.
29. $a^2-10a-56$.
30. $m^2+5m-84$.
31. $x^2-20x+96$.
32. $x^2-6x-72$.
33. $x^2-25x+84$.
34. $x^2+17x-84$.
35. $a^2-9a-90$.
36. $x^2-22x-48$.
37. x^2-x-42.
38. a^2+a-72.
39. $1-19a-120a^2$
40. $1+23x-78x^2$

8. Type VI(a): Trinomials (continued).

Example 22: Resolve into factors $x^2-2xy-15y^2$.

Sol. The factors will evidently be $(x+py)$, and $(x+qy)$, where p and q are such that their sum $= -2$ and their product $= -15$.

These numbers are -5 and $+3$.

$\therefore\ x^2-2xy-15y^2 = (x-5y)(x+3y)$.

Example 23: Resolve into factors x^4+x^2-20.

Sol. Putting a for x^2, the given expression becomes a^2+a-20, and its factors are $a+5$ and $a-4$.

Hence $x^4+x^2-20 = (x^2+5)(x^2-4)$
$= (x^2+5)(x+2)(x-2)$.

Example 24: Resolve into factors $(a^2+2a)^2-4(a^2+2a)-21$.

Sol. Putting x for a^2+2a, the given expression becomes $x^2-4x-21$, and then $x^2-4x-21 = (x-7)(x+3)$.

Hence the given expression $= (a^2+2a-7)(a^2+2a+3)$.

NOTE: Neither of these can be factorised further.

EXERCISE 63.

Resolve into factors:

1. $p^2-12pq+20q^2$.
2. $m^2-12mn+32n^2$.
3. $a^2+ab-30b^2$.
4. $a^2-4ab-45b^2$.
5. $p^2-13pq-48q^2$.
6. $a^2+20ab+96b^2$.
7. $m^2+3mn-28n^2$.
8. $x^2-11xy-80y^2$.
9. a^4-a^2-12.
10. x^4+3x^2-28.
11. x^4-18x^2-175.
12. a^4-25a^2+136.
13. $x^4-2x^2y^2-63y^4$.
14. a^6+2a^3-3.
15. p^6+26p^3-27.
16. x^6-27x^3+180.
17. $(a^2-2a)^2-(a^2-2a)-30$.
18. $(x^2+x)^2-2(x^2+x)-35$.
19. $(m^2-3m)^2-8(m^2-3m)+12$.
20. $(p^2-5p)^2+5(p^2-5p)-36$.
21. $(a^2-4a)^2+16(a^2-4a)+48$.
22. $(x^2-6x)^2-3(x^2-6x)-180$.
23. $(m^2+7m)^2-(m^2+7m)-156$.
24. $(p^2+4p)^2+21(p^2+4p)+98$.
25. $(x^2-8x)^2-(x^2-8x)-240$.

9. Type VI (*b*): Trinomial (*continued*).

Example 25: Resolve into factors
$$(2a+b)^2+(2a+b)(a-b)-12(a-b)^2.$$

Sol. Putting x for $2a+b$ and y for $a-b$, the given expression becomes $x^2+xy-12y^2$, and when factorised
$$= (x+4y)(x-3y).$$

Hence resubstituting for x and y, the given expression
$$= \{(2a+b)+4(a-b)\}\{2a+b)-3(a-b)\}$$
$$= (2a+b+4a-4b)(2a+b-3a+3b)$$
$$= (6a-3b)(-a+4b)$$
$$= -3(2a-b)(a-4b).$$

EXERCISE 64.

Resolve into factors:

1. $(x+y)^2+7(x+y)(x-y)+12(x-y)^2$.
2. $(x+2y)^2-2(x+2y)(x-y)-15(x-y)^2$.
3. $(2a-3b)^2-11(2a-3b)(a+2b)+24(a+2b)^2$.
4. $(3m-n)^2+3(3m-n)(m+n)-28(m+n)^2$.
5. $(3a+4b)^2-3(3a+4b)(a-2b)-10(a-2b)^2$.
6. $(4m-3n)^2-2(4m-3n)(m+2n)-35(m+2n)^2$.
7. $(2a-5b)^2-5(2a-5b)(a-3b)-6(a-3b)^2$.
8. $(a^2-ab)^2-2(a^2-ab)(ab-b^2)-8(ab-b^2)^2$.
9. $(2a^2+3b^2)^2+5(2a^2+3b^2)(a^2-2b^2)-14(a^2-2b^2)^2$.
10. $(a^2-4ab)^2-9(a^2-4ab)(ab-4b^2)+18(ab-4b^2)^2$.

Simplify by factors:

11. $\dfrac{x^2+7x+10}{x^2+5x}$.

12. $\dfrac{x^2+2x+1}{x^2+5x+4}$.

13. $\dfrac{x^2-4x-5}{x^2-2x-15} \div \dfrac{x^2-x-2}{x^2-x-12}$.

14. What is the quotient if $x^2+7x+12$ is divided by $x+4$?

15. If $x+2$ is a factor of $x^2+mx+14$, what is the value of m?

16. If $x-3$ is a factor of $x^2-px-15$, what is the value of p?

17. If $x+4$ is a factor of x^2+9x+c, what is the value of c?

10. **Type VII:** Factors of expressions of the form $\mathbf{ax^2+bx+c}$.

Analysis. $ax^2+bx+c = a\left(x^2+\dfrac{b}{a}x+\dfrac{c}{a}\right)$.

Let $(x+p)(x+q)$ be the factors of $x^2+\dfrac{b}{a}x+\dfrac{c}{a}$.

$p \times q$ must be equal to $\dfrac{c}{a}$ and $p+q$ must be equal to $\dfrac{b}{a}$

or $a \times p \times q$,, ,, ,, c and $a(p+q)$,, ,, ,, b
or $ap \times aq$,, ,, ,, ac and $ap+aq$,, ,, ,, b

In other words, we have to find two quantities (ap) and (aq) whose product $= ac$ and algebraic sum $= b$.

The following examples will make this clear.

Example 26: Resolve into factors $5x^2+33x-14$.

Sol. We have to find two numbers whose
(i) product $= 5\times(-14)$ and
(ii) algebraic sum $= +33$.

Factors of $5\times(-14)$ with + or − before each	1×70	**2×35**	5×14	7×10
Their algebraic sum with + or − before each	71, 69	37, **33**	19, 9	17, 3

$+33$ is obtained by combining $+35$ and -2. Having discovered these two numbers, proceed as under.

$$\begin{aligned} 5x^2+33x-14 &= 5x^2+35x-2x-14 \\ &= 5x(x+7)-2(x+7) \\ &= (x+7)\,(5x-2). \end{aligned}$$

EXERCISE 65.

Resolve into factors:

1. $2x^2+11x+14$.
2. $3x^2+14x-5$.
3. $5x^2+8x+3$.
4. $6x^2+5x-6$.
5. $6x^2-4x-2$.
6. $8x^2-14x+3$.
7. $8x^2+2x-3$.
8. $12m^2+7m-10$.
9. $14a^2+29a+15$.
10. $20p^2+44p-15$.
11. $3-5p+2p^2$.
12. $2+7m-15m^2$.
13. $9a^2-15ab+4b^2$.
14. $2m^2-mn-21n^2$.
15. $4p^2+5pq-9q^2$.
16. $15x^2-77xy+10y^2$.
17. $12a^2+28ab-5b^2$.
18. $10x^2-41xy+21y^2$.

Simplify by factors:

19. $\dfrac{6x^2+x-35}{12x^2-19x-21}\times\dfrac{4x^2-5x-6}{2x^2-x-15}$.

20 $\dfrac{6x^2-7x-20}{3x^2-14x-24}\div\dfrac{21x^2+41x+10}{7x^2-40x-12}$.

Resolve into factors:

21. $2(x+y)^2-3z(x+y)-2z^2$. [*Hint*: Put a for $(x+y)$.]
22. $2(a^2+b^2)^2+5ab(a^2+b^2)+2a^2b^2$.
23. $x^2(x+1)^2-14(x+1)x+24$.
24. $8(a+1)^2+2(a+1)(b+2)-15(b+2)^2$.
25. $6x^4-7x^2-20$.
26. $6x^4-7x^2-3$.
27. $5x^6-7x^3-6$.
28. $18x^4+25x^2y^2-3y^4$.
29. $8x^6-65x^3+8$.
30. $4x^8-17x^4y^4+4y^8$.
31. $5x^2-13xy+6y^2$.
32. $5x^2-13xy-6y^2$.
33. $7x^2-25x+12$.
34. $7x^2-25x-12$.
35. $x^3y^3-9x^2y^2+20xy$.
36. $8a^3-2a^2b-15ab^2$.
37. $(a+7)(a-10)+16$.
38. $(x^2-4x)(x^2-4x-1)-20$. [*Hint*: Put a for (x^2-4x).]
39. $(x^2+x-6)(x^2+x-20)-15$.
40. $(x^2+8x+7)(x^2+8x+15)-9$.
41. $\frac{x^2}{4}+2x-5$. [*Hint*: Put a for $\frac{x}{2}$.]

11. **Type VIII:** Factors of expressions of the form $\mathbf{a^3+b^3+c^3-3abc}$.

$$
\begin{aligned}
&a^3+b^3+c^3-3abc\\
&=(a+b)^3-3ab(a+b)+c^3-3abc\\
&=(a+b)^3+c^3-3ab(a+b+c)\\
&=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c)\\
&=(a+b+c)\{(a^2+2ab+b^2-ac-bc+c^2)-3ab\}\\
&=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)
\end{aligned}
$$

This can also be put in the form

$$a^3+b^3+c^3-3abc=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

NOTE: This is only a reversal of the formula XV.

Example 27: Resolve into factors $8a^3-27b^3+1+18ab$.

Sol. Reducing the given expression to the standard form, we have

$$
\begin{aligned}
&(2a)^3+(-3b)^3+(1)^3-3(2a)(-3b)(1)\\
&=(2a-3b+1)\{(2a)^2+(-3b)^2+(1)^2-(2a)(-3b)\\
&\qquad\qquad -(2a)(1)-(-3b)(1)\}\\
&=(2a-3b+1)(4a^2+9b^2+1+6ab-2a+3b).
\end{aligned}
$$

Example 28: Factorise $(x+y)^3+(y+z)^3+(z+x)^3 -3(x+y)(y+z)(z+x)$.

Sol. *Putting a for* $(x+y)$, b for $(y+z)$ and c for $(z+x)$, we get

$$a^3+b^3+c^3-3abc = \tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

Substituting the values of a, b and c, we get the expression

$$= \tfrac{1}{2}\{(x+y)+(y+z)+(z+x)\}\{(x-z)^2+(y-x)^2+(z-y)^2\}$$
$$= (x+y+z)(2x^2+2y^2+2z^2-2xy-2xz-2yz)$$
$$= 2(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$$

EXERCISE 66.

Factorise :

1. $a^3-8b^3+27c^3+18abc$. **2.** $x^3-y^3-z^3-3xyz$.
3. $a^3-b^3-1-3ab$. **4.** $x^3-3xy+y^3+1$.
5. $a^3+b^3+18ab-216$. **6.** $x^3-8y^3-24xy-64$.
7. $64a^3-27b^3+1+36ab$. **8.** $27x^3-125y^3-180xy-64$.
9. $8a^3-27b^3-c^3-18abc$.
10. $x^3-\frac{1}{x^3}-36$. [*Hint.* $-36 = (-3)^3-3\times 3$.]
11. a^6+32a^3-64. [*Hint.* $32a^3 = (2a)^3+3a^2\times 2a\times 4$.]
12. $(x+y)^3+(y+z)^3-(z+x)^3+3(x+y)(y+z)(z+x)$.
13. $(a-b)^3-(b-c)^3+(c-a)^3+3(a-b)(b-c)(c-a)$.
14. $(x+1)^3+(x+2)^3+(x+3)^3-3(x+1)(x+2)(x+3)$.
15. $(x+y+2z)^3+(y+z+2x)^3+(z+x+2y)^3 -3(x+y+2z)(y+z+2x)(z+x+2y)$.

12. **Type IX.** Factors of expressions of the form $(\mathbf{x+a})(\mathbf{x+b})(\mathbf{x+c})(\mathbf{x+d})+\mathbf{k}$.

Example 29: Factorise $(x+1)(x+3)(x-4)(x-6)+13$.

Sol. The expression $= \{(x+1)(x-4)\}\{(x+3)(x-6)\}+13.$
$= (x^2-3x-4)(x^2-3x-18)+13.$

Putting a for x^2-3x, we get
The expression $= (a-4)(a-18)+13$
$= a^2-22a+72+13$
$= a^2-22a+85$
$= (a-5)(a-17).$
Substituting x^2-3x for a, the expression
$= (x^2-3x-5)(x^2-3x-17).$

Example 30: Factorise $(a+4)(a-6)(a+2)(a-3)-6a^2$.

Sol. The expression $= \{(a+4)(a-3)\}\{(a-6)(a+2)\}-6a^2$
$= (a^2+a-12)(a^2-4a-12)-6a^2$
$= (a^2-12+a)(a^2-12-4a)-6a^2$.

Putting x for a^2-12, the expression
$= (x+a)(x-4a)-6a^2$
$= x^2-3ax-4a^2-6a^2$
$= x^2-3ax-10a^2$
$= (x-5a)(x+2a)$.

Substituting a^2-12 for x, the expression
$= (a^2-5a-12)(a^2+2a-12)$.

NOTE: In factorising expressions of this type, we take the four factors in the expression in two pairs so that their products may contain at least two same terms, as illustrated in the above two examples.

EXERCISE 67.

Factorise:

1. $(x+1)(x+2)(x+3)(x+4)-3$.
2. $(x-1)(x-2)(x+3)(x+4)+4$.
3. $(x+2)(x+3)(x-5)(x-6)+12$.
4. $(x+1)(x+4)(x+7)(x+10)-40$.
5. $(x-3)(x-2)(x-6)(x-4)-12x^2$.
6. $(x-8)(x+4)(x+2)(x-4)+9x^2$.
7. $x(2x+1)(x-2)(2x-3)-63$.
8. $x(3x+2)(x-2)(3x-4)-21$.
9. $(2x+3)(2x+7)(2x-1)(2x-5)+175$.
10. $3x(3x-1)(3x+1)(3x+2)-3$.
11. $(x+3)(x^2-1)(x+5)-9$.
12. $(x^2-4)(x+1)(x+5)-45$.
13. $4(x+1)(2x+3)(x+2)(2x+5)-360$.
14. $9(3x+1)(x+1)(3x-4)(x-2)+13$.
15. Prove that the product of *any* four consecutive numbers increased by one is a perfect square.
16. Prove that the product of *any* four consecutive even numbers increased by 16 is a perfect square.
17. Prove that the product of *any* four consecutive odd numbers increased by 16 is a perfect square.

13. **Type X.** Factors of expressions which *can be reduced* to the form $\mathbf{a^2-b^2}$.

Example 31: Resolve into factors $4a^2-b^2-9c^2+6bc$.

Sol. Since $6bc$ seems to be twice the product of two terms containing b and c, therefore terms containing b and c must be grouped together.

Thus the expression $= 4a^2-(b^2+9c^2-6bc)$
$= (2a)^2-(b-3c)^2$
$= (2a+b-3c)(2a-b+3c)$.

Example 32: Resolve into factors $a^2-b^2-c^2+d^2-2(ad-bc)$.

Sol. The expression $= a^2-b^2-c^2+d^2-2ad+2bc$.
$= (a^2-2ad+d^2)-(b^2-2bc+c^2)$
$= (a-d)^2-(b-c)^2$
$= \{(a-d)+(b-c)\}\{(a-d)-(b-c)\}$
$= (a+b-c-d)(a-b+c-d)$.

EXERCISE 68.

Resolve into factors:

1. $x^2+y^2-z^2+2xy$.
2. $16x^2-4y^2+12yz-9z^2$.
3. $4x^2-9y^2-25z^2+30yz$.
4. $16x^2+40xz-36y^2+25z^2$.
5. $36z^2-60xz-49y^2+25x^2$.
6. $2(ab+cd)-a^2-b^2+c^2+d^2$.
7. $a^2+4b^2-9c^2-16d^2-2(2ab-12cd)$.
8. $16a^2-9b^2-49c^2+25d^2+2(21bc+20ad)$.
9. $1-4a^2+9b^2-25c^2+20ac-6b$.
10. $4a^2-9b^2-25c^2+30bc-24a+36$.
11. $25c^2-49b^2-a^2+14ab-20c+4$.
12. $49a^2-1+16b^2-64c^2+16c-56ab$.

Example 33: Resolve into factors x^4-11x^2+1.

Sol. $x^4-11x^2+1 = x^4-2x^2+1-9x^2$
$= (x^2-1)^2-(3x)^2$.
$= (x^2-1+3x)(x^2-1-3x)$
$= (x^2+3x-1)(x^2-3x-1)$.

Example 34: Resolve into factors $4x^4+81$.

Sol. $4x^4+81 = (2x^2)^2+(9)^2+2(2x^2)(9)-2(2x^2)(9)$
$= (2x^2+9)^2-(6x)^2$
$= (2x^2+6x+9)(2x^2-6x+9)$.

Example 35: Resolve into factors $(a^2-2ab)-(c^2-2bc)$.

Sol. $(a^2-2ab)-(c^2-2bc) = a^2-2ab-c^2+2bc$
$= a^2-2ab+b^2-c^2+2bc-b^2$
$= (a^2-2ab+b^2)-(c^2-2bc+b^2)$
$= (a-b)^2-(c-b)^2$
$= \{(a-b)+(c-b)\}\{(a-b)-(c-b)\}$
$= (a-b+c-b)(a-b-c+b)$
$= (a-2b+c)(a-c)$.

EXERCISE 69.

Resolve into factors:

1. a^4+2a^2+9.
2. a^4+7a^2+16.
3. $4a^4+8a^2b^2+9b^4$.
4. $9a^4-a^2b^2+16b^4$.
5. $4a^4-37a^2+9$.
6. $9a^4-33a^2b^2+16b^4$.
7. $25a^4-19a^2b^2+9b^4$.
8. $a^4-35a^2b^2+25b^4$.
9. $a^4-16a^2b^2+36b^4$.
10. $49a^4-60a^2b^2+16b^4$.
11. $a^4+a^2b^2+b^4$.
12. $625+4a^4$.
13. a^4+4.
14. a^4+64.
15. $9a^4+36b^4$.
16. $64p^4+81q^4$.
17. a^4+a^2+1.
18. a^8+a^4+1.
19. $a^8+a^4b^4+b^8$.
20. $25a^4+35a^2b^2+36b^4$.
21. $(9p^2-6pq)-(r^2-2qr)$.
22. $(a^2-6ab)-(25c^2-30bc)$.
23. $(25a^2-20ab)-3c(3c-4b)$.
24. $7a(7a-4b)-3c(3c-4b)$.

SELECTED QUESTIONS A.

Factorise:

1. $p^2+2pq+q^2-p-q$.
2. $a^2+b^2+2(ab+ac+bc)$.
3. $4a^2+12ab+9b^2-8a-12b$.
4. $(a+b)^4-(a-b)^4$
5. x^8-y^8.
6. $8x^3+27$.
7. x^4y+8xy^4.
8. $125a^6+64b^6$.
9. x^6-729.
10. $64x^{12}-a^{12}$.
11. $16x^7-16x^4-81x^3+81$.
12. $x^2+17x-630$.
13. $x^2+2x-399$.
14. $2x^2+9x-5$.
15. $12x^2+x-35$.
16. $6x^2+25x+21$.
17. $6x^2-13x+5$.

18. $2x^2-x-6$.
19. $2x^2+x-6$.
20. $2x^2+x(2b-a)-ab$.
21. $6x^2+ax-12a^2$.
22. $15x^2-41xa+14a^2$.
23. $2x^2-5xy-12y^2$.
24. $16x^2-16xy+3y^2$.
25. $(x+1)(x+2)(x+3)(x+4)-8$.
26. $x(x-1)(x-2)(x-3)+1$.
27. $(x+3)(x+4)(x-8)(x-9)-64$.
28. $(x-2)(x-9)(x-6)(x-3)-15x^2$.
29. $a^2-b^2+c^2+2ac$.
30. p^4+4.
31. x^4+64.
32. x^4+4y^4.
33. $4x^4+1$.
34. $x^4-x^2y^2+16y^4$.
35. $81a^4+64b^4$.
36. y^4+y^2+1.
37. a^4+4a^2+16.
38. $x^4-7x^2y^2+y^4$.
39. $a^4+2a^2b^2+9b^4$.
40. x^8+x^4+1.
41. x^8+14x^4+81.
42. $(a^2-b^2)(c^2-d^2)+4abcd$.
43. $a^2-b^2-c^2+d^2-2\ (ad-bc)$.

CHAPTER VI

REMAINDER THEOREM, APPLICATIONS

1. It is known that in any division,

Divisor $\times$ *Quotient* $+$ *Remainder* $=$ *Dividend.*

Consider the case in which $2x^2+3x+5$ is divided by $x-2$.

Actual division

$$
\begin{array}{r|l}
x-2 & 2x^2+3x+5 \\
 & 2x^2-4x \\
\hline
 & 7x+5 \\
 & 7x-14 \\
\hline
 & +19 \\
\hline
\end{array}
\quad (2x+7
$$

Remainder $= +19$

Denoting quotient by Q and remainder by R, we have

$(x-2) \times Q+R = 2x^2+3x+5$

Putting $x = 2$, we get

$(2-2) \times Q+R = 2(2)^2+3\times 2+5$

$0 \times Q+R = 8+6+5$

$\therefore \quad R = 19$

Take another case in which $3x^2-2x-6$ is divided by $x+2$.

Actual division

$$
\begin{array}{r|l}
x+2 & 3x^2-2x-6 \\
 & 3x^2+6x \\
\hline
 & -8x-6 \\
 & -8x-16 \\
\hline
 & +10 \\
\hline
\end{array}
\quad (3x-8
$$

Remainder $= +10$

Denoting quotient by Q and remainder by R, we have

$(x+2) \times Q+R = 3x^2-2x-6$

By putting $x = -2$, we get

$(-2+2) \times Q+R = 3\,(-2)^2-2\,(-2)-6$

$0 \times Q+R = 3\times 4+4-6$

$\therefore \quad R = 10.$

Note that the same remainder is obtained by both methods in each case.

In general, let px^3+qx^2+rx+s be divided by $x-a$ and let Q be the quotient and R the remainder.

Then $(x-a) \times Q+R = px^3+qx^2+rx+s.$

Putting $x = a$, $(a-a) \times Q+R = pa^3+qa^2+ra+s$

$0 \times Q+R = pa^3+qa^2+ra+s$

$\therefore \quad R = pa^3+qa^2+ra+s.$

NOTE: It is immaterial whether Q contains x or not, since $0\times Q$ is always 0. But R does not contain x, for, the remainder must be of a lower degree than the divisor which itself is of the first degree.

From these results we can deduce the following theorem.

Theorem: *When an expression in x is divided by $x-a$, the remainder can be obtained by putting a for x in the given expression.*

This is known as the **Remainder Theorem.**

NOTE: (*i*) If the divisor is $x+a$, the remainder can be obtained by putting $-a$ for x in the expression.

(*ii*) If the divisor is $2x-3$, then put the divisor $= 0$ and we get $x = \frac{3}{2}$. The remainder is obtained by putting $x = \frac{3}{2}$.

2. If $2x^3-5x^2+5x-2$ is divided by $x-1$, the remainder $= 2(1)^3-5(1)^2+5(1)-2 = 0$. This means that $x-1$ is a factor of $2x^3-5x^2+5x-2$. Similarly in a division when the remainder is zero, the divisor is a factor of the dividend. So we say,

If an expression in x is reduced to zero on substituting $\pm a$ for x in it, then $x \mp a$ is a factor of the expression.

This is known as the **Factor Theorem.**

Rule (1): *$x-1$ is a factor of an expression in x if the sum of the coefficients of all powers of x including the independent term is zero.*

Rule (2): *$x+1$ is a factor of an expression in x if the sum of the coefficients of odd powers of x is equal to the sum of the coefficients of even powers of x.* (The independent term is counted as an even power of x).

Example 1: Prove that $x-1$, $x+1$, $x-2$ are factors of $2x^4-5x^3+5x-2$.

Sol. (i) When $x-1$ is the divisor,

$$R = 2(+1)^4-5(+1)^3+5(+1)-2$$
$$= 2-5+5-2=0.$$

Since in this case the remainder is 0,
$x-1$ is a factor of the given expression.

(ii) When $x+1$ is the divisor,

$$R = 2(-1)^4-5(-1)^3+5(-1)-2$$
$$= 2+5-5-2 = 0.$$

Since in this case the remainder is 0,
$x+1$ is a factor of the given expression.

(iii) When $x-2$ is the divisor,

$$R = 2(+2)^4-5(+2)^3+5(+2)-2$$
$$= 32-40+10-2 = 0.$$

Since in this case the remainder is 0,
$x-2$ is a factor of the given expression.

Example 2: Resolve into factors $x^3-6x^2+12x-7$.

Sol. The sum of the co-efficients $= 1-6+12-7 = 0$;
$\therefore$ by rule (1), $x-1$ is a factor of the expression.

$$\begin{aligned} x^3-6x^2+12x-7 &= x^3-x^2-5x^2+5x+7x-7 \\ &= x^2(x-1)-5x(x-1)+7(x-1) \\ &= (x-1)(x^2-5x+7). \end{aligned}$$

Example 3: Resolve into factors $x^3+8x^2+19x+12$.

Sol. Here, the sum of the co-efficients of the 1st and third terms $= 1+19 = 20$, and the sum of the co-efficients of the 2nd and the fourth terms $=8+12=20$;
$\therefore$ by rule (2), $x+1$ is a factor of the expression.

$$\begin{aligned} x^3+8x^2+19x+12 &= x^3+x^2+7x^2+7x+12x+12 \\ &= x^2(x+1)+7x(x+1)+12(x+1) \\ &= (x+1)(x^2+7x+12) \end{aligned}$$

As $x^2+7x+12$ can be factorised into $(x+3)(x+4)$,
$\therefore \quad x^3+8x^2+19x+12 = (x+1)(x+3)(x+4)$.

Example 4: Resolve into factors $x^5+x^3-3x^2-2x+3$.

Sol. When the missing term in the form of $0.x^4$ is added, the expression $= x^5+0.x^4+x^3-3x^2-2x+3$ and the sum of the co-efficients of the 1st, 3rd and 5th terms $= 1+1-2 = 0$, whereas the sum of the co-efficients of the 2nd, 4th and 6th terms $= 0-3+3 = 0$.

As these sums are equal, $x+1$ is a factor of the expression.

$$\begin{aligned} & x^5+x^3-3x^2-2x+3 \\ &= x^5+x^4-x^4-x^3+2x^3+2x^2-5x^2-5x+3x+3 \\ &= x^4(x+1)-x^3(x+1)+2x^2(x+1)-5x(x+1)+3(x+1) \\ &= (x+1)(x^4-x^3+2x^2-5x+3). \end{aligned}$$

Again, as the sum of the co-efficients of $x^4-x^3+2x^2-5x+3$ is 0,

$x-1$ is a factor of $x^4-x^3+2x^2-5x+3$.

$$\therefore \quad \begin{aligned} x^4-x^3+2x^2-5x+3 &= x^3(x-1)+2x(x-1)-3(x-1) \\ &= (x-1)(x^3+2x-3). \end{aligned}$$

Again, as the sum of the co-efficients of x^3+2x-3 is 0,

$x-1$ is a factor of x^3+2x-3.

$$\therefore \quad \begin{aligned} x^3+2x-3 &= x^2(x-1)+x(x-1)+3(x-1) \\ &= (x-1)(x^2+x+3). \end{aligned}$$

Since x^2+x+3 cannot be factorised

$$x^5+x^3-3x^2-2x+3 = (x+1)(x-1)(x-1)(x^2+x+3).$$

EXERCISE 70.

By the Remainder Theorem, find the remainder when

1. x^3-5x^2+6x-3 is divided by $x-1$.
2. $2x^3-3x^2+4x-1$ is divided by $x+2$.
3. $4x^3-2x^2+5x-3$ is divided by $x-a$.
4. $5x^3-x^2+3x-4$ is divided by $x+m$.
5. $2x^4+7x^3a-x^2a^2+11xa^3-6a^4$ is divided by $x+2a$.

Prove that:

6. $x-1$ is a factor of $x^4-3x^3+x^2+3x-2$.
7. $x+2$ and $x-3$ are the factors of $x^3-3x^2-4x+12$.
8. $x-2$ and $x+4$ are the factors of $2x^3+7x^2-10x-24$.
9. $(a+b)$ and $(b+c)$ are the factors of $(a+b+c)^3-a^3-b^3-c^3$.

(*Hint*: *Substitute in the expression* $-b$ *for* a *in the first case and* $-c$ *for* b *in the second case*).

10. a is a factor of $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(b+a-c)^3$. [*Hint.* $a = (a-0)a = 0$.]

Resolve into factors using the factor theorem rules:

11. x^3+x^2+x-3.
12. $x^3-6x^2+11x-6$.
13. $x^3+9x^2+23x+15$.
14. $3x^3+x^2+x-5$.
15. $4x^3+3x^2+x+2$.
16. $2x^4+3x^3+2x^2-3x-4$.
17. $x^4+2x^3-5x^2+2$.
18. x^4-x^3+7x+5.
19. $x^4+x^3-7x^2-x+6$.
20. $2x^4-x^3-8x^2+11x-4$.

Example 5: For what value of m will $x-2$ be a factor of x^3-5x^2+mx+m ?

Sol. $x-2$ will be a factor of the given expression

if $(2)^3-5(2)^2+m(2)+m = 0$
or if $-12+3m = 0$
or if $m = 4$.

EXERCISE 71.

1. For what value of p is $x^2-(p+2)x+6$ exactly divisible by $x-p$?

2. Find the condition that ax^3+bx^2+cx+d may be exactly divisible by $x-1$.

3. Find the value of a, b for which x^2+x-6 may be a factor of x^3-ax^2-bx-6.

4. What number must be added to $5x^4-7x^3+6$ so that the result may be divisible by $x+3$?

5. If $x^2-4ax-19a^2-b$ is exactly divisible by $x-7a$, prove that $b = 2a^2$.

6. The expressions px^3+2x^2-3 and x^4-px+4 leave the same remainder when they are divided by $x-2$. Find the value of p.

3. **Trial Factors.**

By actual multiplication, we have

$$(2x+3)(5x+2)(3x-1) = 30x^3+47x^2-x-6$$

Putting $x = 1$ in the above identity, we get

$$(2+3)(5+2)(3-1) = 30+47-1-6$$

or $$5\times7\times2 = 70.$$

Each of the numbers 5, 7, 2 is a factor of 70. But all these are the algebraic sums of the co-efficients of the factors and the expression. Hence the following rule:

The algebraic sum of the co-efficients of each factor of an expression is a factor of the algebraic sum of the co-efficients of the expression.

NOTE: The converse of this statement, however, is not true. The algebraic sum of the co-efficients of $5x+2$ and $2x+5$ is the same. But $5x+2$ is a factor of the above expression while $2x+5$ is not.

When the number of trial factors is large, this rule enables us to reject some of them and to simplify the process of factorisation.

Example 6: Factorise x^3-5x^2-5x-6.

Sol. As the expression is of the 3rd degree, it can have either three linear factors or one linear and one quadratic ; *i.e., it must have* at least *one linear factor.*

As the first term of the expression is x^3, the first term of the linear factor is x.

As the last term of the expression is -6, the last term of the linear factor may be one of

$$+1,\ -1,\ +2,\ -2,\ +3,\ -3,\ +6,\ -6.$$

Or the linear trial factors could be

$(x+1)$, $(x-1)$, $(x+2)$, $(x-2)$, $(x+3)$, $(x-3)$, $(x+6)$ or $(x-6)$.

Since, in this case, neither rule (1) holds good nor rule (2), $(x+1)$ or $(x-1)$ cannot be a factor of the expression,

Again, the sum of the co-efficients of the expression is

$$1-5-5-6 = -15,$$

whereas the sum of the co-efficients of $x+2$ is 3 and it is a factor of 15, therefore $x+2$ may be a factor of the expression: the sum of the co-efficients of $x-2$ is -1 and it is a factor of 15, therefore $x-2$ may be a factor of the expression; the sum of the co-efficients of $x+3$, $x-3$, $x+6$ is 4, -2, 7 respectively and none of them is a factor of 15, therefore none of $x+3$, $x-3$, $x+6$ can be a factor of the expression; the sum of the co-efficients of $x-6$ is -5 and it is a factor of 15, therefore $x-6$ may be a factor of the expression.

Hence $x+2$, $x-2$, $x-6$ are the only binomials left for trial.

Substituting -2 or $+2$ for x in the given expression, we find that it does not vanish, but on substituting $+6$ for x, we see that it vanishes.

$\therefore$ $x-6$ is one of the factors.

The given expression can be put as

$$x^2(x-6)+x(x-6)+(x-6) = (x-6)(x^2+x+1).$$

As x^2+x+1 cannot be further factorised,
$(x-6)(x^2+x+1)$ are the required factors.

Example 7: Factorise $3x^4+4x^3-57x^2-18x-40$.

Sol. Since, in this case, rules (1), (2) do not hold good, $(x-1)$ or $(x+1)$ is not a factor of this expression.

As the last term of the expression is -40, the last term of the linear factor (if any) would be 2, 4, 5, 8, 10 or 20.

Hence the possible trial factors are
$x+2$, $x-2$, $x+4$, $x-4$, $x+5$, $x-5$, $x+8$, $x-8$, $x+10$, $x-10$, $x+20$, $x-20$.

The sum of the co-efficients of the expression

$$= 3+4-57-18-40 = -108.$$

Thesums of the co-efficients of the trial factors are respectively 3, -1, $+5$, -3, $+6$, -4, $+9$, -7, $+11$, -9, $+21$ and -19.

As 5, 9, -7, 11, 21 and -19 are not the factors of 108, we reject $x+4$, $x+8$, $x-8$, $x+10$, $x+20$ and $x-20$.

On substituting -2 or $+2$ for x in the given expression, we find that it does not vanish, therefore we reject $x+2$ and $x-2$ as well.

On substituting $+4$ for x in the given expression, we find that it vanishes.

$\therefore$ $x-4$ is one of its factors.

Hence $3x^4+4x^3-57x^2-18x-40$

$= 3x^3(x-4)+16x^2(x-4)+7x(x-4)+10(x-4)$

$= (x-4)(3x^3+16x^2+7x+10)$.

If we substitute -5 for x in $3x^3+16x^2+7x+10$, it vanishes.

$\therefore$ $3x^3+16x^2+7x+10 = 3x^2(x+5)+x(x+5)+2(x+5)$

$= (x+5)(3x^2+x+2)$

Since $3x^2+x+2$ cannot be further factorised

$(x-4)(x+5)(3x^2+x+2)$ are the required factors.

EXERCISE 72.

Factorise :

1. $x^3+3x^2-10x-24$.
2. $x^3-5x^2-2x+24$.
3. $6x^3+x^2-19x+6$.
4. $x^3-19x-30$.
5. $x^3-39x+70$.
6. x^3-x^2+12.
7. $x^3-5x+12$.
8. $x^4-2x^3-20x^2+39x+36$.

Factorise the following expressions, applying Remainder Theorem :

9. $2a^3+9a^2-8a-15$.
10. $4a^3-15a^2+3a+18$.
11. $a^3+a^2b+ab^2-3b^3$.
12. $a^3-4a^2b+5ab^2-2b^3$.
13. $a^3+3a^2b-18ab^2-40b^3$.
14. $2a^3-9a^2-27a+54$.
15. $2a^3-7a^2-7a+12$.
16. $2a^3-3a^2b+7ab^2-3b^3$.
17. $6a^3-17a^2b+6ab^2+8b^3$.
18. $12a^3+5a^2b-11ab^2-6b^3$.

4. Divisibility of $x^n \pm a^n$ by $x \pm a$.

(i) When x^n-a^n is divided by $x-a$, the remainder is $a^n-a^n=0$,

$\therefore$ x^n-a^n is *always* divisible by $x-a$.

(ii) When x^n-a^n is divided by $x+a$, the remainder is $(-a)^n-a^n$ and is equal to 0 or $-2a^n$, according as n is even or odd.

$\therefore$ x^n-a^n is divisible by $x+a$ only when n is *even*.

(iii) When x^n+a^n is divided by $x+a$, the remainder is $(-a)^n+a^n$ and is equal to 0 or $2a^n$, according as n is *odd* or *even*.

$\therefore$ x^n+a^n is divisible by $x+a$ only when n is *odd*.

(iv) When x^n+a^n is divided by $x-a$, the remainder is $a^n+a^n=2a^n$.

$\therefore$ x^n+a^n is *never* divisible by $x-a$.

Example 8: Show that 10^n-1 is *always* divisible by 9.

Sol. Since x^n-a^n is always divisible by $x-a$,
by putting $x = 10$ and $a = 1$, we find that
10^n-1^n is always divisible by $10-1$,
or 10^n-1 is always divisible by 9.

Example 9: Simplify $\dfrac{8x^3+27y^3}{2x+3y}$.

Sol. Since x^n+a^n is divisible by $x+a$ when n is *odd*,

$$\frac{8x^3+27y^3}{2x+3y} = \frac{(2x)^3+(3y)^3}{2x+3y} = (2x)^2-(2x)(3y)+(3y)^2$$
$$= 4x^2-6xy+9y^2.$$

Example 10: Simplify $\dfrac{x^5+32y^5}{x+2y}$.

Sol. Since x^n+a^n is divisible by $x+a$ when n is *odd*,

$$\frac{x^5+32y^5}{x+2y} = \frac{(x)^5+(2y)^5}{x+2y}$$
$$= x^4-x^3(2y)+x^2(2y)^2-x(2y)^3+(2y)^4$$
$$= x^4-2x^3y+4x^2y^2-8xy^3+16y^4.$$

EXERCISE 73.

Show that

1. 9^n-1 is *always* divisible by 8.
2. 10^n+1 is *never* divisible by 9.
3. 10^n-1 is divisible by 11 when n is *even*.
4. 10^n+1 is divisible by 11 when n is *odd*.
5. $4^{3n}-3^{3n}$ is *always* divisible by 37.
6. $8^{2m}-1$ is *always* divisible by 63.
7. 17^8-5^8 is divisible by 6 and 11.

Write down by *inspection* the quotient of:

8. $(8a^3-x^3)\div(2a-x)$. **9.** $(a^3+64b^3)\div(a+4b)$.
10. $(64a^6-x^6)\div(2a+x)$. **11.** $(27m^3-1)\div(3m-1)$.
12. $(x^4-256y^4)\div(x-4y)$. **13.** $(1-16m^4)\div(1+2m)$.
14. $(32a^5+b^5)\div(2a+b)$.

SELECTED QUESTIONS B

1. If the expression x^3+3x^2+4x+p contains $x+6$ as a factor, find p.

2. The expressions px^3+2x^2-3 and x^4-px+4 leave the same remainder when they are divided by $x-2$. Find the value of p.

Resolve into factors:

3. $x^3+7x^2+14x+8$. **4.** $x^3+7x^2-21x-27$.
5. a^3+a^2+2a+8. **6.** $x^3-13x-12$.
7. $x^3-19x+30$. **8.** x^3-7x+6.
9. $x^3-52x+96$. **10.** x^3-3x-2.

SECTIONAL REVISION II

TEST PAPERS

PAPER I

1. (i) State $(a\pm b)^2=a^2\pm 2ab+b^2$ in words and apply it in expanding $(5x\pm 4y)^2$.

(ii) Simplify $(3a-b)^2+(a+3b)^2$.

2. Find the value of $x^2+\frac{1}{x^2}$ when $x+\frac{1}{x}=6$.

3. Prove that (i) the difference between two numbers which are formed by the same two digits is exactly divisible by 9; (ii) the difference of their squares is exactly divisible by 99.

4. Factorise the following:

(i) $(2a-1)^2-(a-2)^2$; (ii) $1-4a^2-b^2-4ab$;
(iii) $24x^2-4x-48$; (iv) $(x-1)(x-5)(x-9)+(x-3)(x-4)(x-5)$.

5. Find the value of:

$a^3+b^3-c^3+3abc$ when $a=\cdot 457$, $b=\cdot 236$, $c=\cdot 693$.

6. Simplify by factors: $\dfrac{\cdot72\times\cdot72-\cdot48\times\cdot48}{1\cdot2}$.

PAPER 2

1. (i) Find the missing term in the perfect squares:
(a) $81x^2-126xy+(\quad)$; (b) $36x^2+(\quad)+121y^2$.
(ii) Find by $(a\pm b)^2=a^2\pm2ab+b^2$, the value of
(a) $(504)^2$, (b) $(495)^2$.

2. Find the value of $x^2+\dfrac{1}{x^2}$ when $x-\dfrac{1}{x}=8$.

3. (i) A class is required to work out questions 2, 5, 8, 11, 14, 17, 20, etc. in a certain exercise. Find a formula which gives these numbers.
(ii) Write down the quotient of x^5-y^5 by $x-y$.

4. (i) Show that $(4x^2-5x+7)^2-(5x^2+14x+2)^2$ is divisible by x^2+x+1 and find the quotient.
(ii) Prove the identity
$(x-1)(2x+3)+(2x+1)(x-5)\equiv(3x+2)(x-3)+(x-2)(x+1)$.

5. Factorise (i) x^3+x^2-42x.
(ii) $(a+b)^2-2(a^2-b^2)-15(a-b)^2$.

6. Find the value of
$a^3+b^3+c^3-3abc$ when $a=357$, $b=367$, $c=377$.

PAPER 3

1. State $(a+b)(a-b)=a^2-b^2$ in words; apply it in finding the product of $(p-2q+3r)(p-2q-3r)$ and the value of $(328)^2-(323)^2$.

2. Find the value of $x^4+\dfrac{1}{x^4}$ when $x-\dfrac{1}{x}=4$.

3. Prove the identity $(n+3)^2\equiv3(n+2)^2-3(n+1)^2+n^2$ and use it to express the value of d^2 in terms of a^2, b^2, c^2 where a, b, c, d are four consecutive numbers, of which a is the least.

4. Show that $(4x^2-8x-1)^2-(2x^2-5x+7)^2$ is divisible by $2x^2-3x-8$, and express the quotient as the product of two factors.

5. Simplify the following:
(i) $\dfrac{(2a-9)^2-(a-6)^2}{(a-5)^2}$

(ii) $\dfrac{3(a-8)^3(a-9)^2-(a-8)^2(a-9)^3}{2a^2-31a+120}$.

6. Find the value of
$x^3+8y^3-27+18xy$ when $x+2y=3$,

PAPER 4

1. (i) State in words the formula
$(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$, and complete the following:
$(p+2)(\ldots-3)(p+\ldots) = p^3+(\quad)p^2+(\ldots)p\quad-24$.
(ii) Write down the product of $(x+3)$, $(x-4)$, $(x+5)$.

2. Find the value of $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=4$.

3. If $a+b+c=0$, show that
(i) $(a+b)(b+c)(c+a) = -abc$,
and (ii) $a^3+b^3+c^3 = 3abc$.

4. If $a=1$, $b=2$, $c=3$, find the value of
(i) $\dfrac{(a+b+c)^3}{a^3+b^3+c^3}$ (ii) $a^{bc}+b^{ca}+c^{ab}$
(iii) $(bc)^a+(ca)^b+(ab)^c$.

5. Factorise (i) $16x^8-1$, (ii) $27a^5b^2-8a^2b^5$
(iii) $4(2a+3b)^2-9(a-b)^2$,
(iv) $(2x^2-5x+3)(2x^2-5x+4)-2$.

6. Simplify by factors:

$$\frac{62\times62-38\times38}{62\times62+2\times62\times38+38\times38.}$$

PAPER 5

1. State in words $(a\pm b)^3 = a^3\pm3a^2b+3ab^2\pm b^3$
$= a^3\pm b^3\pm3ab(a\pm b)$,
and fill in the blanks in the following
$(5m-\ldots.)^3 \equiv (\ldots.)-3(\ldots.)(\ldots.)+3(5m)(\ldots.)-(2n)^3$.

2. Find the value of:
(i) a^3+b^3 if $a+b=11$ and $ab=30$,
(ii) a^3-b^3 if $a-b=3$ and $ab=40$,
(iii) $8m^3-36m^2+54m-50$ when $m=3$

3. (i) Show that x^2+y^2 is less than $(x+y)^2$

(ii) Find v in terms of u and k from $\frac{1}{u}-\frac{1}{v}=\frac{1}{k}$.

4. Factorise (i) $(x-3)(x+1)^3+(x-1)^4-2(x-1)^3$.
(ii) $x^4+x^3-30x^2$,
(iii) $x^2-xz+yz-y^2$.

5. (i) Make n the subject of the formula $\mathrm{I}=\frac{E}{\frac{r}{n}+R}$.

(ii) Find the co-efficient of x^3 in the expansion of $(6x^3-5x^2-4x+2)(3x^3+2x^2+5x-1)$.

6. (i) Find the value of $a^2+b^2+c^2$ when $a+b+c=15$ and $ab+ac+bc=85$.

(ii) Simplify by factors $\frac{(2\cdot35)^3-(1\cdot65)^3}{(2\cdot35)^2-(1\cdot65)^2}$.

PAPER 6

1. State in words $(a+b)(a^2+ab+b^2)=a^3+b^3$; apply it in writing down the product of
$(3+4a)(9-12a+16a^2)$ and simplify
$(p^2+5)(p^4-5p^2+25)-(p^2-2)(p^4+2p^2+4)$.

2. Find the value of $x^3-\frac{1}{x^3}$ when $x-\frac{1}{x}=p$.

3. Factorise (i) $(2x+y)^2+6y(2x+y)+9y^2$,
(ii) $x^2(x+a)-x(x+a)(a+b)+ab(x+a)$.

4. (i) Find k when $(x-a)(x-3a)(x+a)(x+3a)+k$ is a perfect square.

(ii) Substitute $(m-1)$ for a in the expression $2a^3-3a^2+4a-5$ and arrange the result in descending powers of m.

5. (i) Find the value of
$8x^3+y^3-1+6xy$ when $x=\cdot357$, $y=\cdot286$.

(ii) Simplify by factors
$$\frac{\cdot73\times\cdot73\times\cdot73+\cdot59\times\cdot59\times\cdot59}{\cdot073\times\cdot073-\cdot073\times\cdot059\times\cdot059\times\cdot059.}$$

6. Simplify $\frac{x^2-13x+40}{x^2-5x+4}\times\frac{x^2-4x}{x^2-2x-48}\div\frac{x^2-5x}{x^2+5x-6}$.

PAPER 7

1. State in words $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$; apply it in expanding $(3x-2y+4z)^2$, and simplify $(2p-q+r)^2 -(p+2q-r)^2-(2r+q-p)^2$.

2. Find the value of $a^2+b^2+c^2$ when $a+b+c=12$ and $ab+ac+bc=47$.

3. (i) Find the number which is half-way between $a-b$ and $b-a$.

(ii) Given any three consecutive numbers, prove that the difference of the squares of the greatest and the least is equal to four times the other number.

4. Factorise (i) $81a^4+64b^4$, (ii) $4x^2-9y^2-6x-9y$,
(iii) $35(x-y)^2-41(x-y)+12$.
(iv) $(2x+1)(2x+3)(2x+5)(2x+7)+16$.

5. If $x = b+c-2a$, $y = c+a-2b$, $z = a+b-2c$, find the value of $y^2+z^2-x^2+2yz$.

6. Find the value of x which will make the expression $x^5-8x^3+11x^2+7x-1789$ exactly divisible by x^2+7x-1.

PAPER 8

1. State in words $(a+b+c)(a^2+b^2+c^2-ab-ac-bc) = a^3+b^3+c^3-3abc$; apply it in finding the product of $(2m-n-3)(4m^2+n^2+2mn+6m-3n+9)$.

2. Use the identity $a^2-b^2 = (a+b)(a-b)$ to find the difference between the squares of 587047 and 312953.

3. Resolve into factors:
(i) $512(x-\frac{1}{8})^3-8(x-\frac{1}{2})^3$,
(ii) $1+2x+x(x+2)+(x+1)\ (x+2)$,
(iii) $4a^2+12ab+9b^2-8a-12b$,
(iv) $(a^2-1)(a+2)+(a^2+2a)(a+1)$.

4. Show by means of a formula that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)(x+z)$.

5. Simplify $\dfrac{a^4+a^2b^2+b^4}{a^2-4ab-21b^2} \times \dfrac{a^2+2ab-3b^2}{a^3-b^3} \div \dfrac{1}{a-7b}$.

6. Find the value of $27\times51+27\times49+73\times51+73\times49$.

PAPER 9

1. (i) Simplify by factors $\dfrac{\cdot254\times\cdot254-\cdot16\times\cdot16}{\cdot094}$.

(ii) Write by inspection the continued product of $(x-a)$, (x^2+a^2), (x^4+a^4), $(x+a)$.

2. If $2s = a+b+c$, prove that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c) = \tfrac{1}{2}(a^3+b^3+c^3-3abc).$$

3. Divide the product of $2m^2+11m-21$ and $3m^2-20m-7$ by m^2-49.

4. Find the value of $x^2-y^2+4x+14y-45$ when $x+y = 23$ and $x-y = 7$.

[*Hint*: *Split* -45 *into* $+4$ *and* -49 *and express the whole expression as the difference of two squares.*]

5. Factorise (i) a^4+4. (ii) a^3+a^2-a-1

(iii) a^8+a^4+1.

6. Express: a^3+b^3 in terms of x and y if $x = a+b$, $y = a-b$.

OBJECTIVE TYPE ITEMS—II

1. Rewrite the following, filling up the blanks :—

(i) $(-3)^2-(-2)^3 =$ .....

(ii) $(3\cdot4)^2+(8\cdot6)^2+(6\cdot8)(8\cdot6) =$ .....

(iii) If $(x-2)$ is a factor of x^2-ax+8, then $a =$ .....

(iv) If $x+\dfrac{1}{x} = 2$, then $x^2+\dfrac{1}{x^2} =$ .....

(v) $5x^2-\frac{1}{5}y^2 = 5($.....$)($.....$)$

(vi) $\dfrac{(1\cdot53)^2-(\cdot47)^2}{1\cdot06} =$ .....

(vii) $(2\cdot475)^3+(1\cdot525)^3+12(2\cdot475)(1\cdot525) =$ .....

(viii) $(2a-b)(2a+b)(4a^2+b^2)(16a^4+b^4) =$ .....

2. Put an x in the space provided against the correct answer in each of the following :

(i) If $m-\dfrac{1}{m} = 3$, then $m^3-\dfrac{1}{m^3}$ is equal to

.....A. 9 .....B. 18 .....C. 27 .....D. 36

(ii) The expression $8x^2-6xy-9y^2$ is factorised as

.....A. $(2x+3y)(4x+3y)$ .....B. $(2x-3y)(4x+3y)$
.....C. $(2x+3y)(4x-3y)$ .....D. $(2x-3y)(4x-3y)$

(iii) If $(x-3)$ is a factor of the expression $x^3-5x^2-2x+8k$, then k is equal to

.....A. 11 .....B. 5 .....C. 3 .....D. -3

(iv) If x^3+3x^2-6x-8 is divided by $x+2$, the remainder is

.....A. -8 .....B. $+8$ .....C. Zero .....D. -10

(v) If $a = 3$, $b = 5$, $c = 7$, then the value of $a^2+b^2+c^2-2ab+2bc-2ca$ is equal to

.....A. 1 .....B. 25 .....C. 81 .....D. 225

CHAPTER VII

H. C. F. AND L. C. M.

1. H.C.F. by Factors.

The meaning of the expression H.C.F. and very simple cases of finding the H.C.F. were dealt with earlier in chapter I. In the case of expressions containing two or more terms also, the process of finding the H.C.F. is the same as in arithmetic. *To find the H.C.F. of such expressions, factorise them completely and pick out the factor or factors common to all the expressions.*

Example 1: Find the highest common factor of
$a^3+2a^2b+ab^2$, $6a^3+4a^2b-2ab^2$, $3(a^2+ab)^2$.

Sol.
$$a^3+2a^2b+ab^2 = a(a^2+2ab+b^2) = a(a+b)^2$$
$$6a^3+4a^2b-2ab^2 = 2a(3a^2+2ab-b^2) = 2a(a+b)(3a-b)$$
$$3(a^2+ab)^2 = 3\{a(a+b)\}^2 = 3a^2(a+b)^2$$

The H.C.F. of these expressions is obviously $a(a+b)$.

Example 2: Find the H.C.F. of $5x^2-3x-8$ and x^4-2x^3-4x-7.

Sol.
$$5x^2-3x-8 = 5x^2-8x+5x-8$$
$$= x(5x-8)+(5x-8)$$
$$= (5x-8)(x+1).$$

Obviously $5x-8$ cannot be a factor of the 2nd expression, as it ends in 8, and 8 is not a factor of 7. If there is a common factor at all, it can only be $x+1$.

Test if $x+1$ is a factor of the 2nd expression by putting $x = -1$ according to the Remainder Theorem.

The remainder $= (-1)^4-2(-1)^3-4(-1)-7$
$= 1+2+4-7 = 0$

$\therefore$ $x+1$ is a factor of the expression.

Hence, the H.C.F. is $x+1$.

EXERCISE 74.

Find by factors the H. C. F. of:

1. a^2-b^2, $a(a-b)$.
2. a^3-3a^2b, $3ab^2-9b^3$.
3. $a^4-4a^2b^2$, ab^2+2b^3.
4. a^3+8, a^2+5a+6.
5. a^3-27, $(a-3)^3$.
6. a^2-a, $(a-1)^2$, a^3-1.

7. $a^2-3a-18$, a^2+5a+6.
8. a^2+5a+6, $a^2+9a+14$, $a^2-7a-18$.
9. $a^2-c^2+b^2-2ab$, $a^2-b^2-c^2-2bc$.
10. $x^2+7x+12$, $x^2+9x+20$, x^3+64.

Find by factors the H.C.F. of:

11. $2a^2-7a+6$, a^3-5a^2+a+10.
12. $3a^2+a-14$, $a^3+2a^2-3a-10$.
13. $a^2-4a-21$, a^3+3a^2-3a-9.
14. a^4-1, $3a^5+2a^4+4a^3+2a^2+a$.
15. a^2+a-6, $a^2+3a-10$, a^3+a^2-5a-2.
16. a^2+a-12, a^2-2a-3, $a^3-4a^2-2a+15$.

2. L. C. M. by Factors.

Factorise the given expressions completely. Then take the highest power of every factor which occurs in any of these expressions and multiply them.

Example 3: Find by factors the L. C. M. of:
$(a^2+2a)^2$, $2a^4+3a^3-2a^2$ and $2a^3-3a^2-14a$.

Sol.
$$(a^2+2a)^2 = \{a(a+2)\}^2 = a^2(a+2)^2,$$
$$2a^4+3a^3-2a^2 = a^2(2a^2+3a-2) = a^2(a+2)(2a-1).$$
$$2a^3-3a^2-14a = a(2a^2-3a-14) = a(a+2)(2a-7).$$
$$\therefore \text{ the L. C. M.} = a^2(a+2)^2(2a-1)(2a-7).$$

EXERCISE 75.

Find by factors the L. C. M. of :

1. $(a-1)^2$, (a^3-a).
2. a^2-4, a^3+8.
3. $(a-1)^3$, a^3-1.
4. a^2+4a+3, a^2+5a+6.
5. $a^2-ab-2b^2$, $a^2-5ab+6b^2$, $a^2-2ab-3b^2$.
6. $x^2+5ax+4a^2$, $x^2+11ax+28a^2$, $x^2+20ax+91a^2$.
7. x^2+3x+2, x^2+4x+3, x^2+5x+6.
8. a^2-1, a^3+1, a^3-1.
9. a^2-1, a^2+1, a^4+1, a^8-1.
10. a^2-1, a^3+1, a^3-1, a^6+1.
11. a^2-4b^2, $(a+2b)^2$, $(a-2b)^2$.
12. $6b^2(a-b)^2$, $9a^3(a-b)^3$, $12ab(a-b)^5$.
13. $18(a+b)^2(a^3-b^3)$, $24(a-b)^2(a+b)^3$, $36(a^2-b^2)^2$.
14. $6a^2b^2(a^3-b^3)$, $15b^4(a-b)^3$, $12a^3b(a-b)(a^2-b^2)$.
15. $a^4+a^2b^2+b^4$, a^3b+b^4, $(a^2-ab)^3$.
16. $x^2+(b-c)x-bc$, $x^2+(c-a)x-ca$, $x^2+(a-b)x-ab$

17. $1+4a+4a^2-16a^4$, $1+2a-8a^3-16a^4$.

18. x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, $x^4-2x^3+2x^2-2x+1$.

3. H. C. F. by Division.

When two expressions cannot be easily factorised, their H. C. F. can be found by the method of division. This method is based on the following principles.

(i) The H. C. F. of two expressions A and B is the same as the H. C. F. of $m \times A$ and $n \times B$ and of $\frac{A}{m}$ and $\frac{B}{n}$ provided that m does not have any common factor with n or B and n does not have any common factor with m or A.

(ii) The H. C. F. of two expressions A and B is also the H. C. F. of $mA+nB$ and $ma-nB$ i.e. the sum and difference of their multiples, provided that the multiplying factors m and n have no common factor.

Example 4: Find the H. C. F. of a^3+a^2-2 and a^3+2a^2-3.

Sol. Proceed as you do in finding out the G. C. M. of large numbers in arithmetic.

a	a^3+a^2-2	a^3+2a^2-3	1
	a^3-a	a^3+a^2-2	
1	a^2+a-2	a^2-1	a
	a^2-1	a^2-a	
	$a-1$	$a-1$	1
		$a-1$	

$\therefore$ the H. C. F. $=a-1$.

Steps in the process: Divide a^2+2a^2-3 by a^3+a^2-2, the remainder $=a^2-1$. This remainder contains the required H.C.F.

Divide a^3+a^2-2 by a^2-1, the remainder in this case is $a-1$. This remainder contains the required H. C. F.

Divide a^2-1 by $a-1$. Since in this case there is no remainder, $a-1$ is the H. C. F.

NOTE: The last *divisor* is the H. C. F.

Example 5: Find the H.C.F. of $a^5-a^3-4a^2-3a-2$ and $2a^5-a^4-a^3-3a^2$.

Sol. The second expression $=a^2(2a^3-a^2-a-3)$.

Since a^2 is a factor of the second expression but not of the first, it may be neglected. The second expression thus left is $2a^3-a^2-a-3$.

Now the first expression, as it is, cannot be divided by $2a^3-a^2-a-3$, unless we multiply it by 2, which is not a factor of the second expression.

Thus we have

$$
\begin{array}{r|l|l|l}
 & & a^5 - a^3 - 4a^2 - 3a - 2 & \\
 & & 2 & \\
\hline
 & 2a^3 - a^2 - a - 3 & 2a^5 - 2a^3 - 8a^2 - 6a - 4 & \\
-2 & 2a^3 + 18a^2 + 18a + 16 & 2a^5 - a^4 - a^3 - 3a^2 & a^2 \\
\hline
-19 & -19a^2 - 19a - 19 & a^4 - a^3 - 5a^2 - 6a - 4 & \\
 & & 2 & \\
\hline
 & & 2a^4 - 2a^3 - 10a^2 - 12a - 8 & \\
 & & 2a^4 - a^3 - a^2 - 3a & a \\
\hline
 & & -a^3 - 9a^2 - 9a - 8 & \\
 & & -a^3 - a^2 - a & -a \\
\hline
 & & -8a^2 - 8a - 8 & \\
 & & -8a^2 - 8a - 8 & -8 \\
\hline
\end{array}
$$

$\therefore$ the H. C. F. $= a^2+a+1$.

NOTE: In this method it is *necessary* to arrange the terms in the expressions according to ascending or descending powers of a certain letter.

EXERCISE 76.

Find by the method of division the H. C. F. of:

1. $a^3+6a^2+13a+12$, $a^3+7a^2+16a+16$.
2. $a^3+7a^2+17a+15$, $a^3+8a^2+19a+12$.
3. $a^3-10a^2+26a-8$, $a^3-9a^2+23a-12$.
4. $6a^3-5a^2+4a-1$, $3a^3+2a^2+5a-2$.
5. $2a^3-7a^2-8a-35$, $2a^3+9a^2+16a+21$.
6. $3a^3+4a^2-6a-8$, $36a^3+27a^2-16a+16$.
7. $24a^4-72a^3+54a^2$, $16a^5-48a^4+36a^3$.
8. $2a^3+5a^2+a-2$, $2a^3-5a^2-4a+3$.

9.	$a^3+4a^2b-8ab^2+24b^3$,	$a^5-a^4b+8a^2b^3-8ab^4$.
10.	a^3-4a^2+2a+3,	$2a^4-9a^3+12a^2-7$.
11.	a^4+a^2-6,	a^4-3a^2+2.
12.	$a^4-9a^2-30a-25$,	$a^5+a^4-7a^2+5a$.
13.	$2a^4-6a^3+3a^2-3a+1$,	$a^7-3a^6+a^5-4a^2+12a-4$.
14.	$2a^4+9a^3+14a+3$,	$3a^4+15a^3+5a^2+10a+2$.
15.	$12a^4-30a^3+156a-210$,	$15a^4-25a^3+145a-75$.

4. H. C. F. of more than two expressions.

To find the H. C. F. of more than two expressions by division method, first we find the H. C. F. of two of the expressions, and then find the H. C. F. of this H. C. F. and the third expression ; and so on. The last H. C. F. is the one required.

EXERCISE 77.

Find by the method of division the H. C. F. of:

1. x^3-4x^2+4x, x^3+x^2-7x+2, $2x^3-x^2-7x+2$.

2. x^3-x^2-7x+3, x^3-5x^2+9x-9, x^3-4x^2+4x-3.

3. $2x^3-7x^2+7x-2$, $4x^3-13x^2+11x-2$, x^4-3x^3+6x-4.

4. $x^3+6x^2y+9xy^2+4y^3$, $x^4+9x^3y+28x^2y^2+36xy^3+16y^4$ and $x^4+8x^3y+21x^2y^2+22xy^3+8y^4$.

5. $3x^3+28x^2y+52xy^2-48y^3$, $3x^3+4x^2y-28xy^2+16y^3$ and $3x^3+10x^2y-44xy^2+24y^3$.

5. H. C. F. by the method of alternately cancelling the highest and the lowest terms.

Example 6: Find the H. C. F. of x^5-x^3+8 and x^5-x^2+4.

Sol.

Cancelling the highest term	Cancelling the lowest term
Let $A = x^5 - \quad -x^3+8$ and $B = x^5 \quad -x^2+4$.	$A = x^5-x^3+8$ $2B = 2x^5-2x^2+8$
$\therefore A-B = \quad -x^3+x^2+4$. Multiply by -1 and call it C $C = \quad x^3-x^2-4$	$A-2B = -x^5-x^3+2x^2$ Divide by $-x^2$ and call it D $D = x^3+x-2$

The H. C. F. of A and B is the same as that of C and D.

$C = x^3-x^2-4$ $D = x^3+x-2$	$C = x^3-x^2-4$ $2D = 2x^3+2x-4$
$C-D = -x^2-x-2$ Multiply by -1 and call it E $E = x^2+x+2$	$C-2D = -x^3-x^2-2x.$ Divide by $-x$ and call it F. $F = x^2+x+2$

Now the H. C. F. of E and F is the same as that of C and D or that of A and B; but E and F are indentical.

Therefore, the required H. C. F. is x^2+x+2.

NOTE: The method is of universal application and *especially useful* when the co-efficients of the given expressions are cross-wise equal or related as multiples and sub-multiples.

Example 7: Reduce to the lowest terms $\frac{x^3+x^2+x-3}{x^3+3x^2+5x+3}$.

Sol. First find the H. C. F. of the numerator and denominator. It is x^2+2x+3.

Then divide both by the H. C. F.

$$\therefore \text{ The expression} = \frac{(x^3+x^2+x-3)\div(x^2+2x+3)}{(x^3+3x^2+5x+3)\div(x^2+2x+3)}$$

$$= \frac{x-1}{x+1}.$$

EXERCISE 78.

Find by the method of alternate destruction of the highest and the lowest terms, the H. C. F. of:

1. $2x^4+x^3-6x^2-2x+3$ and $2x^4-3x^3+2x-3$.
2. $x^4-3x+20$ and $5x^4-3x^3+64$.
3. $x^4+x^3-5x^2-3x+2$ and $x^4-3x^3+x^2+3x-2$.
4. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
5. $2x^5-5x^2+3$ and $3x^5-5x^3+2$.
6. x^5+11x^3-54 and $x^5+11x+12$.
7. $6x^3-11x^2+5x-3$ and $9x^3-9x^2+5x-2$.

Reduce to the lowest terms:

8. $\frac{2x^2-13x+18}{2x^3-11x^2+3x+27}$.

9. $\frac{x^3-3x+2}{x^3+3x^2-4}$.

10. $\dfrac{x^3-13x+12}{x^3+x-2}$.

11. $\dfrac{4x^3-8x^2+5x-3}{12x^3+4x^2+x+5}$.

12. $\dfrac{x^4-4x^3+4x^2-4x+3}{x^4-2x^3-2x^2-2x-3}$.

13. $\dfrac{x^4-13x^2+36}{x^4-x^3-7x^2+x+6}$.

14. $\dfrac{3x^4-14x^3-9x+2}{2x^4-9x^3-14x+3}$.

15. $\dfrac{x^4-2x^3-25x^2+26x+120}{x^4-4x^3-19x^2+46x+120}$.

6. L.C.M. by H.C.F.

Let A and B stand for two expressions whose H.C.F. is H. Divide A and B by H and let a and b be their respective quotients.

Then $A = a.\ H$ and $B = b.\ H$.

Since a and b have no common factor, the L.C.M. of A and B is obviously $a \times H \times b$.

Let L stand for the L.C.M. of A and B. Then we have

$$L = a.\ H.\ b$$

or $$L \times H = (a.H) \times (b.H) = A \times B.$$

In words, **the product of the L.C.M. and H.C.F. of two expressions is equal to the product of the expressions.**

Since $L \times H = A \times B$, we have

$$L = \frac{A \times B}{H} = \frac{A}{H} \times B = A \times \frac{B}{H}.$$

Hence, in order to find the L.C.M. of two expressions,

(i) *multiply the two expressions and divide their product by their H.C.F.*

or (ii) *divide one of the expressions by their H.C.F. and multiply the quotient by the other expression.*

[NOTE : The dot is sometimes used instead of $\times$, to denote multiplication. Thus $a.\ H.\ b. = a \times H \times b$].

Example 8: Find the L.C.M. of $x^3-9x^2+26x-24$ and $x^3-6x^2+11x-6$.

Sol. First we find the H.C.F. of these expressions.

$$\begin{array}{r|l|l|l} x & x^3-6x^2+11x-6 & x^3-9x^2+26x-24 & 1 \\ & x^3-5x^2+6x & x^3-6x^2+11x-6 & \\ \hline -1 & -x^2+5x-6 & -3x^2+15x-18 & \\ & -x^2+5x-6 & x^2-5x+6 & -3 \end{array}$$

$\therefore$ the H.C.F. $= x^2-5x+6$ and when the second expression is divided by the H.C.F., the quotient $= x-1$.

$\therefore$ the L.C.M. $=(x-1)(x^3-9x^2+26x-24)$.

Example 9: The H.C.F. of two expressions is $x-7$ and their L.C.M. is $x^3-10x^2+11x+70$. One of the expressions is $x^2-5x-14$. Find the other.

Sol. $\because \quad A\times B=L\times H, \qquad B=\dfrac{L\times H}{A}.$

$$\therefore \quad \text{the 2nd expression} = \frac{(x^3-10x^2+11x+70)(x-7)}{x^2-5x-14}$$

$$= \frac{(x^3-10x^2+11x+70)(x-7)}{(x-7)(x+2)}$$

$$= \frac{x^3-10x^2+11x+70}{x+2}$$

$$= x^2-12x+35.$$

EXERCISE 79.

Find the H.C.F. and L.C.M. of the following expressions:

1. $x^3-5x^2+9x-9,\ x^3+x^2-3x+9.$

2. $x^3-2x^2-13x-10,\ x^3-x^2-10x-8.$

3. $4x^3-10x^2-18x+45,\ 6x^3+8x^2-27x-36.$

4. $2x^4+x^3-9x^2+8x-2,\ 2x^4-7x^3+11x^2-8x+2.$

5. $x^4+x^3-9x^2-3x+18,\ x^5+6x^2-49x+42.$

6. $8x^4-6x^3-8x^2+9x-6,\ 16x^4-12x^3+20x^2-9x+6.$

7. $x^4-2x^3+5x^2-4x+3,\ 2x^4-x^3+6x^2+2x+3.$

8. $15x^3-31ax^2+5a^2x+2a^3,\ 6x^4-25ax^3+26a^2x^2-a^4.$

9. The H.C.F. of two expressions is $5x^2-2x-1$, their L.C.M. is $30x^4+13x^3-11x^2-7x-1$, and one expression is $10x^3+x^2-4x-1$; find the other.

10. The H.C.F. of two expressions is x^3+1, their L.C.M. is $(x^3+1)(3x-1)(4x-1)$, and one expression is $3x^4-x^3+3x-1$; find the other.

11. The H.C.F. of two expressions is $1+x^3-x^4$, their L.C.M. is $(1+x)(1-x^3)(1+x^3-x^4)$, and one expression is $1+x+x^3-x^5$; find the other.

12. The H.C.F. of two expressions is x^2-2x+4, their L.C.M. is $x^6-2x^5+7x^4-6x^3-6x^2+36x-72$, and one expression is $x^4-2x^3+x^2+6x-12$; find the other.

7. L.C.M. of more than two expressions.

To find the L.C.M. of more than two expressions, first we find the L.C.M. of two expressions, then the L.C.M. of that L.C.M. and the third expression, and so on. The last L.C.M. is the one required.

EXERCISE 80.

Find the L.C.M. of the following:

1. x^3+x+2, x^3+3x+4 and x^3+x^2+4.
2. $x^5+x^3+x^2+1$, x^4-x^3+x-1 and x^5+2x^4-x-2.
3. $x^5+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9 and $x^4+2x^3-5x^2-6x+9$.

SELECTED QUESTIONS—C.

Find the H.C.F. of:

1. $2y^3+3y^2x-x^3$ and $4y^3+yx^2-x^3$.
2. $3-4x-16x^2-9x^3$ and $4-7x-19x^2-8x^3$.
3. $8x^4+4x^3-18x^2+11x-2$ and $32x^3+12x^2-36x+11$.
4. $32x^4+16x^3-72x^2+44x-8$ and $32x^3+12x^2-36x+11$.
5. $6(a^3+a^2-a-1)$ and $15(a^5+9a^4-2a^3-6a^2+a-3)$.
6. $6x^4+7x^3+5x^2+2x$ and $4x^5-18x^4-8x^3-10x^2$.
7. x^7+y^7 and x^5+y^5.
8. $x^3-7x^2+5x-35$, x^4+8x^2+15 and $x^3(x^2+8)-7(x^4+15)+15x-56x^2$.

Find the H.C.F. and L.C.M. of

9. $2x^2-6x$, $6x^2-54$ and $4x^2-8x-12$.
10. $a^2-b^2-c^2-2bc$, $b^2-c^2-a^2-2ac$ and $c^2-a^2-b^2-2ab$.
11. $x^3+6x^2+11x+6$ and x^3+2x^2-x-2.
12. $4x^3+8x^2-3x-9$ and $12x^3+28x^2+13x-3$.
13. $1+x^4+x^2$ and $1+x^4-x-x^3$.

14. x^4+2x^2+9 and x^3-x+6.

15. x^4-2x^3+x-1 and x^4-3x^2+1.

16. $4a^4+11a^2+9$ and $2a^4+3a^3+6a^2+4a+3$.

CHAPTER VIII

FRACTIONS

1. Addition and Subtraction.

The processes of addition and subtraction of fractions in algebra are the same as in arithmetic. But before proceeding to addition and subtraction, the first essential thing is *to reduce the given fractions to equivalent fractions with a common denominator*, which is always the L.C.M. of the several denominators.

While finding the L.C.M., the following points must be noted.

(i) $\because$ $x-y=-(y-x)$, their L.C.M. is either $+(x-y)$ or $-(x-y)$.

(ii) x^2-a^2 is divisible by $(x-a)$ as well as by $(a-x)$.

$\therefore$ The L.C.M. of x^2-a^2 and $a-x$ may be taken as $+(x^2-a^2)$ or $-(x^2-a^2)$ *i.e.* (a^2-x^2).

(iii) If there are two or more letters in each of the several denominators, they should be arranged in one particular order throughout.

Denominators like $x-a$, $a+x$, x^2+a^2-ax, a^2+x^2+ax should be rearranged so as to read either $x-a$, $x+a$, x^2-xa+a^2, x^2+xa+a^2 or $-(a-x)$, $a+x$, a^2-ax+x^2, a^2+ax+x^2.

Example 1: Reduce to the lowest common denominator

$$\frac{a}{x-a},\quad \frac{x}{a-x},\quad \frac{a^2}{x^2-a^2},\quad \frac{ax}{a^2-x^2}.$$

Sol. Here we have to arrange the denominators in the same order.

Thus $$\frac{x}{a-x}=\frac{-1\times x}{-(a-x)}=\frac{-x}{x-a},$$

and $$\frac{ax}{a^2-x^2}=\frac{-1\times ax}{-(a^2-x^2)}=\frac{-ax}{x^2-a^2}.$$

The L.C.M. of the denominators when arranged as above is x^2-a^2 and the quotients obtained by dividing it by the denominators are $x+a$, $x+a$, 1, and 1 respectively.

Hence we have

$$\frac{a}{x-a}=\frac{a(x+a)}{x^2-a^2}\ ;\ \frac{x}{a-x}=\frac{-x}{x-a}=\frac{-x(x+a)}{x^2-a^2}\ ;$$

$\frac{a^2}{x^2-a^2}$ stands as it is, $\frac{ax}{a^2-x^2}=\frac{-ax}{x^2-a^2}$.

Example 2: Find the value of $\frac{x}{x-y}+\frac{y}{y-x}$.

Sol. Since $\frac{y}{y-x}=\frac{y\times(-1)}{(y-x)\times(-1)}=\frac{-y}{x-y}$.

we have $\frac{x}{x-y}+\frac{y}{y-x}=\frac{x}{x-y}+\frac{-y}{x-y}$

$$=\frac{x+(-y)}{x-y}=\frac{x-y}{x-y}=1.$$

Example 3: Find the value of $\frac{x}{x-y}-\frac{y}{x+y}$.

Sol. The L. C. M. of the denominators $=x^2-y^2$.

$$\therefore\ \frac{x}{x-y}-\frac{y}{x+y}=\frac{x(x+y)}{x^2-y^2}-\frac{y(x-y)}{x^2-y^2}$$

$$=\frac{x(x+y)-y(x-y)}{x^2-y^2}$$

$$=\frac{x^2+y^2}{x^2-y^2}.$$

EXERCISE 81.

Reduce to the lowest common denominator:

1. $\frac{2}{a-b}$, $\frac{3}{b-a}$, $\frac{4}{a+b}$.

2. $\frac{a+b}{a-b}$, $\frac{a-b}{a+b}$, $\frac{2ab}{b^2-a^2}$.

3. $\frac{4a-b}{1-4ab}$, $\frac{4a+b}{1+4ab}$, $\frac{4b(1-8a^2)}{16a^2b^2-1}$.

4. $\frac{3}{x^2-5x+6}$, $\frac{4}{x^2-4x+3}$, $\frac{5}{3x-2-x^2}$.

5. $\frac{1}{(a-b)^2-c^2}$, $\frac{1}{(b-c)^2-a^2}$, $\frac{1}{(c-a)^2-b^2}$.

6. $\frac{a}{b(a-b-c)}$, $\frac{b}{a(a-b+c)}$, $\frac{c}{a^2+b^2-c^2-2ab}$.

Find the value of:

7. $\frac{2}{x-2}+\frac{3}{x+3}$.

8. $\frac{4}{x-3}+\frac{5}{x+4}$.

9. $\frac{x+a}{x-a}-\frac{x-a}{x+a}$.

10. $1+\frac{4xy}{(x-y)^2}$.

11. $\frac{x-1}{x+1}-\frac{x-2}{x+2}$.

12. $(1+x)+\frac{x^2}{1-x}$.

13. $\frac{1}{a^2+ab}+\frac{1}{b^2+ab}$.

14. $\frac{x}{x^2-y^2}-\frac{y}{y^2-x^2}$.

15. $\frac{1}{2x+3y}-\frac{(2x-3y)^2}{8x^3+27y^3}$.

16. $\frac{1}{(a-b)(a-c)}+\frac{1}{(a-c)(b-c)}$

17. $\frac{a+b}{a-b}-\frac{a-b}{a+b}+\frac{2ab}{b^2-a^2}$.

18. $\frac{1}{a^2-b^2}-\frac{2}{(a+b)^2}+\frac{3}{(a-b)^2}$.

Example 4: Simplify $\frac{1}{x^2-5x+6}-\frac{1}{x^2-6x+8}-\frac{2}{x^2-7x+12}$.

Sol. Factorising the denominators, we have

the expression $=\frac{1}{(x-2)(x-3)}-\frac{1}{(x-2)(x-4)}-\frac{2}{(x-3)(x-4)}$.

The L. C. M. of the denominators is $(x-2)(x-3)(x-4)$.

$\therefore$ The given expression $=\frac{(x-4)-(x-3)-2(x-2)}{(x-2)(x-3)(x-4)}$

$=\frac{x-4-x+3-2x+4}{(x-2)(x-3)(x-4)}$

$=\frac{-2x+3}{(x-2)(x-3)(x-4)}$.

Example 5: Simplify $\frac{2a^2+a-1}{a+1}+\frac{3a^2+5a+2}{3a+2}+\frac{4-a^2}{a+2}$.

Sol. Factorising the numerators of these fractions, we have the expression

$=\frac{(2a-1)(a+1)}{a+1}+\frac{(3a+2)(a+1)}{3a+2}+\frac{(2-a)(2+a)}{a+2}$

$=(2a-1)+(a+1)+(2-a)$

$=2a+2=2(a+1)$.

EXERCISE 82.

Find the value of:

1. $\frac{1}{x^2-5x+6}+\frac{1}{x^2-9x+20}$.

2. $\frac{1}{x^2+3x-10}-\frac{1}{2x^2-x-6}$.

3. $\frac{1}{x^2-3x+2}+\frac{1}{x^2-5x+6}+\frac{2}{x^2-8x+15}$.

4. $\frac{1}{x^2+7x+12}+\frac{1}{x^2+9x+20}-\frac{2}{x^2+8x+15}$.

5. $\frac{1}{x^2+3x+2}+\frac{1}{x^2+4x+3}-\frac{2}{x^2+5x+6}$.

6. $\frac{2x-3}{x^2+2x-3}-\frac{3x-4}{x^2+x-2}+\frac{x+5}{x^2+5x+6}$.

7. $\frac{1}{x^2+5ax+4a^2}+\frac{1}{x^2+11ax+28a^2}+\frac{2}{x^2+20ax+91a^2}$.

Simplify :

8. $\frac{x^2-1}{x-1}+\frac{x^2-7x+12}{x-4}+\frac{x^2+5x+6}{x+3}$.

9. $\frac{x^2-2x-3}{x^2-4x+3}+\frac{5x^2+5x-30}{x^2+2x-3}+\frac{x^2+x-2}{x^2-2x+1}$.

10. $\frac{a^4+a^2b^2+b^4}{a^3+b^3}-\frac{a^2+ab}{a-b}+\frac{b^3}{a^2-b^2}$.

11. $\frac{a^2-1}{(a+1)^2}+\frac{a^2-5a+6}{a^2-4a+4}+\frac{1}{a+1}$.

12. $\frac{x^3+3x^2-4}{x+2}+\frac{x^3+1}{x+1}+\frac{2x^3+6x^2+x+3}{x+3}$

13. $\frac{3x^3-4x^2-2x+6}{x^2-1}-\frac{3x^3-4x^2+4x-3}{x^2+1}$

Example 6: Simplify $\frac{1}{a^2-1}+\frac{1}{a^2-2}-\frac{1}{a^2+1}-\frac{1}{a^2+2}$.

Sol. Combining fractions having almost similar denominators,

we have the expression $= \left(\frac{1}{a^2-1} - \frac{1}{a^2+1}\right) + \left(\frac{1}{a^2-2} - \frac{1}{a^2+2}\right)$

$= \frac{(a^2+1)-(a^2-1)}{a^4-1} + \frac{(a^2+2)-(a^2-2)}{a^4-4}$

$= \frac{2}{a^4-1} + \frac{4}{a^4-4}$

$= \frac{2(a^4-4)+4(a^4-1)}{(a^4-1)(a^4-4)}$

$= \frac{6(a^4-2)}{(a^4-1)(a^4-4)}$.

Example 7: Simplify $\frac{1}{a+3} + \frac{1}{a-3} + \frac{6}{9-a^2}$.

Sol. Since the first two fractions have almost similar denominators, we combine them together and change the signs of the third.

$\therefore$ The expression $= \left(\frac{1}{a+3} + \frac{1}{a-3}\right) - \frac{6}{a^2-9}$

$= \frac{a-3+a+3}{a^2-9} - \frac{6}{a^2-9}$

$= \frac{2a}{a^2-9} - \frac{6}{a^2-9}$

$= \frac{2(a-3)}{a^2-9} = \frac{2}{a+3}$.

EXERCISE 83.

Simplify :

1. $\frac{1}{a-1} - \frac{1}{a+1} + \frac{1}{a-2} - \frac{1}{a+2}$.

2. $\frac{1}{a-b} - \frac{2}{2a+b} + \frac{1}{a+b} - \frac{2}{2a-b}$.

3. $\frac{1}{a-3b} + \frac{3}{a+b} + \frac{1}{a+3b} + \frac{3}{a-b}$.

4. $\frac{1}{a^2-1} + \frac{1}{a^2-3} - \frac{1}{a^2+1} - \frac{1}{a^2+3}$.

5. $\frac{3}{1+a}-\frac{2}{1-a}-\frac{5a}{a^2-1}$.

6. $\frac{3-x}{1-3x}-\frac{3+x}{1+3x}-\frac{1-16x}{9x^2-1}$.

7. $\frac{a+b}{a-b}-\frac{2ab}{a^2-b^2}-\frac{2a^2b^2}{a^4-b^4}$.

8. $\frac{1}{a+b}+\frac{3b}{ab-a^2}+\frac{4b-a}{a^2-b^2}$.

9. $\frac{1}{1-a}+\frac{1}{1+a}+\frac{2}{1+a^2}+\frac{4}{1+a^4}$.

10. $\frac{1}{a-1}-\frac{1}{a+1}+\frac{a-2}{a^2-a+1}-\frac{a+2}{a^2+a+1}$.

Example 8: Simplify $\frac{1}{a-1}-\frac{a}{a^2-1}-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$.

Sol. In exercises of this type, it is useful to combine the first fraction with the second, the result so obtained with the third, and so on.

$$\text{The expression} = \left(\frac{1}{a-1}-\frac{a}{a^2-1}\right)-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$$

$$= \frac{(a+1)-a}{a^2-1}-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$$

$$= \frac{1}{a^2-1}-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$$

$$= \left(\frac{1}{a^2-1}-\frac{a^2}{a^4-1}\right)-\frac{a^4}{a^8-1}$$

$$= \frac{(a^2+1)-a^2}{a^4-1}-\frac{a^4}{a^8-1}$$

$$= \frac{1}{a^4-1}-\frac{a^4}{a^8-1}$$

$$= \frac{a^4+1-a^4}{a^8-1} = \frac{1}{a^8-1}.$$

Example 9: Simplify $\frac{1}{x+1}-\frac{3}{x+2}+\frac{3}{x+3}-\frac{1}{x+4}$.

Sol. Combining the first and the fourth terms and the second and the third, we have the expression

$$= \left(\frac{1}{x+1}-\frac{1}{x+4}\right)-3\left(\frac{1}{x+2}-\frac{1}{x+3}\right)$$

$$= \frac{x+4-x-1}{(x+1)(x+4)} - 3 \times \frac{x+3-x-2}{(x+2)(x+3)}$$

$$= \frac{3}{(x+1)(x+4)} - \frac{3}{(x+2)(x+3)}$$

$$= 3\left\{\frac{1}{(x+1)(x+4)} - \frac{1}{(x+2)(x+3)}\right\}$$

$$= 3\left\{\frac{(x+2)(x+3)-(x+1)(x+4)}{(x+1)(x+2)(x+3)(x+4)}\right\}$$

$$= \frac{3(x^2+5x+6-x^2-5x-4)}{(x+1)(x+2)(x+3)(x+4)}$$

$$= \frac{6}{(x+1)(x+2)(x+3)(x+4)}.$$

EXERCISE 84.

Simplify :

1. $\frac{1}{x+a} + \frac{a}{x^2-a^2} + \frac{x}{x^2+a^2} + \frac{2x^3}{x^4+a^4}$.

2. $\frac{1}{a-1} - \frac{a}{a^2-1} - \frac{a^2}{a^4-1} - \frac{a^4}{a^8-1}$.

3. $\frac{1}{x-a} + \frac{2x}{x^2+a^2} + \frac{4x^3}{x^4+a^4} - \frac{8x^7.}{x^8-a^8}$.

[*Hint. Add and subtract* $\frac{1}{x+a}$.]

4. $\frac{a-b}{a^2-ab+b^2} + \frac{a+b}{a^2+ab+b^2} + \frac{2b^3}{a^4+a^2b^2+b^4}$.

5. $\frac{1}{a^2-ab+b^2} - \frac{2ab}{a^4-a^2b^2+b^4} + \frac{4a^3b^3}{a^8-a^4b^4+b^8} - \frac{8a^7b^7}{a^{16}+a^8b^8+b^{16}}$.

6. $\frac{1}{x+3} + \frac{x+1}{x^2-3x+9} - \frac{2x^2+x+12}{x^3+27}$.

7. $\frac{1}{(n-1)n} + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}$.

8. $\dfrac{1}{x+2}-\dfrac{3}{x+3}+\dfrac{3}{x+4}-\dfrac{1}{x+5}$.

9. $\dfrac{1}{x+4}-\dfrac{3}{x+2}+\dfrac{3}{x}-\dfrac{1}{x-2}$.

10. $\dfrac{1}{x+1}-\dfrac{1}{x+3}-\dfrac{1}{x+5}+\dfrac{1}{x+7}$.

11. $\dfrac{1}{x+a}-\dfrac{1}{x+b}+\dfrac{1}{x-a}-\dfrac{1}{x-b}$.

12. $\dfrac{1}{a^2-2b^2}-\dfrac{1}{a^2+2b^2}+\dfrac{1}{a^2-3b^2}-\dfrac{1}{a^2+3b^2}$.

2. Complex and Continued Fractions.

The process for the simplification of complex fractions in Algebra is similar to the one in Arithmetic. In Algebra, however, the process is generally shortened by the application of factors.

In the case of continued fractions, we begin from the bottom and go upward, step by step.

A few examples given below will further illustrate the method.

Example 10: Simplify $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}$

Sol. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}=\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}=\dfrac{4ab}{a^2-b^2}$

$\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}=\dfrac{(a+b)^2+(a-b)^2}{a^2-b^2}=\dfrac{2(a^2+b^2)}{a^2-b^2}$.

Hence the given fraction $=\dfrac{4ab}{a^2-b^2}\div\dfrac{2(a^2+b^2)}{a^2-b^2}$

$=\dfrac{4ab}{a^2-b^2}\times\dfrac{a^2-b^2}{2(a^2+b^2)}$

$=\dfrac{2ab}{a^2+b^2}$.

Example 11: Simplify $\dfrac{a}{a-\dfrac{1}{a-\dfrac{a}{1+a}}}$.

Sol. The expression $= \dfrac{a}{a-\dfrac{1}{\dfrac{a+a^2-a}{1+a}}} = \dfrac{a}{a-\dfrac{1}{\dfrac{a^2}{1+a}}}$

$= \dfrac{a}{a-1\times\dfrac{1+a}{a^2}} = \dfrac{a}{a-\dfrac{1+a}{a^2}}$

$= \dfrac{a}{\dfrac{a^3-1-a}{a^2}} = \dfrac{a\times a^2}{a^3-a-1}$

$= \dfrac{a^3}{a^3-a-1}$.

EXERCISE 85.

Simplify :

1. $\dfrac{1-\dfrac{1}{1+x}}{\dfrac{1}{1+x}}$.

2. $\dfrac{\dfrac{1}{x-1}+\dfrac{x}{x+1}}{\dfrac{x}{x-1}-\dfrac{1}{x+1}}$.

3. $\dfrac{\dfrac{1+x}{1-x}-\dfrac{2x}{1+x}}{\dfrac{1-x}{1+x}+\dfrac{2x}{1-x}}$.

4. $\dfrac{\dfrac{x}{x-1}-\dfrac{x+1}{x}}{\dfrac{x}{x+1}-\dfrac{x-1}{x}}$.

5. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x}{x+y}+\dfrac{y}{x-y}}$.

6. $\dfrac{\dfrac{x+2}{x+1}-\dfrac{1}{x+3}}{\dfrac{1}{x+1}+\dfrac{x+2}{x+3}}$.

7. $\dfrac{\left(\dfrac{x+a}{x-a}\right)^2-x^2}{\dfrac{x+a}{x-a}-x}$.

8. $\dfrac{\dfrac{x^3-y^3}{x-y}+\dfrac{x^3+y^3}{x+y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$.

9. $\dfrac{1+\dfrac{x-y}{x+y}}{1-\dfrac{x-y}{x+y}}\times\dfrac{1+\dfrac{y-z}{y+z}}{1-\dfrac{y-z}{y+z}}\times\dfrac{1+\dfrac{z-x}{z+x}}{1-\dfrac{z-x}{z+x}}$.

10. $\cfrac{1}{x-\cfrac{1}{x+\cfrac{1}{x-\cfrac{1}{x}}}}$.

11. $\cfrac{x}{x-\cfrac{x-1}{1-\cfrac{1}{x+1}}}$.

12. $2-\cfrac{2}{1-\cfrac{3}{2-\cfrac{3}{1-x}}}$.

13. $\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1+\cfrac{2b-a}{a-b}}}}$.

14. $\cfrac{2x^3}{x^2+y^2-\cfrac{4xy(x-y)}{(x+y)-\cfrac{(x-y)^2}{x+y}}}$.

15. $\cfrac{x}{1+\cfrac{x}{2-x+\cfrac{4x}{3-2x}}}$.

It is sometimes useful to decompose a given fraction into two parts, one integral and the other fractional, as illustrated below:

$\dfrac{x+3}{x-2}$ can be written as $1+\dfrac{5}{x-2}$ by actual division.

$\dfrac{2x-3}{x+5}$ can be written as $2-\dfrac{13}{x+5}$.

Example 12: Prove that $\dfrac{2x^2-5x-7}{x-3}\equiv 2x+1-\dfrac{4}{x-3}$.

Sol. $x-3\overline{)2x^2-5x-7}\,(2x+1$

$\underline{2x^2-6x}$

$x-7$

$\underline{x-3}$

-4

By actual division we obtain $2x+1$ as the quotient and -4 as the remainder.

If the numerator be of a degree lower than that of the denominator, even then the decomposition is possible, as illustrated below:

Example 13: Prove that $\frac{3x}{2x^2+1} \equiv 3x-6x^3+12x^5-\frac{24x^7}{2x^2+1}$.

Sol. $\frac{3x}{2x^2+1} \equiv \frac{3x}{1+2x^2}$

$\equiv 3x-6x^3+12x^5-\frac{24x^7}{1+2x^2}$ [*By actual division.*]

$\equiv 3x-6x^3+12x^5-\frac{24x^7}{2x^2+1}$.

In such cases, the division can be carried on to any number of terms and the process can be stopped at any stage by annexing to the quotient the fraction whose numerator is the last remainder and whose denominator is the divisor.

Thus if we had carried the quotient to four terms, in the above example, we would have got

$$\frac{3x}{2x^2+1} \equiv 3x-6x^3+12x^5-24x^7+\frac{48x^9}{2x^2+1}.$$

EXERCISE 86.

Express the following fractions in a form partly integral and partly fractional:

1. $\frac{x+7}{x+3}$. **2.** $\frac{x+7}{x-3}$. **3.** $\frac{2x+5}{x-1}$. **4.** $\frac{3x-4}{x+2}$.

5. $\frac{6x-7}{3x-2}$. **6.** $\frac{x^4-1}{x+a}$. **7.** $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3}$.

Prove that

8. $\dfrac{1}{1-x} \equiv 1+x+x^2+x^3+\dfrac{x^4}{1-x}$.

9. $\dfrac{1}{x-1} \equiv \dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^3(x-1)}$.

10. $\dfrac{1}{1+2x} \equiv 1-2x+4x^2-\dfrac{8x^3}{1+2x}$.

11. $\dfrac{1}{2x+1} \equiv \dfrac{1}{2x}-\dfrac{1}{4x^2}+\dfrac{1}{8x^3}-\dfrac{1}{8x^3(2x+1)}$.

12. $\dfrac{1+x^2}{(1-x)^2} \equiv 1+2x+4x^2+6x^3+\dfrac{8x^4-6x^5}{(1-x)^2}$.

13. $\dfrac{x+6}{x+4}+\dfrac{x+2}{x-4} \equiv 2+\dfrac{4x}{x^2-16}$.

14. $\dfrac{4x^2+7}{2x-1}+\dfrac{6x^2-8x+11}{3x-1} \equiv 4x-1+\dfrac{42x-17}{6x^2-5x+1}$.

15. $\dfrac{3x+6}{(x+1)(x+2)(x+3)} \equiv \dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+1)(x+3)}+\dfrac{1}{(x+2)(x+3)}$

16. $\dfrac{3x^2-14}{(x-1)(x-2)(x-3)} \equiv \dfrac{x+1}{(x-2)(x-3)}+\dfrac{x+2}{(x-1)(x-3)}+\dfrac{x+3}{(x-1)(x-2)}$.

17. Apply the identity

$\dfrac{1}{1-x} \equiv 1+x+x^2+\dfrac{x^3}{1-x}$ in finding the approximate value of $\dfrac{1}{0{\cdot}9}$ to 3 decimal places.

SELECTED QUESTIONS—D.

Simplify :

1. $\dfrac{x+3}{x^2-3x+2}+\dfrac{x+2}{x^2-4x+3}+\dfrac{x+1}{x^2-5x+6}$.

2. $\left(x+\frac{1}{x}\right)^2-\left(y+\frac{1}{y}\right)^2-\left(xy-\frac{1}{xy}\right)\left(\frac{x}{y}-\frac{y}{x}\right)$.

3. $\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}+\frac{2x^3}{a^4+a^2x^2+x^4}$.

4. $\frac{1}{1-x}-\frac{1}{1+x}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4}$.

5. $\frac{1}{1-x}-\frac{1}{1+x}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4}-\frac{8x^7}{1+x^8}$.

6. $\frac{1}{x+a}+\frac{2x}{x^2+a^2}+\frac{4x^3}{x^4+a^4}-\frac{8x^7}{x^8-a^8}$.

7. $\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}\times\left\{1+\frac{b^2+c^2-a^2}{2bc}\right\}$.

8. $\frac{\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}$.

CHAPTER IX

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE AND PROBLEMS

1. Consider the equation $x+y=7$.

This contains two unknowns, x and y. Any values of x and y chosen at random, say $x=8$, $y=5$, will not satisfy it. But if we assign any value to x, a corresponding value can be found for y which will satisfy the equation. A few such pairs of values are given in the table below:

$x=$	0	1	2	3	4	**5**	6	7	8	9	10
$y=$	7	6	5	4	3	**2**	1	0	−1	−2	−3

Clearly there is no limit to the number of pairs of values which will satisfy this equation. This is an *indeterminate* equation.

Suppose there is another equation, $x-y=3$.

Here again, the number of pairs of values satisfying the equation is unlimited. However, a few pairs of such values are given below:

$x=$	0	1	2	3	4	**5**	6	7	8	9	10
$y=$	−3	−2	−1	0	1	**2**	3	4	5	6	7

It will be noticed that there is one pair of values, namely $x=5$ and $y=2$ for which both equations are true *at the same time*.

The two equations are then called **simultaneous equations.**

Equations which are true at the same time for particular values of the variables involved in them are said to be simultaneous equations.

2. **Solution of Simultaneous Equations.**

The first step in the solution of these equations is to obtain a *simple equation* from them by eliminating one of the two variables.

This can be done in three ways as illustrated below:

Example 1: Solve the equations $3x+y = 17$ ... (i)
$8x+11y = 37$ ... (ii)

First Method:

Express one variable in terms of the other in both equations, equate the results and solve.

From (i) we have $y = 17-3x$

,, (ii) ,, $y = \frac{37-8x}{11}$

Equating these values of y, we have

$$17-3x = \frac{37-8x}{11}$$

or $187-33x = 37-8x$

or $-25x = -150$

or $x = 6.$

Substituting the value of x in (i), we have

$3\times6+y = 17$

or $y = -1.$

Thus $x = 6$ and $y = -1.$

[*Verify by substituting these values in equation (ii)*].

Second Method:

Express one variable in terms of the other in one equation only, substitute this value in the other equation and solve.

From (i), we have $y = 17-3x.$

Substituting this in (ii), we have

$8x+11(17-3x) = 37$

or $8x+187-33x = 37$

or $8x-33x = 37-187$

or $-25x = -150$ $\therefore$ $x = 6$

Then $y = 17-3\times6$ *i.e.* $y = -1.$

Thus $x = 6$, $y = -1.$

This is not much different from the first method.

EXERCISE 87.

Solve the following equations by the first method:

1. $\left.\begin{aligned} x-2y &= 2 \\ 5x+5y &= 4. \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x-y &= 6 \\ 3x-2y &= 7. \end{aligned}\right\}$

3. $\left.\begin{aligned} x+5y &= 13 \\ 2x-3y &= 0. \end{aligned}\right\}$

4. $\left.\begin{aligned} 3x-4y &= 0 \\ 2x+3y &= 17. \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x+5y &= 7 \\ 8y+11x &= 19. \end{aligned}\right\}$

6. $\left.\begin{aligned} 3x+4y &= 19 \\ 6x-5y &= 25. \end{aligned}\right\}$

7. $\left.\begin{aligned} 5x+3y &= 47 \\ 3x-5y &= 1. \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x-2y &= 10 \\ 2x+4y &= 12. \end{aligned}\right\}$

9. $\left.\begin{aligned} x+ay &= b \\ ax-by &= c. \end{aligned}\right\}$

10. $\left.\begin{aligned} \tfrac{1}{3}(x-y) &= \tfrac{1}{4}(y-1) \\ \tfrac{1}{7}(4x-5y) &= x-7. \end{aligned}\right\}$

Solve the following equations by the second method:

11. $\left.\begin{aligned} 3x-2y &= 9 \\ x+3y &= 10. \end{aligned}\right\}$

12. $\left.\begin{aligned} 7x-y &= 5 \\ 2x+3y &= 8. \end{aligned}\right\}$

13. $\left.\begin{aligned} 4x+7y &= 62 \\ 3y-2x &= 8. \end{aligned}\right\}$

14. $\left.\begin{aligned} 2x+5y+7 &= 0 \\ 3x+7y+10 &= 0. \end{aligned}\right\}$

15. $\left.\begin{aligned} 7x+4y &= 5 \\ 5x+6y &= 2. \end{aligned}\right\}$

16. $\left.\begin{aligned} 3y &= 7(x-10) \\ 2(x-5) &= 3(y+10). \end{aligned}\right\}$

17. $\left.\begin{aligned} 3x+9y-42 &= 0 \\ 2x-4y+2 &= 0. \end{aligned}\right\}$

18. $\left.\begin{aligned} \frac{7+x}{5}-\frac{2x-y}{4} &= 3y-5 \\ \frac{5y-7}{2}+\frac{4x-3}{6} &= 18-5x. \end{aligned}\right\}$

19. $\left.\begin{aligned} \tfrac{1}{2}(3x-2y) &= \tfrac{1}{4}(2x-y)+3 \\ \tfrac{1}{2}(5x-4y) &= \tfrac{1}{3}(4x-3y)+3. \end{aligned}\right\}$

20. $\left.\begin{aligned} ax+by &= c \\ a^2x+b^2y &= c^2. \end{aligned}\right\}$

Third Method:

Suppose A buys 1 table and 5 Chairs for Rs. 100 and B buys 1 table and 3 chairs for Rs. 70.

We can at once say that A paid more money because he bought more chairs. In other words, the cost of 2 extra chairs = Rs. 30. Then each chair costs Rs. 15 and so a table costs Rs. 25.

Suppose A buys 3 tables and 5 chairs for Rs. 150 and B buys 2 tables and 3 chairs for Rs. 95.

Here the numbers of tables are different and the numbers of chairs are also different. We cannot at once find out the price of each article.

If A buys two sets of (3 tables and 5 chairs), he will have to to pay Rs. 300.

If B buys three sets of (2 tables and 3 chairs), he will have to pay Rs. 3×95 = Rs. 285.

Now A has 6 tables and 10 chairs for Rs. 300.

B has 6 tables and 9 chairs for Rs. 285.

It is at once clear that one extra chair means Rs. 15 extra. Hence, the cost of a chair and of a table are found.

In symbols, this problem reduces to the solution of the equations :

$$3t+5c = 150 \text{ and } 2t+3c = 95.$$

Here *we equalise the co-efficients of one and the same variable in both the equations ; then subtract or add and derive a simple equation containing the other variable only.*

This is the **third method.**

Example 2: Solve the equations

$$3x+4y = 20 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$5x-3y = 14 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. Multiplying (i) by 3 and (ii) by 4, we have

$$9x+12y = 60 \quad \ldots \quad \ldots \quad \text{(iii)}$$
$$20x-12y = 56 \quad \ldots \quad \ldots \quad \text{(iv)}$$

Adding (iii) and (iv), we have

$$29x = 116$$
$$\therefore \quad x = 4.$$

Substituting this value of x in (i), we have

$$3 \times 4+4y = 20$$
or $$4y = 8$$
or $$y = 2.$$
$$\therefore \quad \left.\begin{matrix} x = 4 \\ y = 2 \end{matrix}\right\} \text{ is the required solution.}$$

EXERCISE 88.

Solve the following equations by the third method :

1. $\left.\begin{matrix} x + y = 14 \\ x - y = 6. \end{matrix}\right\}$

2. $\left.\begin{matrix} x - y = 12 \\ x + y = 26. \end{matrix}\right\}$

3. $\left.\begin{matrix} 2x + 3y = 28 \\ 3x - 4y = -9. \end{matrix}\right\}$

4. $\left.\begin{matrix} 4x - 2y = 12 \\ 5x + 3y = 26. \end{matrix}\right\}$

5. $\left.\begin{matrix} 6x + 7y = 32 \\ 9x + y = 29. \end{matrix}\right\}$

6. $\left.\begin{matrix} 7x - 8y = -22 \\ 11x-10y = 4. \end{matrix}\right\}$

7. $\left.\begin{aligned} 32x+15y &= 62 \\ 18x-13y &= -8. \end{aligned}\right\}$

8. $\left.\begin{aligned} 13x-11y &= 32 \\ 15x+7y &= 96. \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x+\cdot 4y &= 1\cdot 2 \\ 3\cdot 4x-\cdot 02y &= \cdot 01 \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{1}{6}x+\frac{1}{16}y &= 6 \\ \frac{1}{12}y-\frac{1}{9}x &= 2 \end{aligned}\right\}$

11. $7x-3y = 5x+y = 22.$

12. $14x+3y = 6x+15y = 48.$

13. $6x+5y = 7x+3y+1 = 2(x+6y-1).$

14. $\left.\begin{aligned} \frac{x-y}{3} &= \frac{y-1}{4} \\ \frac{4x-5y}{7} &= x-7. \end{aligned}\right\}$

15. $\left.\begin{aligned} \frac{32+x}{5}-\frac{2x-y}{4} &= 3y \\ \frac{5y-43}{2}+\frac{4x-3}{6} &= -5x. \end{aligned}\right\}$

16. $\frac{x}{4}+\frac{y}{5} = \frac{x}{5}+\frac{y}{4}-1 = 22.$

17. $3x-2y+2 = 5x-3y+1\frac{5}{6} = 6x-y-4\frac{1}{2}.$

18. $7(x-5) = y-2, \ \frac{4y-3}{2} = \frac{x+10}{6}.$

19. $7(2y-3x)-2 = 2(9x-y), \ 4x-2y = 2(x+3)-12.$

20. $ax+by = c, \ (a-b)x+(a-c)y = d.$

Simultaneous equations involving the reciprocals of the variables are solved as illustrated below :

Example 3 : Solve the equations

$$\frac{2}{x}+\frac{3}{y} = 2 \ \ldots \qquad \ldots \quad \text{(i)}$$

$$\frac{5}{x}+\frac{2}{y} = 3\frac{1}{6} \qquad \ldots \quad \text{(ii)}$$

Sol. Multiplying (i) by 2 and (ii) by 3, we have

$$\frac{4}{x}+\frac{6}{y} = 4 \ \ldots \qquad \ldots \quad \text{(iii)}$$

$$\frac{15}{x}+\frac{6}{y} = \frac{19}{2} \qquad \ldots \quad \text{(iv)}$$

Subtracting (iv) from (iii), we have

$$-\frac{11}{x} = -\frac{11}{2} \quad \therefore \quad x = 2.$$

Substituting this value of x in (i), we have

$$\frac{2}{2}+\frac{3}{y}=2$$

or $$\frac{3}{y}=1 \quad \therefore \quad y=3.$$

$\therefore$ $\left.\begin{matrix} x=2 \\ y=3 \end{matrix}\right\}$ is the required solution.

Aliter: If we put p for $\frac{1}{x}$ and q for $\frac{1}{y}$, $\frac{2}{x}=2\times\frac{1}{x}=2p$ and $\frac{3}{y}=3\times\frac{1}{y}=3q$.

$\therefore$ The equations become $2p+3q=2$ ... ... (iii)

$$5p+2q=\frac{19}{6} \quad \ldots \quad \ldots \quad \text{(iv)}$$

Multiplying (iii) *by* 5 and (iv) *by* 2,

$$10p+15q=10 \quad \ldots \quad \ldots \quad \text{(v)}$$

$$10p+4q=\frac{19}{3} \quad \ldots \quad \ldots \quad \text{(vi)}$$

Subtracting (vi) from (v), we get $11q=10-\frac{19}{3}$

or $11q=\frac{11}{3}$ $\quad \therefore \quad q=\frac{1}{3}$

By substituting in (iii), $2p+3\times\frac{1}{3}=2$

or $2p=2-1$ $\quad \therefore \quad p=\frac{1}{2}$

If $p=\frac{1}{2}$, $\frac{1}{x}=\frac{1}{2}$ or $x=2$

If $q=\frac{1}{3}$, $\frac{1}{y}=\frac{1}{3}$ or $y=3$

$\therefore \quad x=2, \quad y=3.$

EXERCISE 89.

Solve the equations:

1. $\left.\begin{matrix} 2x+\frac{3}{y}=5 \\ 5x-\frac{2}{y}=3. \end{matrix}\right\}$

2. $\left.\begin{matrix} \frac{2}{x}+\frac{1}{y}=\frac{11}{12} \\ \frac{1}{x}+\frac{3}{y}=1\frac{1}{12}. \end{matrix}\right\}$

3. $\left.\begin{aligned}\frac{2}{x}+\frac{5}{y}&=1\\ \frac{3}{x}+\frac{2}{y}&=\frac{19}{20}.\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{6}{x}+\frac{4}{y}&=3\\ \frac{9}{x}-\frac{1}{y}&=2\tfrac{3}{4}.\end{aligned}\right\}$

5. $\left.\begin{aligned}\frac{1}{3x}-\frac{1}{7y}&=\frac{2}{3}\\ \frac{1}{2x}-\frac{1}{3y}&=\frac{1}{6}.\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{2}{3x}-\frac{5}{4y}&=\frac{1}{12}\\ \frac{3}{4x}-\frac{7}{6y}&=\frac{1}{3}.\end{aligned}\right\}$

7. $\frac{m}{x}-\frac{n}{y}=a,\ px=qy.$

When the co-efficients of x and y are interchanged in two simultaneous equations, it is useful to employ the method of **addition** and **subtraction,** as illustrated below:

Example 4: Solve the equations

$4x+7y=49$ ... ... (i)

$7x+4y=61$ ... ... (ii)

Sol. Adding (i) and (ii), we have

$11x+11y=110$

or $x+y=10$ ... ... (iii)

Subtracting (i) from (ii), we have

$3x-3y=12$

or $x-y=4$ ... ... (iv)

Adding (iii) and (iv), $2x=14$ $\therefore\ x=7$

Subtracting (iv) from (iii), $2y=6$ $\therefore\ y=3$

Thus $x=7,\ y=3$ is the solution.

EXERCISE 90.

Solve the equations:

1. $\left.\begin{aligned}10x+11y&=20\\ 11x+10y&=22.\end{aligned}\right\}$

2. $\left.\begin{aligned}9x+5y&=60\\ 5x+9y&=52.\end{aligned}\right\}$

3. $\left.\begin{aligned}11x+3y&=-21\\ 3x+11y&=35.\end{aligned}\right\}$

4. $\left.\begin{aligned}17y+6x&=28\\ 17x+6y+5&=0.\end{aligned}\right\}$

5. $\left.\begin{aligned}3x+20&=4y-10\\ 4(x-1)&=3(y-3).\end{aligned}\right\}$

6. $\left.\begin{aligned}\tfrac{1}{6}x+\tfrac{1}{4}y&=\tfrac{13}{6}\\ \tfrac{1}{4}x+\tfrac{1}{6}y&=2.\end{aligned}\right\}$

7. $\left.\begin{aligned}ax+by&=m\\ bx+ay&=n.\end{aligned}\right\}$

8. $\left.\begin{aligned}49x-57y&=172\\ 57x-49y&=252.\end{aligned}\right\}$

3. Problems involving simultaneous equations.

(Problems on Numbers)

Example 5: Find two numbers such that four times the first added to three times the second is 93 and the excess of three times the first over twice the second is 6.

Sol. Let x be the first number, and y the second.

Four times the first + three times the second $= 4x+3y$,

$\therefore$ the first equation is $4x+3y=93$ ... (i)

The excess of three times the first over twice the second $= 3x-2y$,

$\therefore$ the second equation is $3x-2y=6$ ... ... (ii)

Multiplying (i) by 2 and (ii) by 3, we have

$8x+6y = 186$ ... ... (iii)

$9x-6y = 18$ ... ... (iv)

Adding (iii) and (iv), we have

$17x = 204$, $\therefore$ $x = 12$.

Substituting this value of x in (i), we have

$48+3y = 93$, $\therefore$ $y = 15$.

Hence the two numbers are 12 and 15.

EXERCISE 91.

1. Find two numbers whose sum is 39 and difference is 7.

2. Find two numbers such that three times their sum is 84 and twice their difference is 4.

3. Find two numbers such that four times the first added to the second is equal to 96 and the excess of three times the first over the second is equal to 51.

4. Find two numbers such that three times the first added to four times the second is equal to 114 and four times the first added to three times the second is equal to 117.

5. Two decimal fractions have their sum equal to ·56 and three times their difference equal to ·24. Find them.

6. A straight line 27cm in length is divided into two parts such that six times the first part exceeds four times the other by 12cm. Find each part.

(Problems on Fractions)

Example 6: A fraction becomes equal to $\frac{1}{3}$ when 1 is subtracted from its numerator and it becomes equal to $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction.

Sol. Let x be the numerator and y the denominator of the fraction.

Then $$\frac{x-1}{y}=\frac{1}{3} \quad \text{or} \quad 3x-y=3 \quad \ldots \quad \text{(i)}$$

and $$\frac{x}{y+8}=\frac{1}{4} \quad \text{or} \quad 4x-y=8 \quad \ldots \quad \text{(ii)}$$

Subtracting (i) from (ii), we have

$$x = 5.$$

Substituting this value of x in (i), we have

$$15-y = 3$$

or $$y = 12.$$

Hence the required fraction is $\frac{5}{12}$.

EXERCISE 92.

1. A certain fraction becomes $\frac{2}{3}$ when its numerator is diminished by 1, and becomes $\frac{1}{2}$ when its denominator is increased by 5. Find the fraction.

2. What fraction is that which becomes 1 if 1 be added to its numerator and becomes $\frac{1}{2}$ if 1 be added to its denominator ?

3. What fraction is that which becomes $\frac{2}{3}$ if its numerator be doubled and denominator increased by 9 and becomes $\frac{1}{2}$ if its numerator be increased by 7 and denominator doubled ?

4. If the denominator of a fraction be added to the numerator and the numerator be subtracted from the denominator, it becomes $2\frac{2}{3}$. The denominator of the fraction exceeds twice its numerator by 1. Find the fraction.

5. The denominator of a fraction exceeds the numerator by 4, and if 5 be taken away from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5. Find the original fraction.

(Problems on Trade, Money)

Example 7: A person sold 9 chairs and 6 tables for Rs. 90. Again he sold 8 chairs and 5 tables at the same rate, for Rs. 77. Find the price of one chair and one table.

Sol. Let Rs. x be the price of one chair and Rs. y the price of one table. Since the price of 9 chairs and 6 tables $=$ Rs. $(9x+6y)$,

the first equation is $9x+6y = 90$ ... ... (i)

Since the price of 8 chairs and 5 tables $=$ Rs. $(8x+5y)$,

the second equation is $8x+5y = 77$ ... (ii)

Multiplying (i) by 5 and (ii) by 6, we have

$45x+30y = 450$ ... ... (iii)

$48x+30y = 462$ ... ... (iv)

Subtracting (iii) from (iv), we have

$$3x = 12, \quad \therefore \quad x = 4.$$

Substituting this value of x in (i), we have

$$36+6y = 90, \quad \therefore \quad y = 9.$$

Hence the price of one chair is Rs. 4 and the price of one table is Rs. 9.

EXERCISE 93.

1. 25 horses and 12 cows together cost Rs. 11,750 ; also 18 horses and 15 cows cost Rs. 10,050. Find the cost price of a horse and a cow.

2. 16kg of wheat and 6kg of rice together cost Rs. 29, and 24kg of wheat and 15kg of rice cost Rs. 52·50. Find the rates of wheat and rice per kg.

3. The price of 12 tables and 20 chairs is Rs. 164 and the price of 15 tables and 24 chairs is Rs. 201. Find the price of a table and a chair.

4. A person bought a horse and a carriage for Rs. 810 ; if the price of the horse were 10% more and that of the carriage 10% less, he would have to pay Rs. 799. Find the cost price of the horse and also that of the carriage.

5. A person sells two articles together for Rs. 46, making 10% profit on one and 20% profit on the other. If he had sold each article at 15% profit, the result would have been the same. At what price does he sell each article ?

6. A person has two horses and a saddle worth Rs. 95 ; if the saddle be put on the first horse, his value becomes double that of the second ; but if the saddle be put on the second horse, his value would be less than that of the first by Rs. 125. What is the value of each ?

7. *A* and *B* have Rs. 320 between them. *A* gains from *B* $\frac{1}{3}$ of *B*'s money. Then *B* gains from *A* Rs. 16. Each has now the same amount. Find the original sum of each.

8. A man had a sum of Rs. 20·30 in rupees and paise. If the number of rupees and paise were interchanged, he would gain Rs. 9·90. Find how many rupees and paise he had.

(Problems on Distance, Speed and Time)

Example 8 : Two persons *A* and *B*, 30 kilometres apart, start at the same time and walking in opposite directions meet in $2\frac{1}{2}$ hours. But if they walk in the same direction, starting at the same time, *A* overtakes *B* in 10 hours. Find their speeds per hour.

Sol. Let *A*'s speed be x km per hour and *B*'s speed be y km per hour.

When they walk in *opposite* directions, the distance between them reduces at the rate of $(x+y)$ km per hour.

Since they together cover 30km in $2\frac{1}{2}$ hours,

$$2\tfrac{1}{2}(x+y) = 30$$

or $$x+y = 12 \quad \ldots \quad \ldots \quad \text{(i)}$$

When they walk in the *same* direction, the gap between them is reduced at the rate of $(x-y)$ km per hour.

Since *A* overtakes *B* after making up a gap of 30km in 10 hours, $$10(x-y) = 30$$

or $$x-y = 3 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Solving (i) and (ii), we have

$$x = 7\tfrac{1}{2} \text{ and } y = 4\tfrac{1}{2}$$

$\therefore$ *A*'s speed is 7·5 km per hour and
B's speed is 4·5 km per hour.

NOTE : In problems on streams, if a person rows along a stream, it may be assumed that
his speed down-stream = speed in still water + speed of the current
and speed up-stream = speed in still water − speed of the current.

EXERCISE 94.

1. Two persons, 15km apart, start at the same time. If they walk in opposite directions, they meet in 2 hours; but if they walk in the same direction, they meet in 30 hours. Find their rates of walking.

2. A person rows down-stream 35km in 7 hours and up-stream the same distance in $17\frac{1}{2}$ hours. Find the rate at which he rows and the speed of the stream.

3. A person rows down-stream 26km and up-stream the same distance in $19\frac{1}{2}$ hours. Also the difference of time taken by him to row 39km up and down is 9 hours and 45 minutes. Find the rate at which he rows and the speed of the stream.

4. A boat goes up-stream 30km and down-stream 44km in 10 hours. It also goes up-stream 40km. and down-stream 55km in 13 hours. Find the speeds of the boat and the stream.

5. A person goes a certain distance on bicycle. If he had gone 3km an hour faster, he would have taken $\frac{2}{3}$ hour less ; but if he had gone 2km an hour slower, he would have taken $\frac{2}{3}$ hour longer. Find the distance.

6. A person walks on the first day a certain distance at the rate of 4km an hour and on the second day a certain distance at the rate of 3km an hour, and thus completes a journey of 30km. If he had walked the first distance at 3km an hour and the second distance at 4km an hour, he would have taken $\frac{1}{3}$ hour more. Find the time he took to walk the whole distance.

7. A messenger is allowed 16 hours for going from P to Q and back again, including 4 hours' rest at Q. But if he were to go 2km an hour faster each way, he would be able to take 5 hours' rest at Q. Find the speed and the distance from P to Q.

(*Miscellaneous Problems*)

Example 9: Two persons have together 200kg of luggage and are charged for the excess above the ordinary concession Rs. 10 and Rs. 6 respectively. If all the luggage had belonged

to one man, he would have been charged Rs. 18. How much luggage had each passenger and what was the limit of concession per passenger ?

Sol. Let the first passenger's luggage weigh x kg and let the weight each is allowed to carry free be y kg and let the charge for the excess luggage be Rs. z per kg. The first passenger will be charged for $(x-y)$kg.

$$\therefore \quad (x-y)z = 10 \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

The second passenger will be charged for $(200-x-y)$ kg.

$$\therefore \quad (200-x-y)z = 6 \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

If the entire luggage is carried by one person, he would be charged for $(200-y)$ kg.

$$\therefore \quad (200-y)z = 18 \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Dividing (ii) by (i), $\dfrac{200-x-y}{x-y} = \dfrac{6}{10}$

or $\quad 2000-10x-10y = 6x-6y$

or $\quad -16x-4y = -2000$

or $\quad 4x+y = 500 \qquad \ldots \qquad \ldots \qquad \text{(iv)}$

Dividing (iii) by (i) $\quad \dfrac{200-y}{x-y} = \dfrac{18}{10}$

or $\quad 2000-10y = 18x-18y$

or $\quad 18x-8y = 2000$

or $\quad 9x-4y = 1000 \qquad \ldots \qquad \ldots \qquad \text{(v)}$

Multiplying (iv) by 4 and adding to (v), we have

$$25x = 3000 \qquad \therefore \quad x = 120$$

Then $\quad y = 500-4\times120 \quad$ or $\quad y = 20$

Hence the luggage of one man = 120kg, of the other = 80kg, and the concession allowed per passenger = 20kg.

EXERCISE 95.

1. Two persons have together 270kg of luggage and are charged for the excess above the ordinary concession Rs. 3·40 and Rs. 3 respectively. If all the luggage had belonged to one man, he would have been charged Rs. 6·80. How much luggage did each passenger have and what is the limit of concession per passenger ?

2. The monthly expenses of a family, when wheat is selling at 1·6kg. for a rupee, are Rs. 175 ; when it is selling @ 1·5kg for a rupee, expenses are Rs. 180, other expenses remaining unchanged. What will be the expenses when wheat is selling @ 1·2kg for a rupee ?

3. The perimeter of a rectangle is 72 metres. If its length be decreased by 6 metres and breadth increased by 4 metres, the area is unaltered. Find the area.

4. A merchant has two kinds of tea, one worth Rs. 8 per kg and the other worth Rs. 14 per kg ; from these he prepares a mixture of 120kg worth Rs. 9·50 per kg. How much of each kind must he mix ?

5. Three ladies buy 3 kinds of cloth, A, B and C. The first buys 5 metres of A and 3 metres of B, spending the same amount on each ; the second buys 10 metres of A and 5·5 metres of C, spending Rs. 100 ; while the third buys 3 metres of each kind of cloth for Rs. 66. Find the price per metre of each kind of cloth.

6. A sum of money is divided equally among a certain number of men ; had there been eight more, each would have received Re. 1 less, and had there been four fewer, each would have received Re. 1 more than he did. Find the number of men and the sum of money.

4. Simultaneous Equations involving product and sum or difference of two variables.

Example 10 : Solve $x+y=10$, $xy=24$.

Sol. We know the identity

$$(x-y)^2 \equiv (x+y)^2-4xy$$

Substituting, $(x-y)^2 = 10^2-4\times 24=4$

$$(x-y) = \sqrt{4}=2$$

and $x+y=10$ (given)

Adding, $2x=12$ or $x=6$ and

Subtracting, $-2y=-8$ or $y=4$.

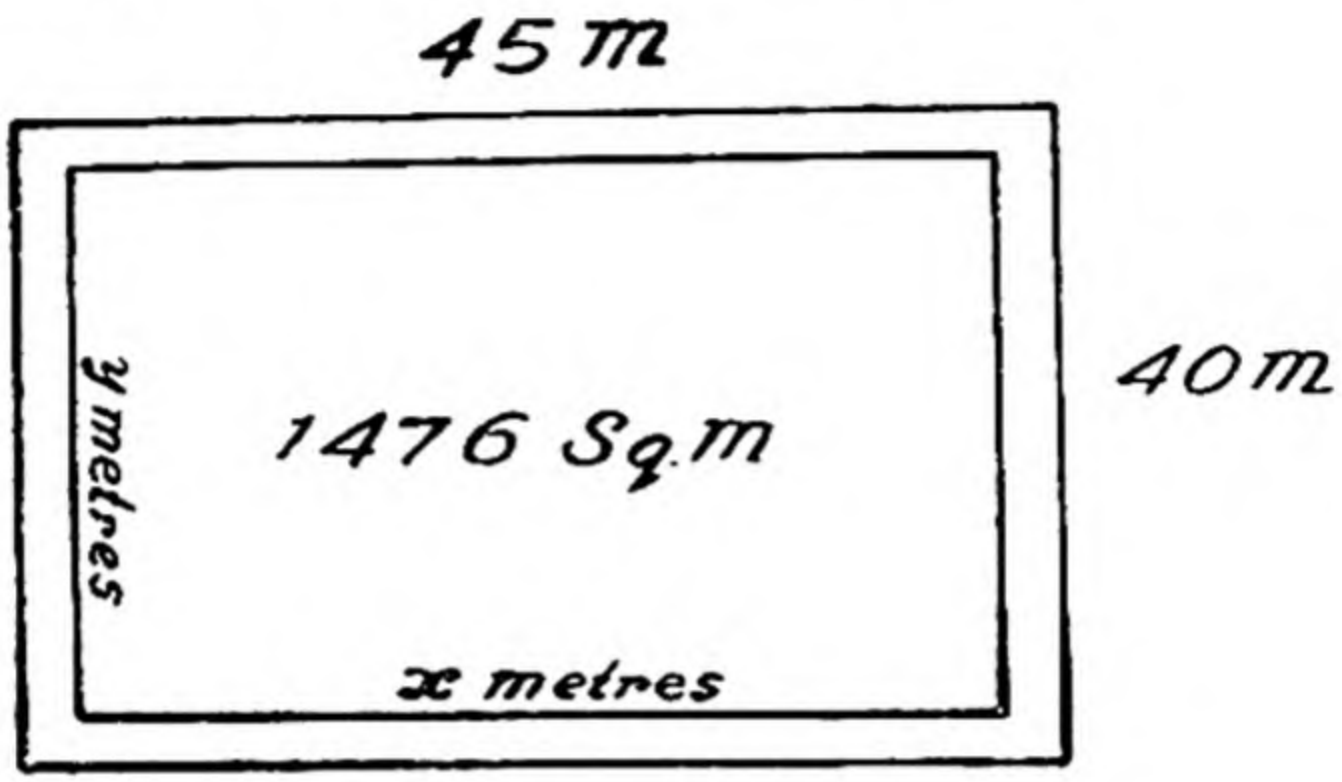
NOTE : It may be recalled that in the factorisation of trinomials, we had to find out two numbers whose sum and product were known. They were found by trial. The above method is the regular method of solving the same.

Example 11: A rectangular garden is 45 metres long and 40 metres broad. A road of uniform width is constructed inside all round it. The area of the remaining garden is 1476 sq. metres. Find the width of the road.

Sol. Let the length of the inner rectangle be x metres and its breadth y metres.

Obviously, $x-y = 45-40 = 5$ ... (i)

and $xy = 1476$ ... (ii)



$$(x+y)^2 \equiv (x-y)^2 + 4xy$$
$$(x+y)^2 = 5^2 + 4 \times 1476$$
$$= 5929 = 77^2$$
$$\therefore \quad x+y = 77$$

Adding (i) & (ii), $2x = 82$ or $x = 41$

Subtracting, $2y = 72$ or $y = 36$

The width of the road $= \frac{1}{2}(45-41) = 2$ metres.

EXERCISE 96.

Solve the equations:

1. $\left.\begin{aligned} x+y &= 11 \\ xy &= 28. \end{aligned}\right\}$

2. $\left.\begin{aligned} x+y &= 17 \\ xy &= 60. \end{aligned}\right\}$

3. $\left.\begin{aligned} x-y &= 7 \\ xy &= 120. \end{aligned}\right\}$

4. $\left.\begin{aligned} x-y &= 5 \\ xy &= 84. \end{aligned}\right\}$

5. $\left.\begin{aligned} x+y &= 15 \\ xy &= 56. \end{aligned}\right\}$

6. $\left.\begin{aligned} x-y &= 4 \\ xy &= 77. \end{aligned}\right\}$

7. $\left.\begin{aligned} x+y &= a \\ xy &= b^2. \end{aligned}\right\}$

8. $\left.\begin{aligned} x-y &= m \\ xy &= n^2. \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{7}{12} \\ \frac{1}{xy} &= \frac{1}{12}. \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{1}{x}-\frac{1}{y} &= \frac{3}{10} \\ \frac{1}{xy} &= \frac{1}{10}. \end{aligned}\right\}$

11. The area of a rectangle is 1728 sq. metres and its semi-perimeter is 84 metres. Find its length and breadth.

12. The area of a triangle is 80 sq. metres, the sum of its base and altitude is 28 metres. Find its base and altitude.

13. The area of a rectangle is 3840 sq. cm. and its length is greater than its breadth by 32 cm. Find its length and breadth.

14. The top of a rectangular table is 40 cm long and 28 cm wide. Leaving out a uniform margin all round, it is covered with oil cloth whose area is 640 sq. cm. Find the width of the margin.

15. The difference between the length and breadth of a rectangle is 14 metres and its area is 275 sq. metres.

(i) Find its semi-perimeter ; (ii) find its length and breadth.

SELECTED QUESTIONS—E.

1. Two positive numbers differ by three and their reciprocals differ by $\frac{3}{40}$. Find them.

2. A number consists of three digits whose sum is 17 ; the middle digit exceeds the sum of the other two by '1' ; if the digits be reversed, the number is diminished by 396. Find the number.

3. Find the fraction which is such that when 15 is added to the numerator, it becomes 3 and when 11 is added to the denominator, it becomes $\frac{1}{3}$.

4. Sohan says to Mohan, "Give me Rs. 900 and I shall be twice as rich as you". Mohan replies, "If you give me Rs. 100 I shall be thrice as rich as you". How much money has each got ?

5. Five years hence a father will be 3 times as old as his son ; 5 years ago he was seven times as old as the son. How old are they now ?

6. A student cycles from Delhi to Okhla and back at the same rate, but the clock at Okhla is 15 minutes slower than that at Delhi ; so he appears to go out at 15 km per hour and to return at 10 km per hour. Find his rate and the distance.

7. Two cyclists ride from A to B, a distance of 55 km, and the first arrives 30 minutes before the second. They then ride from B to A, the first giving the second a start of 4 km and yet arriving 6 minutes before him. Find the rate of each cyclist in km per hour.

8. A man walks a certain distance at a certain rate. Had he walked 0·75 km an hour faster, he would have taken 1 hour less ; but if he had walked 1·5 km an hour slower, he would have taken 3 hours longer. Find the distance and his rate.

9. A and B run a race of 1 kilometre. First A gives B a start of 100 metres and beats him by 30 seconds. Next A gives B a start of 1 minute 30 seconds and is beaten by 125 metres. Find the time in which A and B can run the race separately.

10. A boat goes up-stream 25 km and down-stream 33 km, altogether in 8 hours ; it also goes up-stream 40 km and down-stream 77 km in 15 hours. Find the rate of the stream and of the boat.

11. There are two examination rooms A and B. If 10 candidates are sent from A to B, the number in each is the same, while if 20 are sent from B to A, the number in A is double the number in B. Final the number of candidates in each room.

SECTIONAL REVISION III
TEST PAPERS

PAPER 1

1. Show that the H.C.F. of two expressions A and B divides exactly their sum as well as difference.

2. Simplify (i) $\frac{a}{a^2-b^2}-\frac{b}{b^2-a^2}$.

(ii) $\frac{a^2-3ab+2b^2}{a-2b}+\frac{6a^2-5ab-6b^2}{2a-3b}-\frac{6a^2+ab-2b^2}{3a+2b}$.

3. Solve the equations $5x+\frac{3}{y}=20\frac{1}{2}$

$3x-\frac{4}{y}=11\frac{1}{3}$.

4. Find two numbers such that three times the first added to five times the second is equal to 53, and also such that three times the second subtracted from four times the first is equal to 3.

5. Find the value of $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=4$.

PAPER 2

1. Find the H.C.F. of
$6x^3-4x^2+3x-2$, $6x^3-31x^2+30x-8$ and $12x^3-2x^2-7x+2$.

2. (i) Express $\frac{x^3-1}{x+a}$ in a form partly integral and partly fractional.

(ii) Simplify $\frac{x+1}{2x^3-4x^2}+\frac{x-1}{2x^3+4x^2}-\frac{1}{x^2-4}$.

3. Solve the equations $\frac{12}{x}-\frac{5}{y}=3$ $\frac{9}{x}+\frac{2}{y}=3\frac{2}{5}$.

4. Eight years back A was twice as old as B. After 15 years their united ages will be 82. Find their present ages.

5. A man's monthly income is progressive and varies as below: Rs. 50, Rs. 56, Rs. 62, Rs. 68, Rs. 74, Rs. 80 &c. Find a formula which determines his monthly income *in general*.

PAPER 3

1. Find the H.C.F. of $2x^5-5x^2+3$ and $3x^5-5x^3+2$ by the method of alternate destruction of the highest and the lowest terms.

2. Simplify $\frac{5}{x^2-3x-28}+\frac{3}{x^2+x-12}+\frac{9}{x^2-10x+21}$.

3. Solve the equations $\left.\begin{aligned}13x+9y&=57\\9x+13y&=53.\end{aligned}\right\}$

4. A number of two digits is reversed if it is diminished by 9 ; the sum of its digits is 15. Find the number.

5. Show that $(4x^2-2x-15)^2-(2x^2-5x-9)^2$ is divisible by $2x^2+3x-6$ and express the quotient in the form of factors.

PAPER 4

1. Reduce to the lowest terms $\frac{6x^4+25x^3+25x^2+13x+3}{3x^4+11x^3+13x^2+7x+2}$.

2. Simplify $\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}$.

3. Solve for x and y: $\left.\begin{aligned}x-y&=b\\ax-by&=ab\end{aligned}\right\}$

4. A person does a journey in a motor car at a uniform speed in 6 hours. On his return he stops at the half-way for half an hour, but quickening his speed by 3 km an hour completes it in the same time. Find his original speed and the length of the journey.

5. Find the value of $a^3+b^3+c^3-3abc$ when $a+b+c=9$ and $a^2+b^2+c^2=29$.

PAPER 5

1. If A and B are two algebraic expressions and $A=a.H$ $B=b.H$ prove that the L.C.M. $=\frac{A}{H}.B$.

2. Simplify $\frac{6x^2-5xy-6y^2}{14x^2-23xy+3y^2}-\frac{15x^2+8xy-12y^2}{35x^2+47xy+6y^2}$.

3. Solve for x, y: $\left.\begin{aligned}ax-y&=a^2\\bx-y&=b^2\end{aligned}\right\}$

4. A person goes on a bicycle from A to B and back in a certain time at 12 km an hour. If he had gone 10 km an hour from A to B and returned at 16 km an hour, he would have taken $6\frac{1}{4}$ minutes less for the double journey. Find the distance from A to B.

5. Factorise (i) $(x^2-3x)(x^2-3x-2)-24$.
(ii) $81x^4+64$.

PAPER 6

1. If A and B are two expressions, H and L their H.C.F. and L.C.M. respectively, prove that $A \times B = H \times L$.

2. Simplify $\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\frac{8}{x^8-1}$.

3. Solve the equations $\frac{1}{2x}-\frac{1}{3y}=\frac{1}{12}$
$\frac{1}{6xy}=\frac{1}{72}$.

4. Divide Rs. 1,440 into two parts such that if they are invested at $3\frac{1}{2}\%$ and 5% respectively, they may together yield the same income as if the whole were invested at $4\frac{1}{2}\%$.

5. Find the value of $x^2-y^2+6x+10y-16$ when $x+y=15$ and $x-y=3$.

PAPER 7

1. The H.C.F. of two expressions is $x+2y$ and their L.C.M. is $6x^3+7x^2y-16xy^2-12y^3$ and one expression is $2x^2+xy-6y^2$. Find the other.

2. Simplify $\cfrac{3}{2x+3-\cfrac{3}{1-\cfrac{x}{x+6}}}$

3. What is meant by an indeterminate equation? Illustrate it by an example.

4. A person buys 50 kg of tea and 60 kg of coffee for Rs. 205. By selling the coffee at a loss of 5% and the tea at a gain of 10%, he makes a profit of Rs. 8·50. What was the cost price of tea and coffee per kg?

5. Collect the co-efficient of x^2 from the product of $(2x^2+5x-1)$ and $(3x^2-2x+4)$ without actual multiplication.

PAPER 8

1. Find the H.C.F. and L.C.M. of x^5+x^3, x^6-x^2, $x^4-3x^3+3x^2-3x+2$.

2. Simplify $x+1-\dfrac{x}{x+2-\dfrac{x+1}{x+\dfrac{1}{x+2}}}$.

3. Solve the equations $\left.\begin{aligned} 2x+5y &= 33 \\ 4x-3y &= 1 \end{aligned}\right\}$ by all the three methods.

4. A rectangular garden is 36 metres long and 25 metres broad. A road of uniform width is constructed inside all round it. The area of the remaining garden is 672 sq. metres. Find the width of the road.

5. Factorise $2a^3-7a^2+4a-3$.

PAPER 9

1. Find the L.C.M. of $3x^3+17ax^2-62a^2x+14a^3$ and $7x^3+52ax^2-46a^2x+8a^3$.

2. Simplify $\left\{1+\dfrac{2b^2}{a(a-3b)}\right\}\left\{1+\dfrac{b}{2b-a}\right\}$

$$-\left(\frac{a^2}{b^2}+\frac{b}{a}\right)\left(\frac{a^2-ab}{a^2-ab+b^2}-1\right).$$

3. Solve the equations $\left.\begin{aligned} \frac{4}{a}+\frac{5}{b} &= -4. \\ \frac{10}{b}-\frac{6}{a} &= 13. \end{aligned}\right\}$

4. A number of persons contribute a certain sum towards a charitable fund. If there had been 12 fewer, each would have contributed Rs. 1·60 more ; if there had been 12 more, each would have contributed 96 paise less. Find the number of persons and the contribution of each.

5. Collect the co-efficient of x^3 from the product of $(3x^4+x^3-5x^2+2x-1)(5x^3+4x^2-x+3)$ without actual multiplication.

OBJECTIVE TYPE ITEMS III

1. Fill up the blanks in the following :

(i) If both the numerator and the denominator of a fraction are multiplied by the same number, the value of the fraction is.....

(ii) To solve simultaneous equations involving two unknown quantities, you must be given at least..... equations.

(iii) Every common factor of two expressions is a factor of their.....

(iv) A fraction having a fraction in its numerator or denominator or in both is called a.....fraction.

(v) In addition and subtraction of fractions, they are first expressed with the same.....

2. Which of the following are simultaneous equations ?

A. $3x+17=29,\ y^2=9$
B. $5x-2y=22,\ 2x-5y+8=0$
C. $7y-4=2x,\ 3y-12x+54=0,\ x+y=3.$
D. $x+2y+z=6,\ 2x+2y-z=3,\ x-y-2z=-5.$
E. $4x-y=5,\ 8x-2y=10.$

3. In which of the following cases can the numerical values of the variables be found out ?

A. $x^2-25=(x+5)(x-5).$
B. $13x-7y=9x+11y.$
C. $29(x-y)=7,\ 99x=7(x+17)-27.$
D. $2x+5y=13,\ 6x+y-37=4(x-y)-24.$

4. Put a tick mark ($\surd$) against the correct result in the space provided in the following cases :

(i) If A and B are two expressions and H and L are their H.C.F. and L.C.M. respectively, then

.....A. $A+B=H+L$ .....B. $A-B=H-L$
.....C. $A\times B=H\times L$ .....D. $A\div B=H\div L$

(ii) The value of $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{x+1}\right)\left(1+\frac{1}{x+2}\right)\left(1+\frac{1}{x+3}\right)$ is

.....A. $1+\frac{1}{x+4}$ .....B. $\frac{1}{x+4}$

.....C. $\frac{x+4}{x}$ .....D. 1 .....E. none of these

(iii) x and y are respectively the digits in unit's place and ten's place of a number. If the digits are reversed, the new number is given by

.....A. $x+y$. .....B. $10y+x$.
.....C. $10x+y$. .....D. yx.

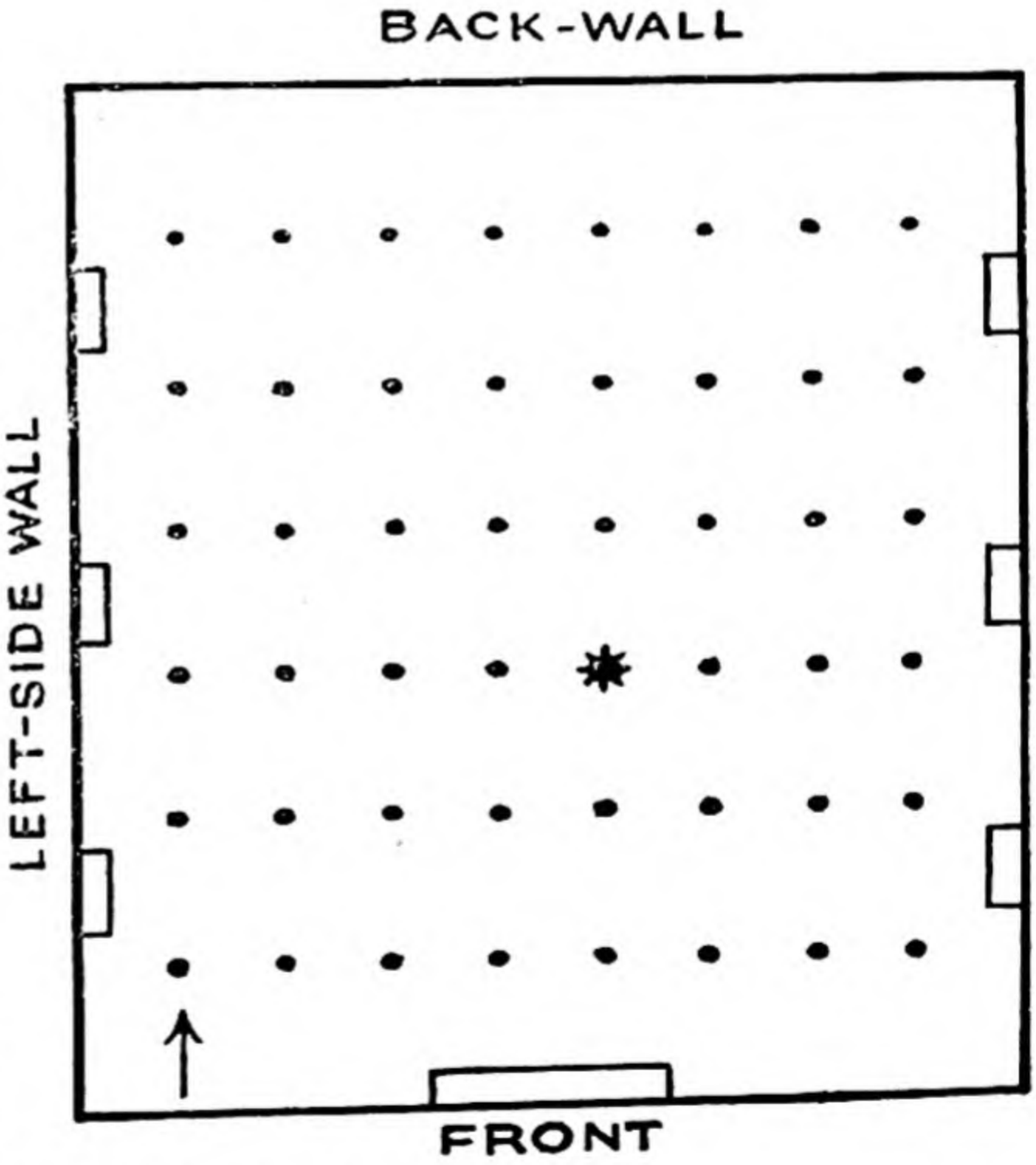
CHAPTER X

LINEAR GRAPHS

1. A graph is a pictorial representation. Its object is to convey quickly as much information as possible about the variations in two related quantities. Generally, a graph is a line made up of a series of points. Therefore it is necessary to know how to fix the positions of points which form the graph.

The adjoining figure is the plan of an examination hall. The position of each candidate is shown by a dot.

Suppose the position * of a particular person has to be defined. To say 'he is in the 3rd row' is not enough. The 3rd row may befrom the backwall or the front wall and it may refer to all the 8 candidates in that row. If we add that he is in the fifth column, even then it is not enough. There are 4 candidates answering that position. If, however, we say that he is in the 3rd row from the front and in the 5th column from the left side wall, his position is definitely fixed.

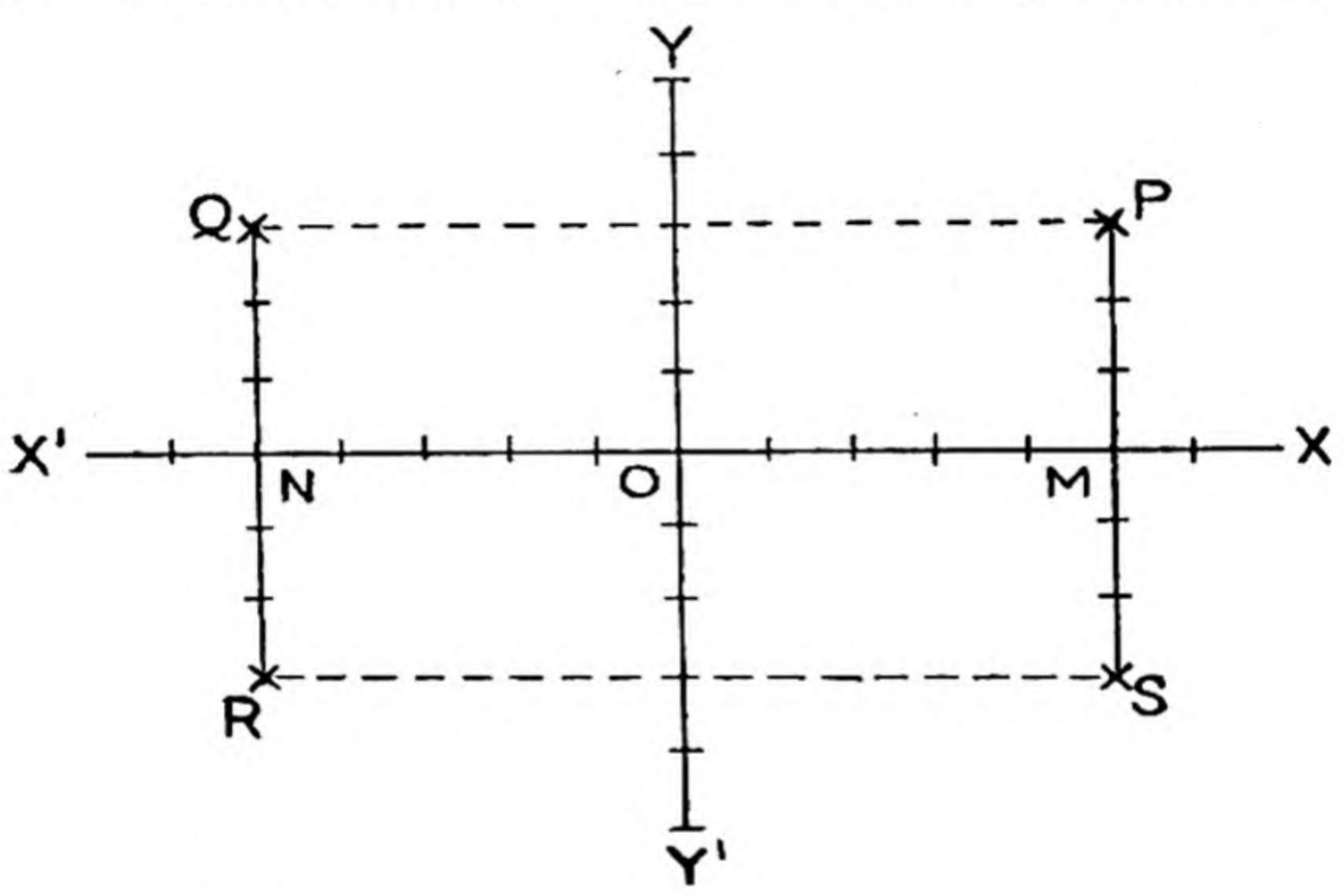


Similar, in order to fix the position of a point in a plane, we have to refer to *two selected straight lines.* These two lines

may well be the number-scales referred to in the chapter on Directed Numbers.

In them, we assumed that all measurements to the east and the north were positive and those in the opposite directions were negative.

We shall consider that XOX' and YOY' are two such graduated straight lines intersecting at O at right angles. We shall also follow the same convention regarding + and — signs to be prefixed to numbers denoting measurements



of lengths. We follow one other convention that in order to reach a point, we start from O and first proceed away from YOY' (*i.e.* east or west) and then away from XOX'. Accordingly, in the figure, P is reached by traversing from O to M and then from M to P. Similarly to reach R, we go from O to N and then from N to R. With these conventions, we can locate any point in a plane. In symbols the position of P is denoted as $(+5, +3)$.

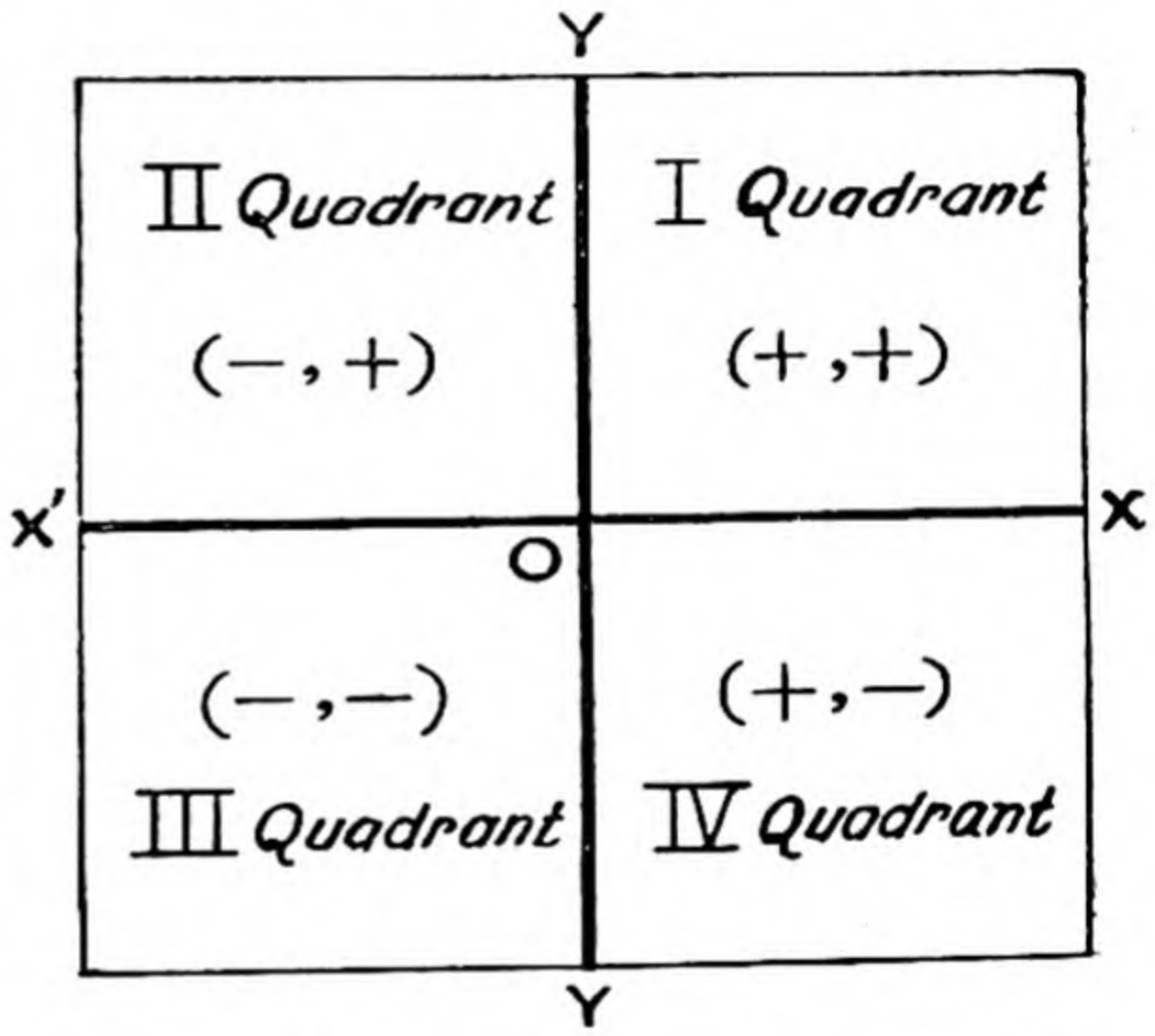
The lines XOX' and YOY' are called the *axes of reference*. The east-west line XOX' is called the *X-axis* and the north-south line YOY' is called the *Y-axis*. O is called the *Origin*. The measurements which define the position of a point are

called the *Co-ordinates* of the point. The co-ordinates of P are of $(+5, +3)$ and not $(+3, +5)$;.

the co-ordinates	Q are	$(-5, +3)$.
,,	R ,,	$(-5, -3)$.
,,	S ,,	$(+5, -3)$.

The first measurement *i.e.* the distance *from* YOY' is called the **abscissa** or *X-co-ordinate* and the second, *i.e.* the distance from XOX' is called the **ordinate** or *Y-co-ordinate*. In general, a point is represented by (X, Y).

The two axes divide the plane into 4 compartments ; each of them is called a quadrant. The quadrants are numbered in the order shown in the adjoining figure.



In I qdrt., x is $+$, y is $+$.
In II ,, x is $-$, y is $+$.
In III ,, x is $-$, y is $-$.
In IV ,, x is $+$, y is $-$.

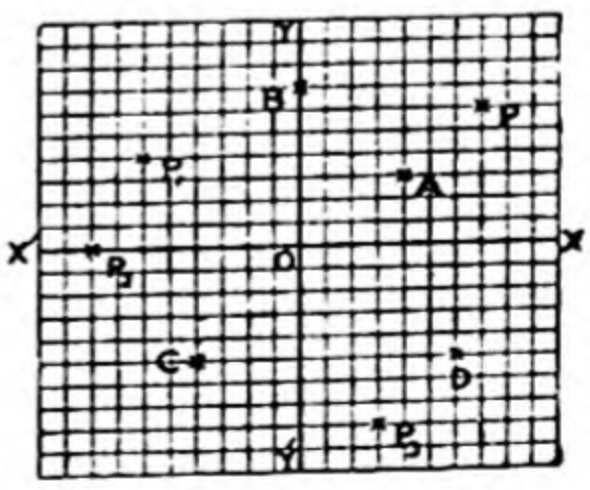
2. Plotting Points.

The process of determining the position of a point in a plane with reference to two axes, is called "*Plotting the point*".

For purposes of graphs, a squared paper is used (cm., mm).

Example 1 : (i) Write down the co-ordinates of the points A, B, C, D, taking one division as a unit.

(ii) Write down the co-ordinates of the points P, P_1, P_2, P_3, taking 2 divisions as a unit.



Sol. (i) The abscissa of A is $+4$ and its ordinate is $+3$.
,, B is 0 ,, ,, $+7$.
,, C is -4 ,, ,, -5.
,, D is $+6$,, ,, -5.

(ii) If we take 2 divisions as a unit,
P is $(+3{\cdot}5,\ +3)$; P_1 is $(-3,\ +2)$,
P_2 is $(-4,\ 0)$; P_3 is $(+1{\cdot}5,\ -4)$.

Example 2 : Take one division as 1 unit and plot the points $(7,\ 10)$, $(-5,\ 6)$ $(-8,\ 0)$, $(-4,\ -5)$, $(0,\ -6)$, $(3,\ -8)$.

Sol. To plot the point $(7,\ 10)$, we take from O, 7 units along OX and from there 10 units parallel to OY. (See fig.).

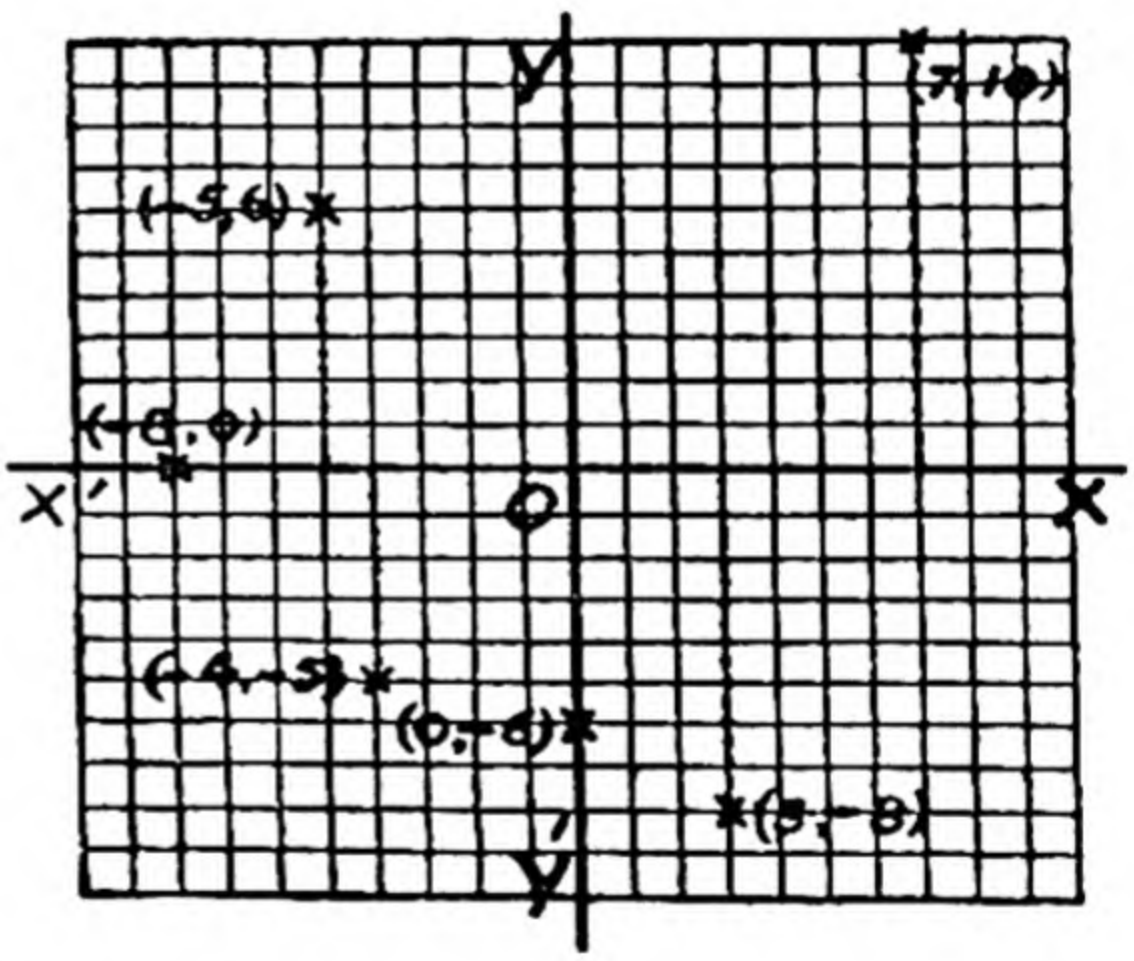
To plot the point $(-5,\ 6)$ we take from O, 5 units along OX' and from there 6 units parallel to OY. (See fig.)

To plot the point $(-8,\ 0)$, we take from O, 8 units along OX' and from there no unit above or below it. (See fig.)

To plot the point $(-4,\ -5)$, we take from O, 4 units along OX' and from there 5 units parallel to OY'. (See fig.)

To plot the point $(0,\ -6)$, we take no units along OX or OX' and 6 units parallel to OY' from O. (See fig.)

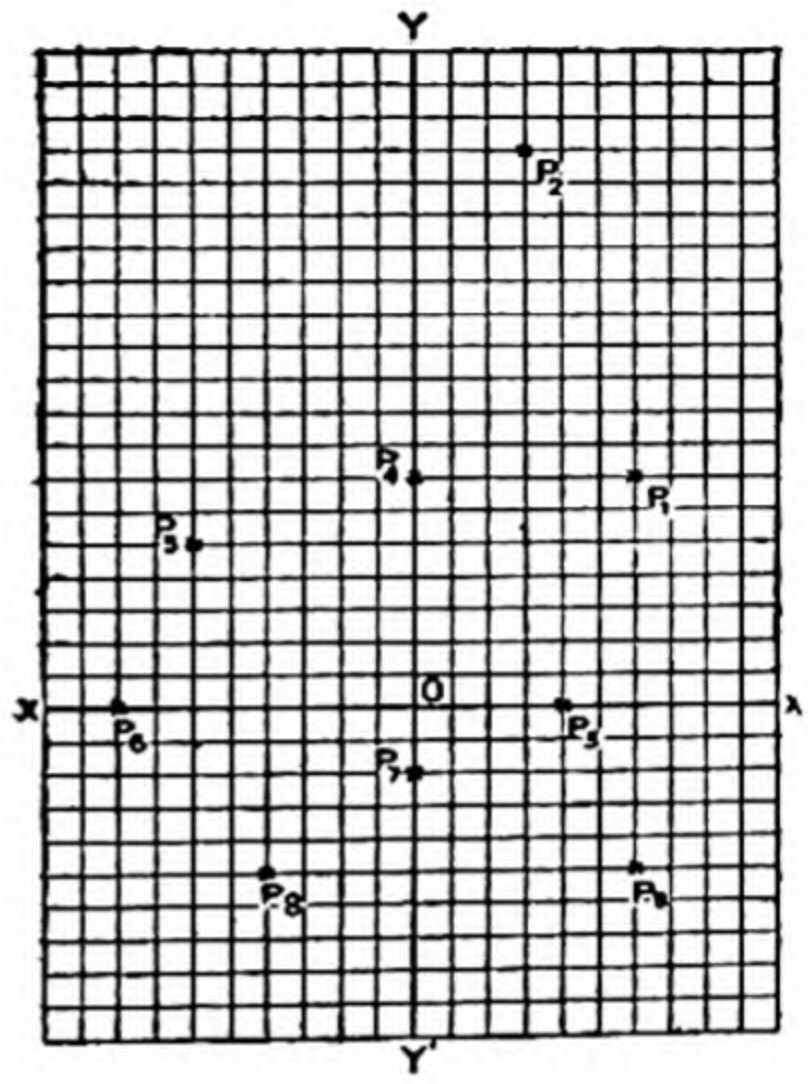
To plot the point $(3,\ -8)$, we take from O, 3 units along OX and from there 8 units parallel to OY'. (See fig.)



EXERCISE 97.

1. *Write down* the co-ordinates of the points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 in the figure on next page,
(i) taking one division as a unit.
(ii) taking 5 divisions as a unit.

2. Take one division as a unit and plot the points.
(9, 7), (0, 6), (−5, 8), (6, 0), (−7, −4). (0, 7), (−8, −5), (−7, +8), (9, −6). (17, −1), (−20, 4), (0, 0), (13, 0), (0, −13), (9, −9), (−9, −9), (0, 11).



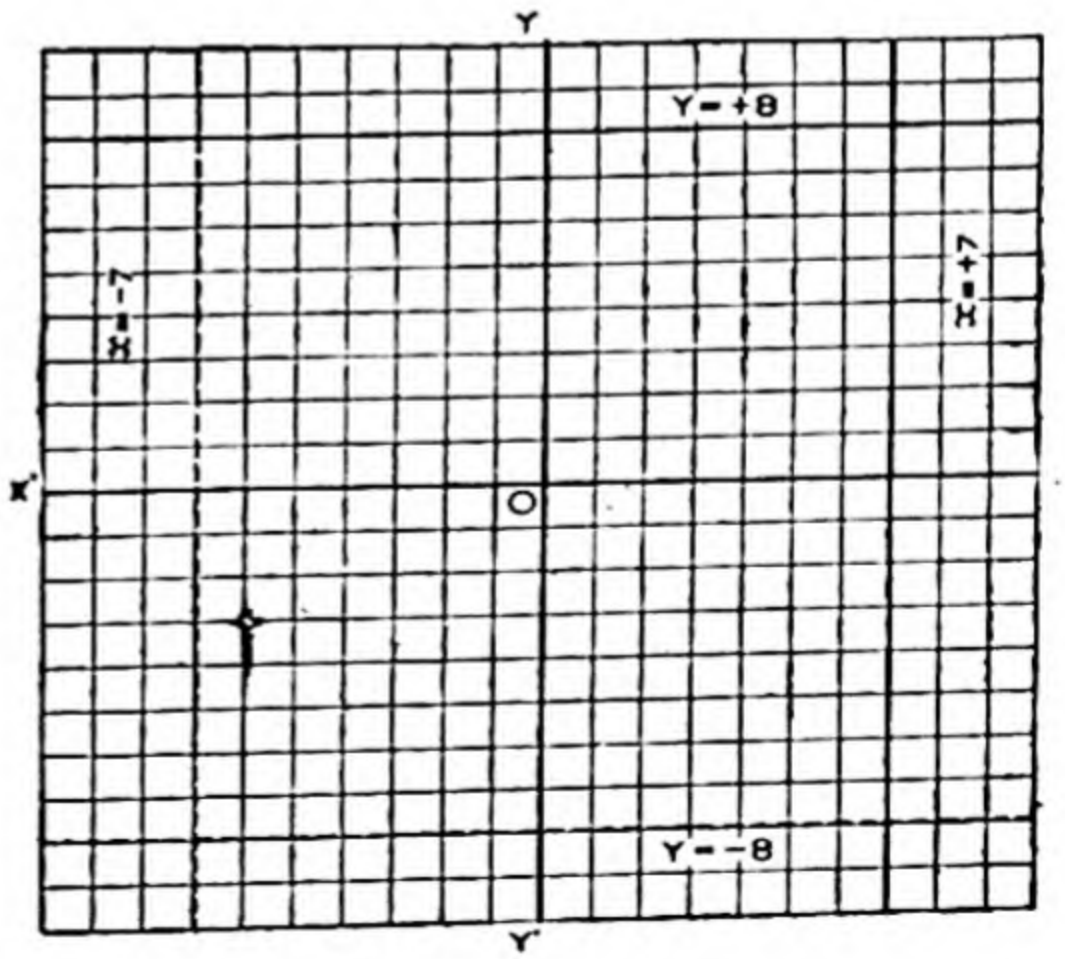
3. Take 1 division as a unit and plot the points:
(1, 0), (0, −1), (−1, −1), (−1, 0), (1·3, 1·5) (3·2, −·6), (3·6, −1·2), (−1·4, 0), (0, −·5), (·6, −·5).

4. Draw on squared paper taking 1 division as a unit, the figures whose vertices are the points:
(i) (0, 7), (5, 0), (−5, 0);
(ii) (5, 9), (−11, 6), (7, −4);
(iii) (6, 8), (−6, 10), (−5 −7), (12, −9);
(iv) (6, 10), (−7, 10), (−12, −4), (14, −4);
(v) (3 6), (−7, 7), (−12, −3), (−4 −6), (9, −8).

3. Graphs of $y = a$, $x = b$, $y = 0$ and $x = 0$.

If we plot points (−6,8), (−5,8), (−4,8) (−3,8), (−2,8), (−1,8), (0,8), (1,8), (2,8), (3,8), (4,8), (5,8), (6,8), and join them, a straight line is formed which is parallel to x-axis and is such that every point on it has its $y = +8$ and this is true for no other points. (See figure.)

The straight line is the **locus** of the points whose $y = +8$, whatever may be the x-co-ordinate, and the equation $y = +8$ represents this straight line.



Similarly, $y = -8$, represents a straight line parallel to x-axis and is such that every point on it has its $y = -8$.

Hence $y = a$, where a is constant, represents *a straight line parallel to x-axis*, and $y = 0$ is the *x-axis* itself.

Again the straight line parallel to the y-axis and at the distance of $+7$ units is such that every point on it has $x = +7$ and this is true for no other points. (See figure.)

Hence $x = +7$ represents the straight line.

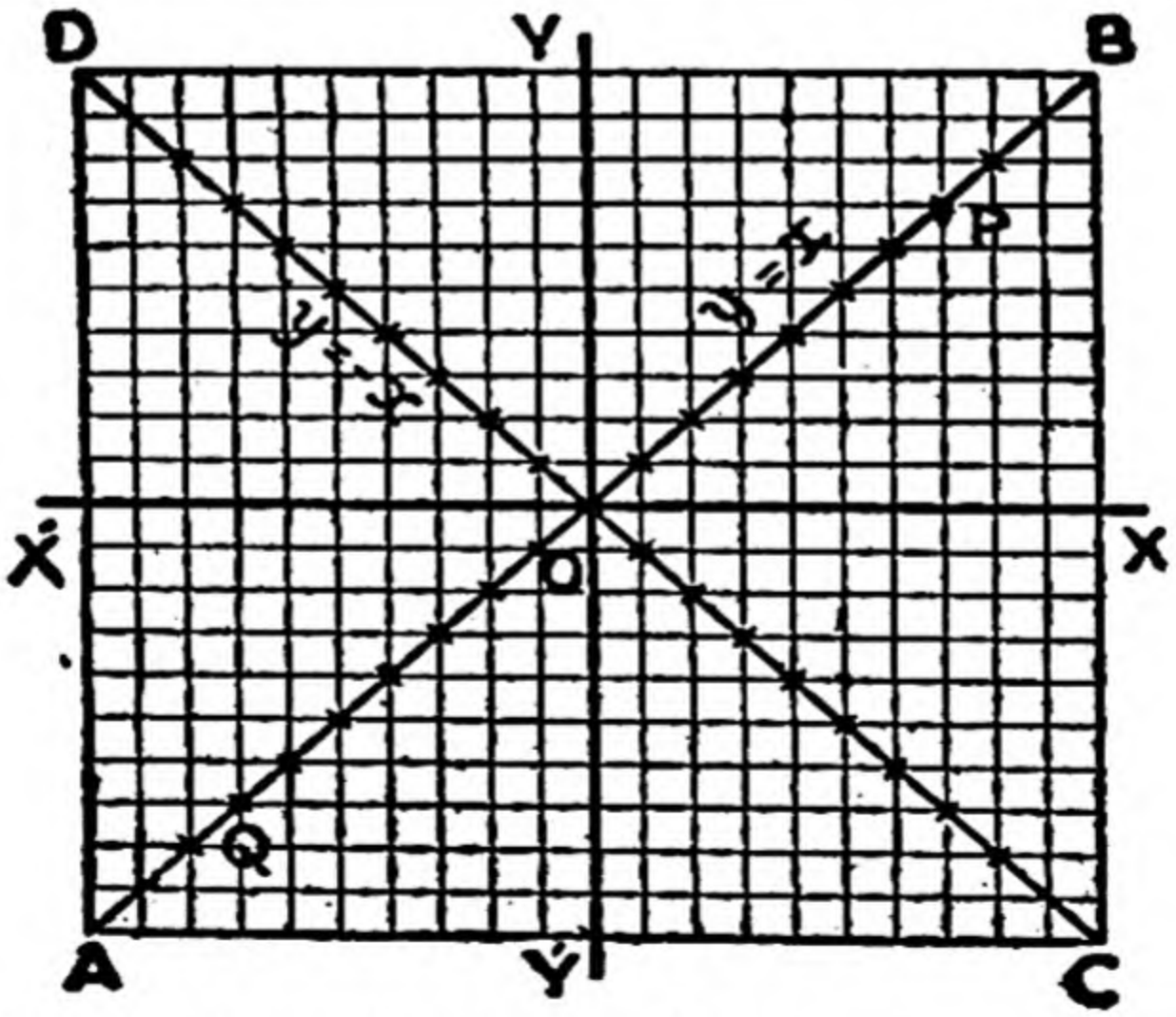
Similarly, $x = -7$ represents a straight line parallel to the y-axis and at a distance of -7 units from it.

Hence $x = b$, where b is constant, represents a *straight line parallel to the y-axis*, and $x = 0$ is the *y-axis* itself.

4. Graphs of (i) $\mathbf{y = x}$, (ii) $\mathbf{y = -x}$, (iii) $\mathbf{y = 2x}$, (iv) $\mathbf{y = -2x}$.

If we tabulate the values of x and the corresponding values of y from the equation $y = x$, we have

$x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y =$	-5	-4	-3	-2	-1	0	1	2	3	4	5



Plotting the points $(-5, -5)$, $(-4, -4)$, $(-3, -3)$, $(-2, -2)$, $(-1, -1)$, $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, etc.,

we find that all these points appear to lie on the straight line AB. (See figure.)

Now take *any* point P on the straight line AB. Its co-ordinates (7,7) satisfy the equation $y = x$. Also take *any* point Q $(-8, -8)$ whose co-ordinates satisfy the equation $y = x$. We find that Q lies on the straight line AB.

Since the co-ordinates of every point on the straight line AB satisfy the equation $y = x$ and every solution of $y = x$ gives us the co-ordinates of a point on the straight line AB, the straight line AB is **the locus** or **the graph** of **the equation** $y = x$.

Similarly, by tabulating the values of x and y from the equation $y = -x$, as shown below :

$x =$	5	4	3	2	1	0	−1	−2	−3	−4	−5
$y =$	−5	−4	−3	−2	−1	0	1	2	3	4	5

and plotting the points $(5, -5)$, $(4, -4)$, $(3, -3)$, $(2, -2)$, $(1, -1)$, $(0, 0)$, $(-1, 1)$, $(-2, 2)$, $(-3, 3)$, $(-4, 4)$, $(-5, 5)$ and joining them, we get the straight line CD.

Since the co-ordinates of every point on DC and of no other point satisfy the equation $y = -x$, the straight line CD is **the graph of the equation** $\mathbf{y = -x}$.

It is important to note that the graphs of $y = x$ and $y = -x$ *pass through the origin.*

In finding the graph of $y = 2x$, we give a series of positive and negative values to x and from the equation find the corresponding values of y and tabulate them as below :

$x =$	0	1	2	6	−1	−3	−4
$y =$	0	2	4	12	−2	−6	−8

Now if we plot the points (0,0), (1,2), (2,4), (6,12), $(-1, -2)$, $(-3, -6)$, $(-4, -8)$, we find that all these points lie on the straight line POQ.

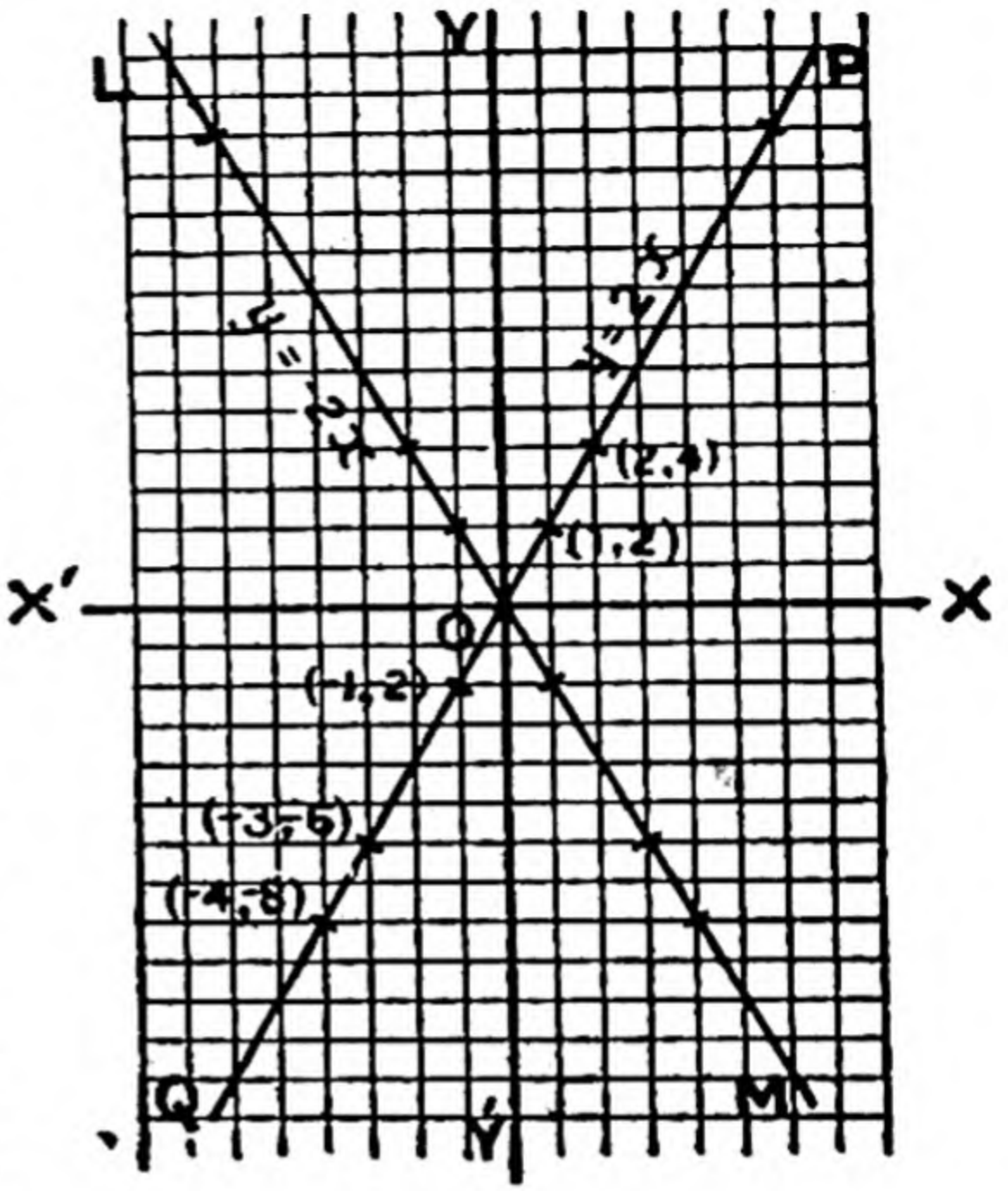
It can be easily verified that the co-ordinates of any point on the straight line POQ satisfy the equation $y = 2x$

and that the co-ordinates of no other point satisfy the equation.

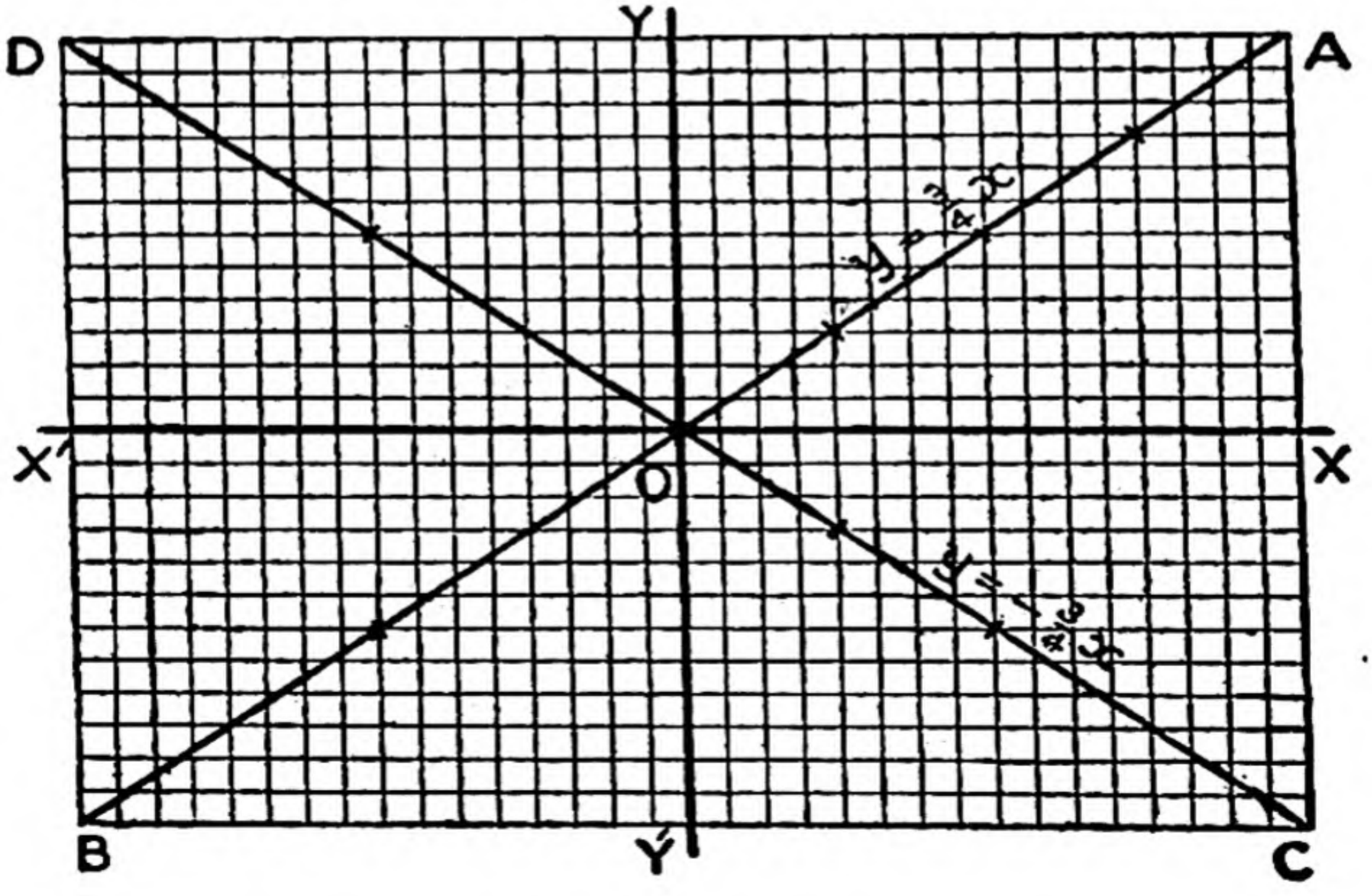
Thus the straight line POQ is **the graph** of **$y = 2x$.**

For the graph of $y = -2x$, we proceed exactly in the same way and find that the straight line LOM is the required graph.

It is important to note that the graphs of $y = 2x$ and $y = -2x$ *pass through the origin.*



5. **Graphs of (i) $4y - 3x = 0$, or $y = \frac{3}{4}x$; (ii) $4y + 3x = 0$, or $y = -\frac{3}{4}x$.**



Tabulating the corresponding values of x and y from the equation $y=\frac{3}{4}x$, we have

$x=$	0	1	2	3	4	-2	-4
$y=$	0	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3	$-\frac{3}{2}$	-3

As these values involve fractions with denominators 2 and 4, it is therefore necessary for accurate drawing to take 4 divisions as the unit instead of one division.

Now we plot points (0, 0), (1, $\frac{3}{4}$), etc., according to this scale, and observe that all these points lie on the straight line AOB.

As it can be easily verified that the co-ordinates of any point on the straight line AOB and the co-ordinates of no other point satisfy the equation $y=\frac{3}{4}x$, therefore the straight line AOB is **the graph** of $\boldsymbol{y=\frac{3}{4}x}$.

Similarly, by tabulating the corresponding values of x and y from the equation $y=-\frac{3}{4}x$ and plotting the points thus got, we can show that the straight line COD is **the graph** of $\boldsymbol{y=-\frac{3}{4}x}$.

It is important to note that the graphs of $y=\frac{3}{4}x$ and $y=-\frac{3}{4}x$ *pass through the origin*.

The following points are worth noting:

(i) An equation connecting x and y can be represented by a *graph*. For every graph, there is a corresponding *equation*.

(ii) The equations $y-x=0$, $y+x=0$, $y-2x=0$, $y+2x=0$, $4y-3x=0$, $4y+3x=0$, $y=x$, $y=-x$, $y=2x$, $y=-2x$, $y=\frac{1}{3}x$, $y=-\frac{1}{3}x$, $y=\frac{3}{4}x$, $y=-\frac{3}{4}x$ (considered above), can be put into the form $\boldsymbol{ax+by=0}$.

Since $ax+by=0$ can again be put into the form $y=\left(-\frac{a}{b}\right)x$ or $y=mx$, therefore each of the above equations can be put into the form $\boldsymbol{y=mx}$.

(iii) **The graph** of $\boldsymbol{y=mx}$ is *a straight line passing through the origin*. All lines are slanting lines.

(iv) In each case m denotes the *slope* of the line.

Slope means the amount of steepness of a line. As we move from left to right, *i.e.* as x increases, if the line goes upward, it has a positive slope ; if the line dips downward, it has a negative slope.

To have an idea of the slope of a line, take any 2 points on it ; note their x and y co-ordinates.

Then the slope $m = \dfrac{\text{Change in } y\text{-co-ordinates}}{\text{Change in } x\text{-co-ordinates}}$.

EXERCISE 98.

1. If a point moves so that its ordinate is always -5, find the path traced out by it.

2. If a point moves so that its abscissa is always -8, find its locus.

3. Find the locus of a point which moves so that the ratio of its ordinate to its abscissa is (i) 3 : 2 (ii) -3 : 2.

Draw the graphs of the following equations :

4. (i) $x = -7$, (ii) $x = 12$, (iii) $x = 0$, (iv) $x = -9$.

5. (i) $y = -6$, (ii) $y = 11$. (iii) $y = 0$, (iv) $y = -10$.

6. (i) $y = -\frac{1}{4}x$, (ii) $y = \frac{1}{5}x$, (iii) $y = -\frac{1}{6}x$.

7. (i) $y = \frac{3}{5}x$, (ii) $y = -\frac{3}{5}x$,.

8. (i) $3x+2y = 0$, (ii) $3x-2y = 0$.

9. (i) $4x+5y = 0$, (ii) $4x-5y = 0$.

10. (i) $3x-5y = 0$, (ii) $3x+5y = 0$.
(iii) $3y-5x = 0$, (iv) $3y+5x = 0$.

6. **Graphs of (i) $y = 2x$ (ii) $y = 2x+6$, (iii) $y = 2x-6$.**

The corresponding values of x and y satisfying

(i) $y = 2x$ are

$x=$	0	1	2	3	4	-3	-4
$y=$	0	2	4	6	8	-6	-8

(ii) $y = 2x+6$ are

$x=$	0	1	2	3	-2	-4
$y=$	6	8	10	12	2	-2

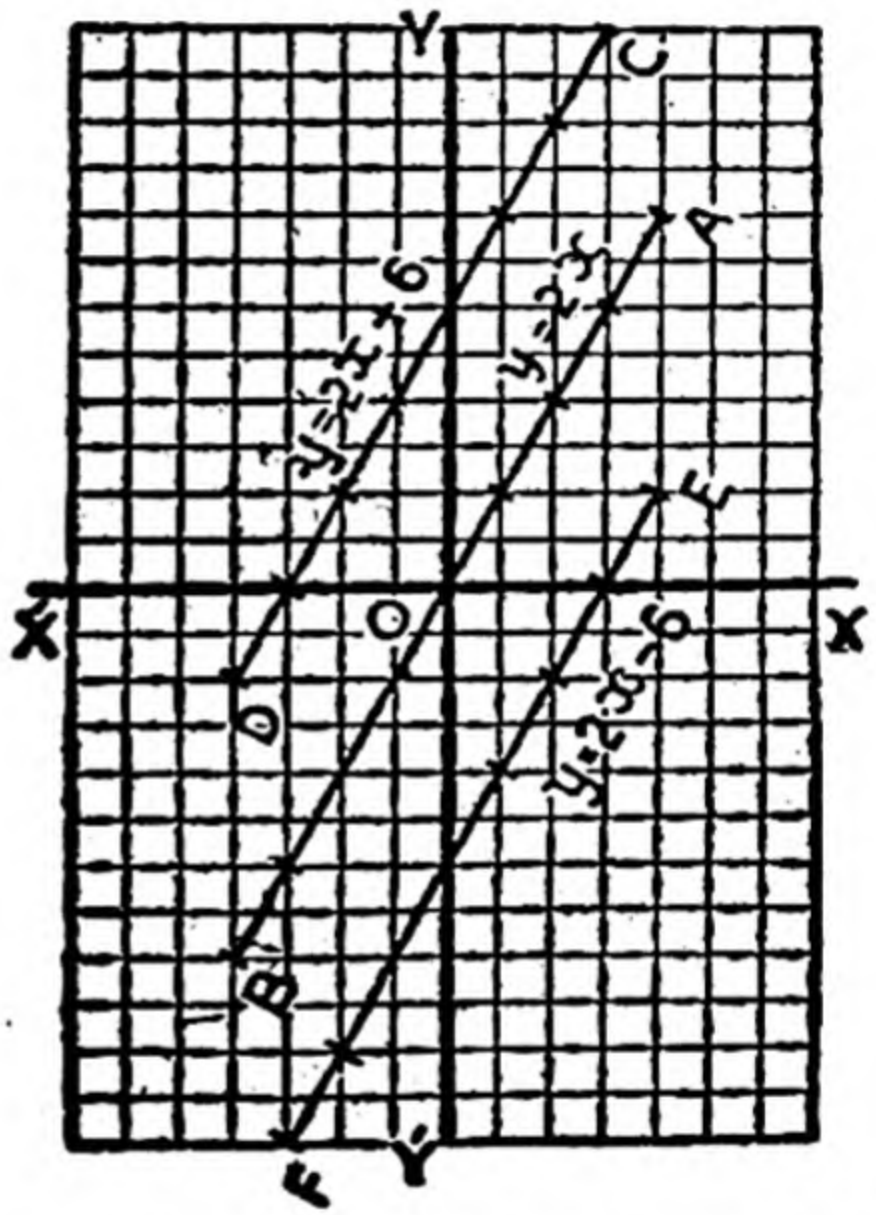
(iii) $y = 2x - 6$ are

$x =$	0	1	2	3	4	-2	-3
$y =$	-6	-4	-2	0	2	-10	-12

Now we plot points from table (i) and find that AOB is the graph of $y = 2x$.

We plot points from table (ii) and find that CD is the graph of $y = 2x + 6$.

We plot points from table (iii) and find that EF is the graph of $y = 2x - 6$.



In each case the graph is a straight line and all these straight lines are *parallel*.

The line $y = 2x + 6$ cuts y-axis at $+6$; $y = 2x + (0)$ cuts y-axis at 0 ; and $y = 2x - 6$ cuts y-axis at -6.

Then $+6$ and -6 are called the intercepts of the lines $y = 2x + 6$ and $y = 2x - 6$ respectively on the y-axis.

or the intercept made by the graph of an equation of the form $\boldsymbol{y = mx + c}$ on y-axis is $\boldsymbol{c}$, and $\boldsymbol{m}$ is the slope.

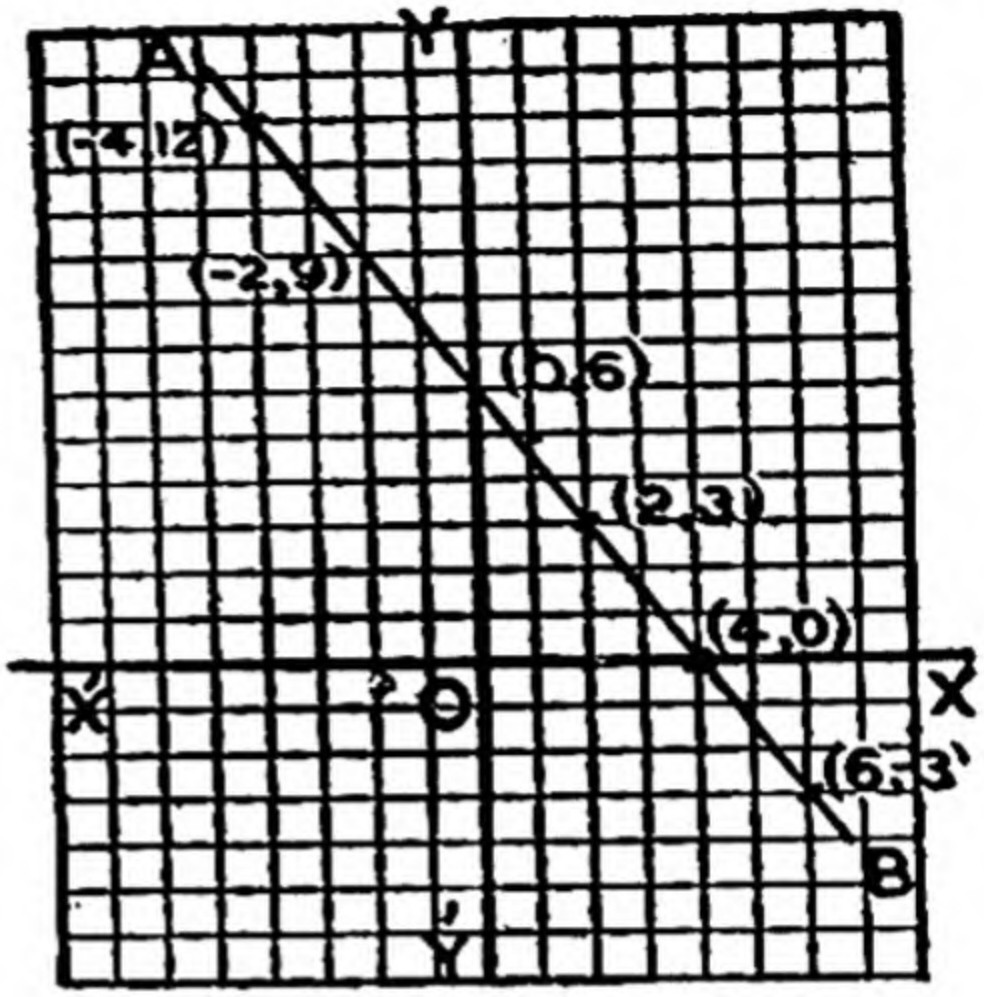
Note that any equation of the form $ax + by + c = 0$ can be put into the form $y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$ or $\boldsymbol{y = mx + c}$.

7. Graph of $\frac{x}{4} + \frac{y}{6} = 1$.

Tabulating the corresponding values of x and y from this equation, we have

$x =$	0	1	2	3	4	6	-2	-4
$y =$	6	$\frac{9}{2}$	3	$\frac{3}{2}$	0	-3	9	12

Plot the points (0, 6), (2, 3), (4, 0), (6, −3), (−2, 9), (−4, 12) (with integral co-ordinates) and join them.



All these points lie on the straight line AB.

Since it can be easily verified that the co-ordinates of any point on AB and the co-ordinates of no other point satisfy the equation $\frac{x}{4}+\frac{y}{6}=1$, AB is the graph of $\frac{x}{4}+\frac{y}{6}=1$.

Draw similarly the graphs of the following equations:

(i) $\frac{x}{3}+\frac{y}{4}=1$, (ii) $\frac{x}{4}+\frac{y}{5}=1$, (iii) $\frac{x}{5}+\frac{y}{6}=1$, and observe the intercepts made by these lines on the axes and compare their length wi h the denominators of x and y in each case.

It is important to note that the straight line $\frac{x}{4}+\frac{y}{6}=1$ cuts x-axis at (4, 0) and y-axis at (0, 6) or the intercept made by it on x-axis is 4 and is the same as the denominator of x in the equation, and the intercept made by it on y-axis is 6 and is the same as the denominator of y in the equation.

Similarly, the intercepts of $\frac{x}{3}+\frac{y}{4}=1$ on the x-axis and y-axis are **3** and **4** respectively.

In *general*, if an equation is of the form $\frac{x}{a}+\frac{y}{b}=1$, its graph is a straight line which makes with x-axis an intercept $= a$ and with y-axis an intercept $= b$.

From the above examples it is clear that **the graph of an equation of the first degree in x and y, whatever be its form, is always a straight line.**

It is for this reason that an equation of the first degree in x and y is generally called a **linear equation.**

Since a straight line is determined by any two points on it, therefore for drawing the graph of a linear equation, theoretically it is sufficient to plot two points on it, but for practical purposes and accuracy it is necessary to plot at least three points on it sufficiently apart. Integral values are more helpful than fractional ones. In most cases the values we get by putting $x = 0$ and $y = 0$ are very convenient.

Summary

(i) (0,0) represents the origin.

(ii) $y = 0$ represents x-axis.

(iii) $x = 0$ represents y-axis.

(iv) $x = a$ represents a straight line parallel to y-axis and at the distance of a from it.

(v) $y = b$ represents a straight line parallel to x-axis and at the distance of b from it.

(vi) $y = mx$ represents a straight line passing through the origin with m as its slope from x-axis.

(vii) $y = mx + c$ represents a straight line with m as its slope from x-axis, and c as its intercept on y-axis.

(viii) Equations $y = mx + c_1$, $y = mx + c_2$, etc., which differ in the absolute term only, represent straight lines parallel to $y = mx$.

(ix) $\frac{x}{a} + \frac{y}{b} = 1$ represents a straight line having intercepts a and b on x-axis and y-axis respectively.

EXERCISE 99.

[Choose a suitable scale wherever necessary, use accurate squared paper and a pencil with a sharp end.]

1. In the same diagram draw the graphs of:

(i) $y = 4x + 5$. (ii) $y = -4x + 5$, (iii) $y = 4x - 5$.

(iv) $y = -4x - 5$.

2. In the same diagram draw the graphs of:
(i) $3y = x+6$, (ii) $3y+x = 6$, (iii) $x-3y=6$,
(iv) $x+3y+6 = 0$.

3. In the same diagram draw the graphs of:
(i) $y = 5x$, (ii) $y = 5x+4$, (iii) $y = 5x-4$,

4. In the same diagram draw the graphs of:
(i) $y+2x = 0$, (ii) $y+2x = 5$, (iii) $y+2x+5 = 0$.

Reduce each of the following equations to the form $y = mx+c$, and state the slope in each case.

5. $x = 2y+5$. **6.** $x+\frac{1}{2}y = 4$. **7.** $3x-4y = 10$.

8. $2y = \frac{1}{4}(x-3)$. **9.** $\frac{x}{3}-\frac{y}{4}+1 = 0$. **10.** $\frac{4}{5}x+\frac{5}{4}y = 1$.

Reduce each of the following equations to the form $y = mx+c$; state the slope in each case.

11. $y = 3$. **12.** $x = 5$. **13.** $y = 0$.

14. Reduce each of the equations of questions 5 to 10 to the form

(i) $\frac{x}{a}+\frac{y}{b} = 1$; (ii) $ax+by+c = 0$.

Draw the graphs of the following equations:

15. $x+3 = 0$. **16.** $y+7 = 0$. **17.** $3y+5x = 0$.
18. $5y-3x = 0$. **19.** $5y+3x = 0$. **20.** $8y+6x = 5$.
21. $8x-6y = 5$. **22.** $x-4y = 9$. **23.** $x-4y+9 = 0$.
24. $4x = 2y+6$. **25.** $x = 11y-4$. **26.** $x+3y = 9$.
27. $\frac{x}{5}+\frac{y}{3} = 1$. **28.** $\frac{x}{5}-\frac{y}{3} = 1$. **29.** $\frac{x}{7}+\frac{y}{9} = 1$.
30. $\frac{x}{7}-\frac{y}{9} = 1$. **31.** $\frac{x-y}{3} = \frac{3}{4}$ **32.** $\frac{y}{3} = \frac{7x-6}{5}$.
33. $\frac{y}{3} = 4\frac{1}{2}-x$.

34. Draw the graphs of the following equations in the same diagram and find the area included by them:
(i) $y-3 = 0$, (ii) $x-2 = 0$, (iii) $4x+5y = 60$.

35. Draw the graph of the equation $3x+4y = 8$. In this graph read the value of x when y is -10. (Punjab, 1918.)

36. Plot the graph of $y = \frac{3x+13}{2}$ and from the graph read off the value of y when $x = 3$. What are the intercepts on the axes of x and y?

Show that the points $(-1, 5)$ and $(1, 8)$ lie on the graph.

8. **Graph of a Linear Function of x.**

An expression which contains a variable x and which has a definite value for every value of x is called **a function of x.**

Thus $3x^2+2x-5$, $3x+5$ are functions of x.

Such expressions as $6x$, $7x+3$, $\frac{3x-5}{4}$ which contain no power of x higher than the first are called **linear functions of x.**

Suppose we have to draw the graph of the linear function $\frac{2x-6}{3}$.

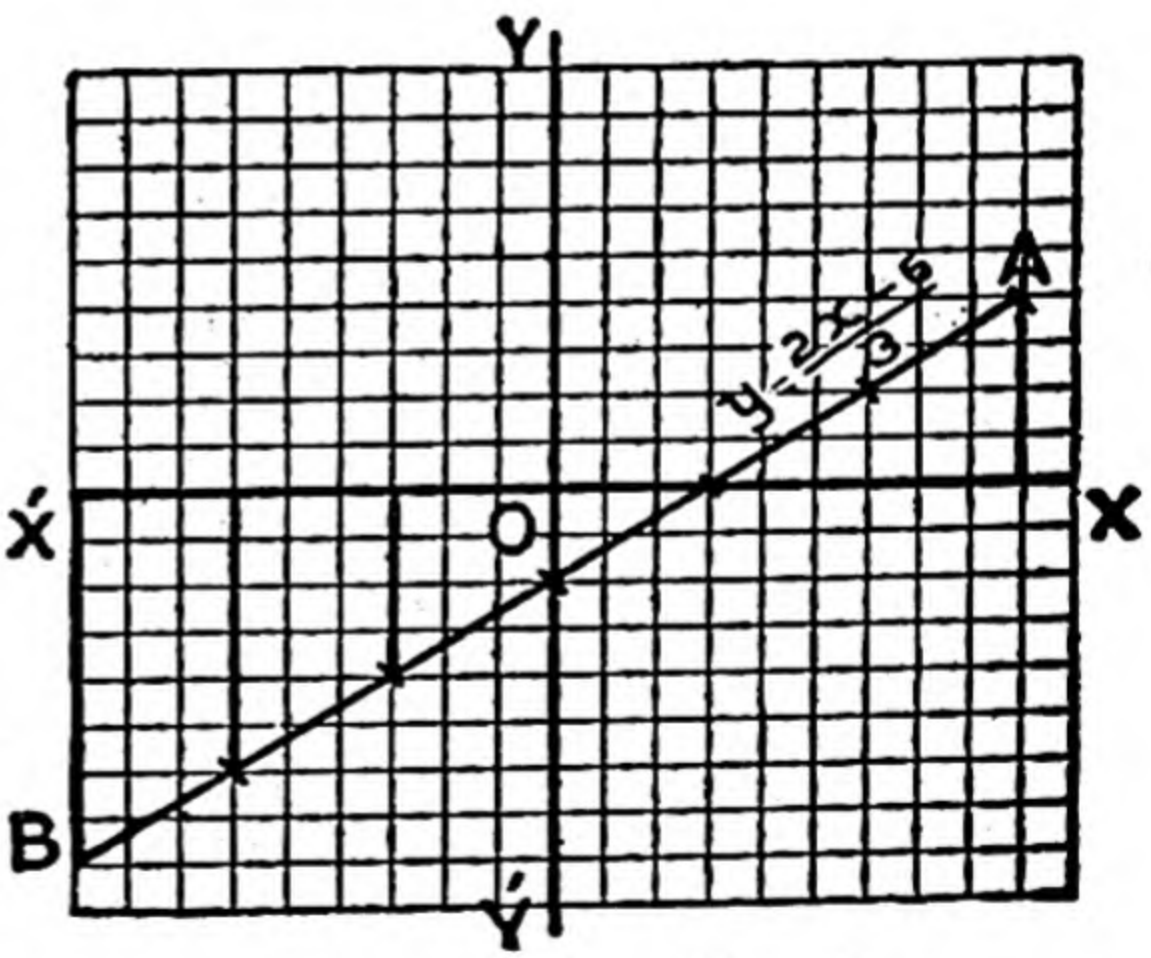
Here we put $y = \frac{2x-6}{3}$ and by giving different values to x find the corresponding values of y.

As this is an equation of the first degree, its graph must be a straight line.

Now we choose 3 convenient points on this straight line as given below :

x =	0	3	6
y =	−2	0	2

Plot the points $(0, -2)$ $(3, 0)$, $(6, 2)$ and join them. The straight line AB is the required graph.



From this graph we can read the value of the function of x for different values of x, *e.g.*, when

(i) $x = 9$, $\frac{2x-6}{3} = 4$; (ii) $x = -3$, $\frac{2x-6}{3} = -4$;

(iii) $x = -6$, $\frac{2x-6}{3} = -6$; (iv) $x = -9$, $\frac{2x-6}{3} = -8$.

9. Graphical Solution of Simultaneous Equations.

Example 3: Solve graphically the equations:

$2x + y = 17$ . . . . (i)

$3x - 2y = 8$ . . . . (ii)

Sol. As both equations are of the first degree in x and y, their graphs must be straight lines. Prepare a table of 3 convenient points from each equation:

From $2x + y = 17$, we have

$x=$	0	5	10	**6**
$y=$	17	7	−3	**5**

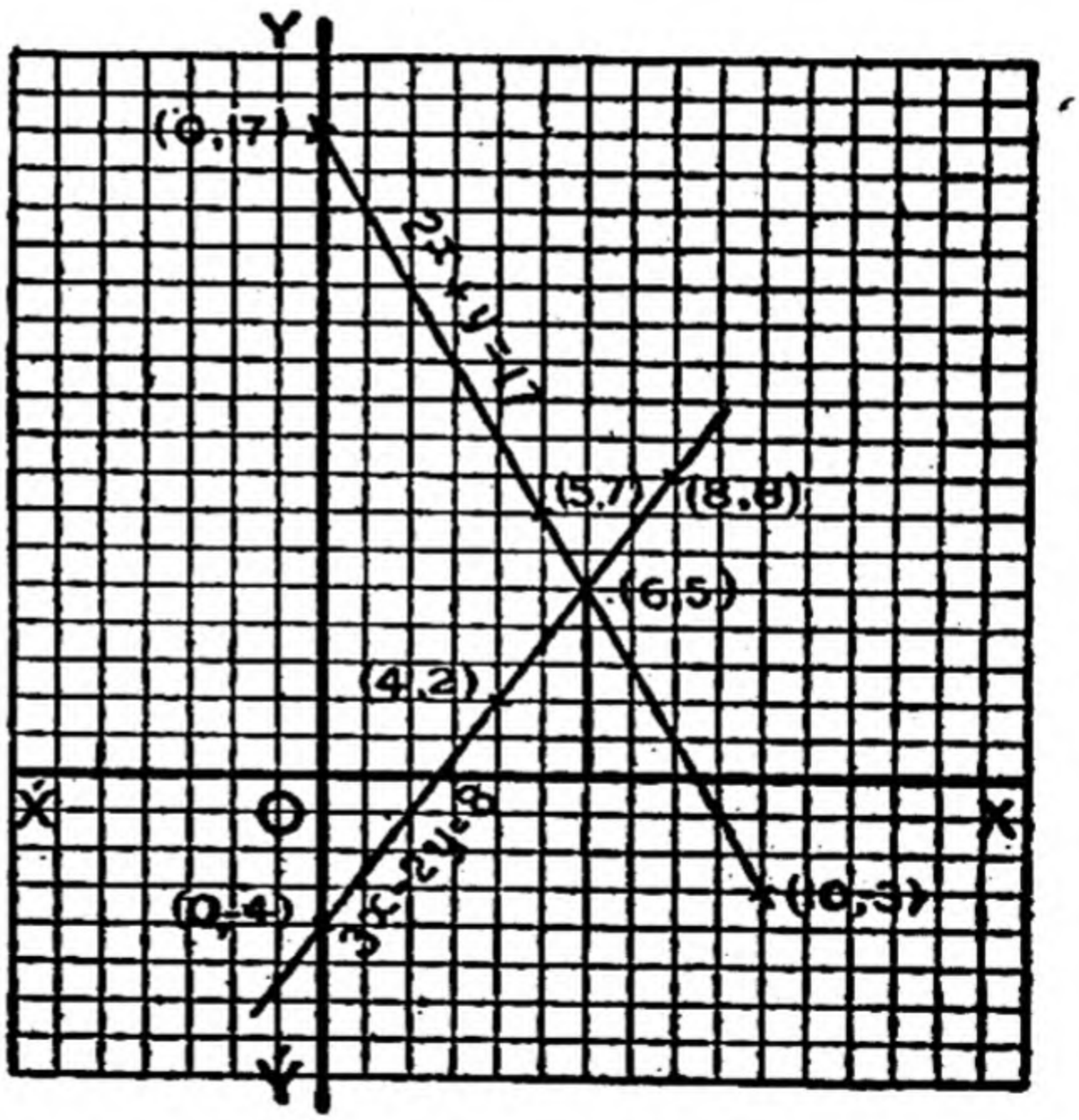
From $3x - 2y = 8$, we have

$x=$	0	4	8	**6**
$y=$	−4	2	8	**5**

Plot the first set of 3 points and join them and write the corresponding equation on the graph.

Similarly, plot the second set of 3 points and join them and write down its equation for the sake of distinction. Note the co-ordinates of the point where the two graphs intersect. They cut at (6, 5).

As the point of intersection is common to both lines, the co-ordinates (6, 5) satisfy both the equations.



Hence $x = 6$ and $y = 5$ is the solution of the given equations.

[The students may verify the result by solving the equations by the ordinary method.]

EXERCISE 100.

Solve graphically and verify the result by the ordinary method :

1. $x+y=7$, $x-y=-1$.

2. $3x-y=11$, $2x+3y=22$.

3. $2x+3y=17$, $3x+2y=18$.

4. $2x-3y=-7$, $3x-2y=2$.

5. $4y=3(x+2)$, $5x=2(y+2)$.

6. $2x=3y$, $\frac{x}{2}+\frac{y}{6}=1\frac{5}{6}$.

7. $x+5=0$, $2x-5y=0$.

8. $2y=3x-16$, $\frac{1}{2}x=3y+16$.

9. $3x+4y=24$, $4x-3y=7$.

10. $5x-3y=29$, $3x+5y+3=0$.

NOTE : In the qns. 9 & 10 above, the co-efficients of x and y in the simultaneous equations are interchanged with a single change in their signs. The graphs of such equations are perpendicular to each other.

11. $x-2y+11=0$, $2x-3y+18=0$.

12. $\frac{x}{5}+\frac{y}{10}=1$ $3x-2y=1$.

Show that the following straight lines are concurrent; also find the co-ordinates of the common points :

13. $x+y=11$, $2x-y-4=0$, $x+2y=17$.

14. $2x+y=13$, $3x-2y-2=0$, $x+11=3y$.

15. $4x-y=5$, $5x+2y=16$, $x+3y=11$.

16. Find the co-ordinates of the vertices and also the lengths of the sides of the triangle formed by the straight lines

$$x-y=0,\ x+5=0,\ x+y=4.$$

SELECTED QUESTIONS—F.

1. Solve the equations graphically :

$x+3y=-4$, $\frac{1}{4}x-3y=-1$.

2. Draw the graphs of $x-2y=1$, $y+2x=-3$ and hence or otherwise solve the equations.

3. Solve graphically the equations :

$$x=\frac{y-1}{3},\quad \frac{x+5}{2}+y=0.$$

4. Solve the equations $11x+12y=58$, $12x+11y=57$ and verify the solution graphically.

5. Draw the graph of the equations $y=\frac{3x+5}{4}$ and $y=\frac{5x-21}{3}$. For what value of x are the values of y equal?

6. Solve graphically the equations $4x+3y=12$ and $2x-y=16$ and read the points of intersection of the latter with the axes of x and y.

7. Solve the equations $148x+231y=527$, $231x+148y=610$ and verify the solution graphically.

8. Solve the equations $\frac{2}{x}+3y=15$, $\frac{5}{x}-4y=3$ and verify the solution graphically.

CHAPTER XI

STATISTICAL GRAPHS

1. The representation of statistical information is one of the most useful and interesting practical applications of graphs. A graph of statistics is much more clearly grasped than a table of numbers. It presents a vivid picture of the results of periodical observations. It also enables us to make comparisons and draw useful conclusions.

2. **Ready-Reckoner.**

Example 1: Sixty oranges sell for Rs. 9. Make a graph to show the cost of any number upto 80, and from it write down the cost of 25 oranges, and the number of whole oranges you would get for Rs. 5.

Sol. Let the price in rupees be represented by x and the number of oranges by y.

$$\text{then } \frac{y}{x} = \frac{60}{9} \text{ or } y = \frac{20}{3}x$$

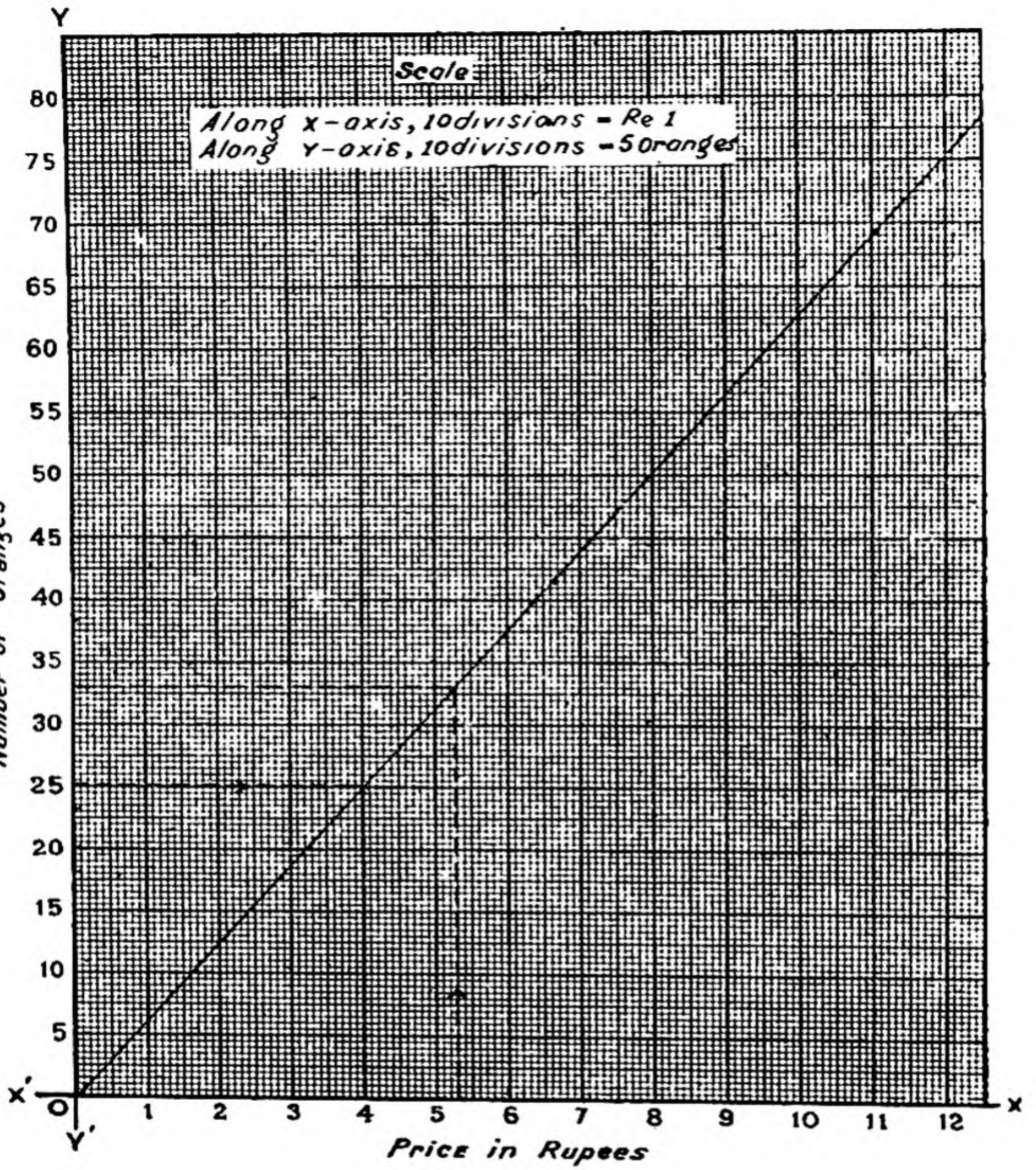
Prepare a table of values for x and y.

Number of rupees	$x=$	0	3	9	12
Number of oranges	$y=$	0	20	60	80

Using a suitable scale, plot the points and draw a graph as shown at page 209.

From the graph read off the cost in rupees corresponding to 25 oranges. It is Rs. 4. Again the number of oranges for Rs. 5·30 lies between 33 and 34. The number of whole oranges is **33.** This can be verified by calculation.

Such a graph is called a *ready-reckoner*. It enables us to know one quantity in terms of the other by sight without the arduous process of calculation. But the limit to which it can be used depends on the size of the graph paper. With a larger paper, greater accuracy is possible.



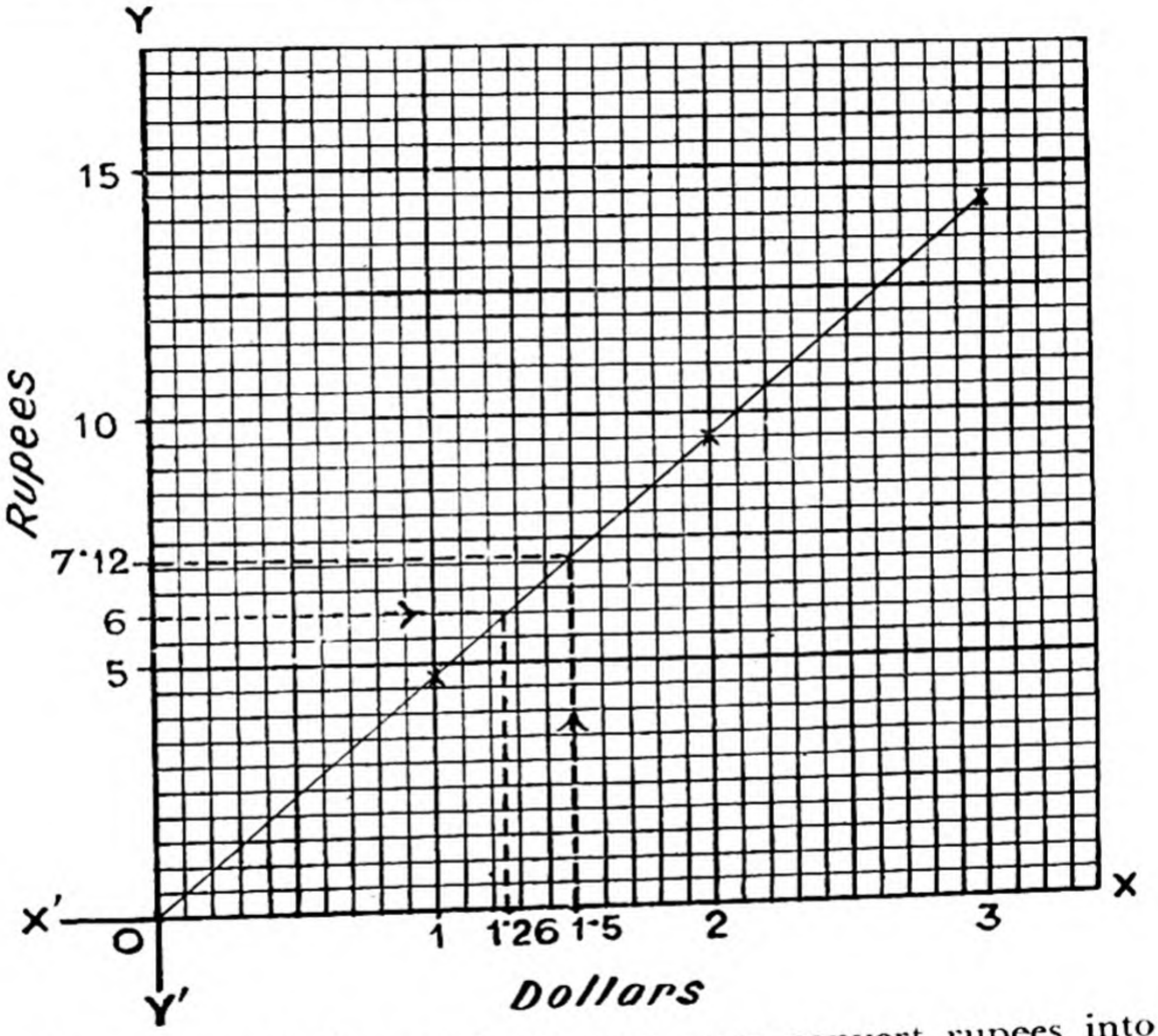
3. **Conversion Graph.** \$1 = Rs. 4·75 (approx.); therefore if y represents the number of rupees and x the number of corresponding dollars, we have

$$y = 4{\cdot}75x$$

Tabulate the values of x and y from this equation.

$x=$	0	1	2	3
$y=$	0	4·75	9·50	14·25

Taking 10 small divisions for \$1 along the x-axis and 2 small divisions for Re. 1 along the y-axis, we have the following graph :



By means of the graph we can now convert rupees into dollars and dollars into rupees with limited accuracy. Thus \$1·5 = Rs. 7·12 (approx.) and Rs. 6 = \$1·26 (approx). By drawing a larger graph, we can increase its range of application. Such graph is called a **conversion graph.**

To illustrate still further the use of conversion graphs we construct here a graph representing the relation between the degrees of Centigrade and Fahrenheit scales.

Since 32°F = 0°C and 212°F corresponds to 100°C and there are (212°−32°) or 180°F for 100°C, ∴ $\frac{9}{5}$°F = 1°C.

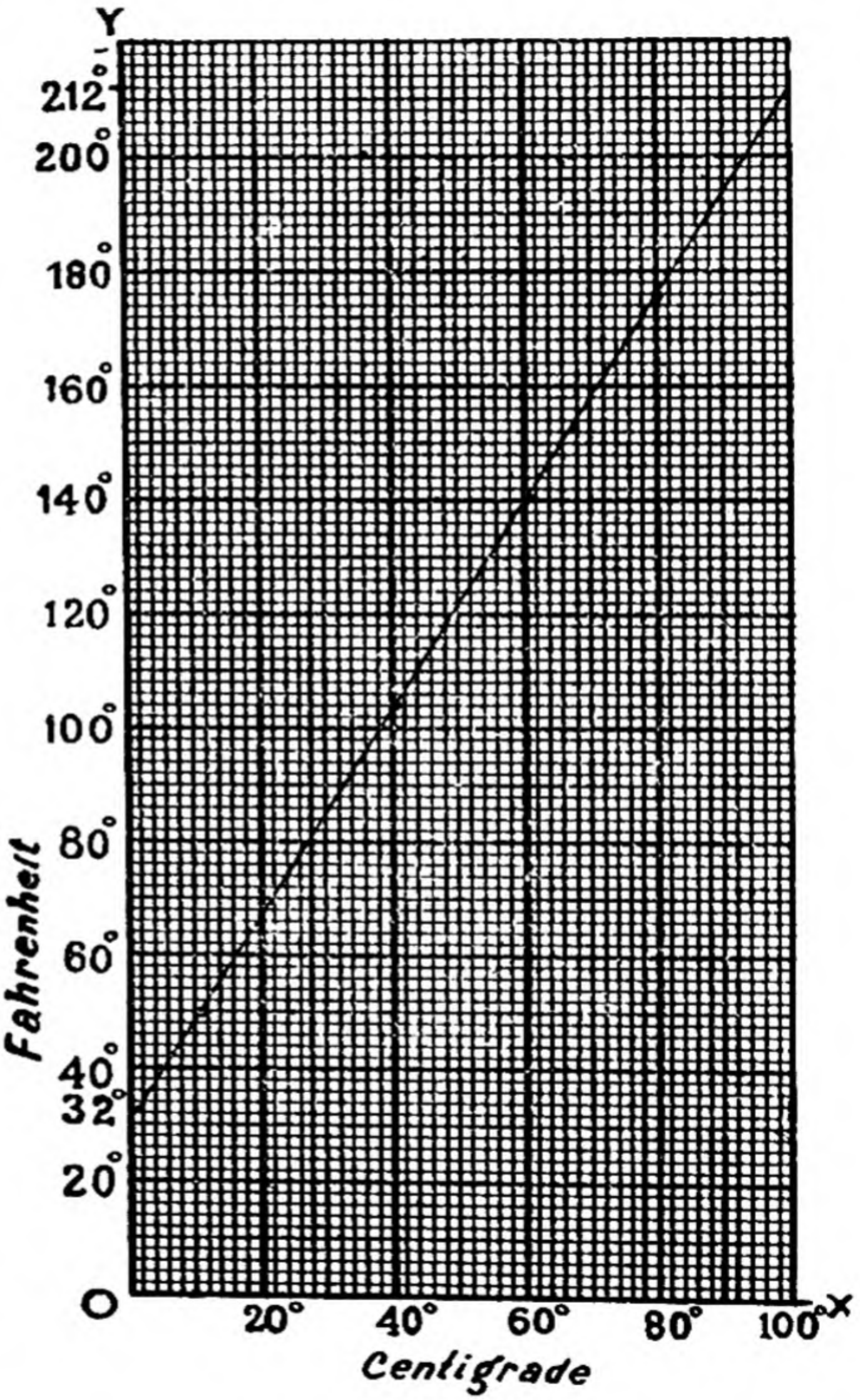
Hence the relation between the number of degrees in the two scales is given by the equation $\frac{y-32}{9} = \frac{x}{5}$ or $y = \frac{9}{5}x + 32$,

where x stands for the number of degrees in the Centigrade scale and y for the number of degrees in the Fahrenheit scale.

Tabulating the values of x and y from the above equation, we have

x=	0	10	40	60	80
y=	32	50	104	140	176

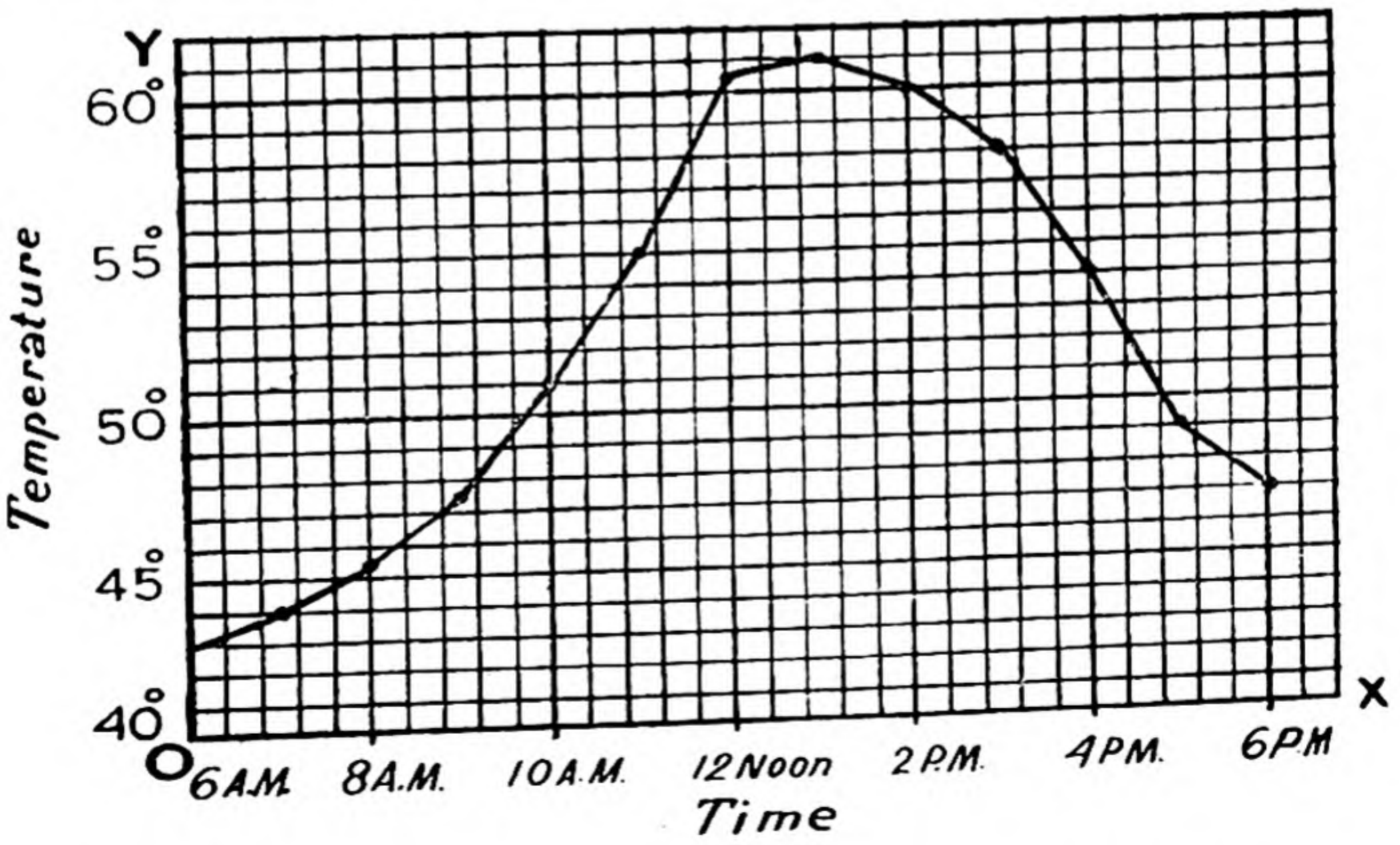
Taking 1 cm to represent 20°C along x-axis and 20°F along y-axis and plotting the points, we get the adjoining graph. From this graph read the value of 50°C in Fahrenheit scale and that of 120°F in Centigrade scale.



4. Discontinuous and Continuous Graphs.

Sometimes the corresponding values of two *variables* cannot be determined from a given relation or an equation but are obtained by actual observation or experiments, *e.g.*, the temperature at different intervals of time throughout a given period. In such cases we draw the graph representing the relation between the two variables with the help of the data thus collected and by joining the successive points by *straight lines*. The graph drawn consists of *separate lines* and is not continuous. To illustrate this point we have drawn here a graph which represents the relation between the hours of a day and the temperature of a place at such hours in Fahrenheit scale.

	Forenoon							Afternoon					
Time	6	7	8	9	10	11	12	1	2	3	4	5	6
Temp.	43°	44°	45·5°	47·5°	51°	55°	60·5°	61°	60°	58°	54°	49°	47°



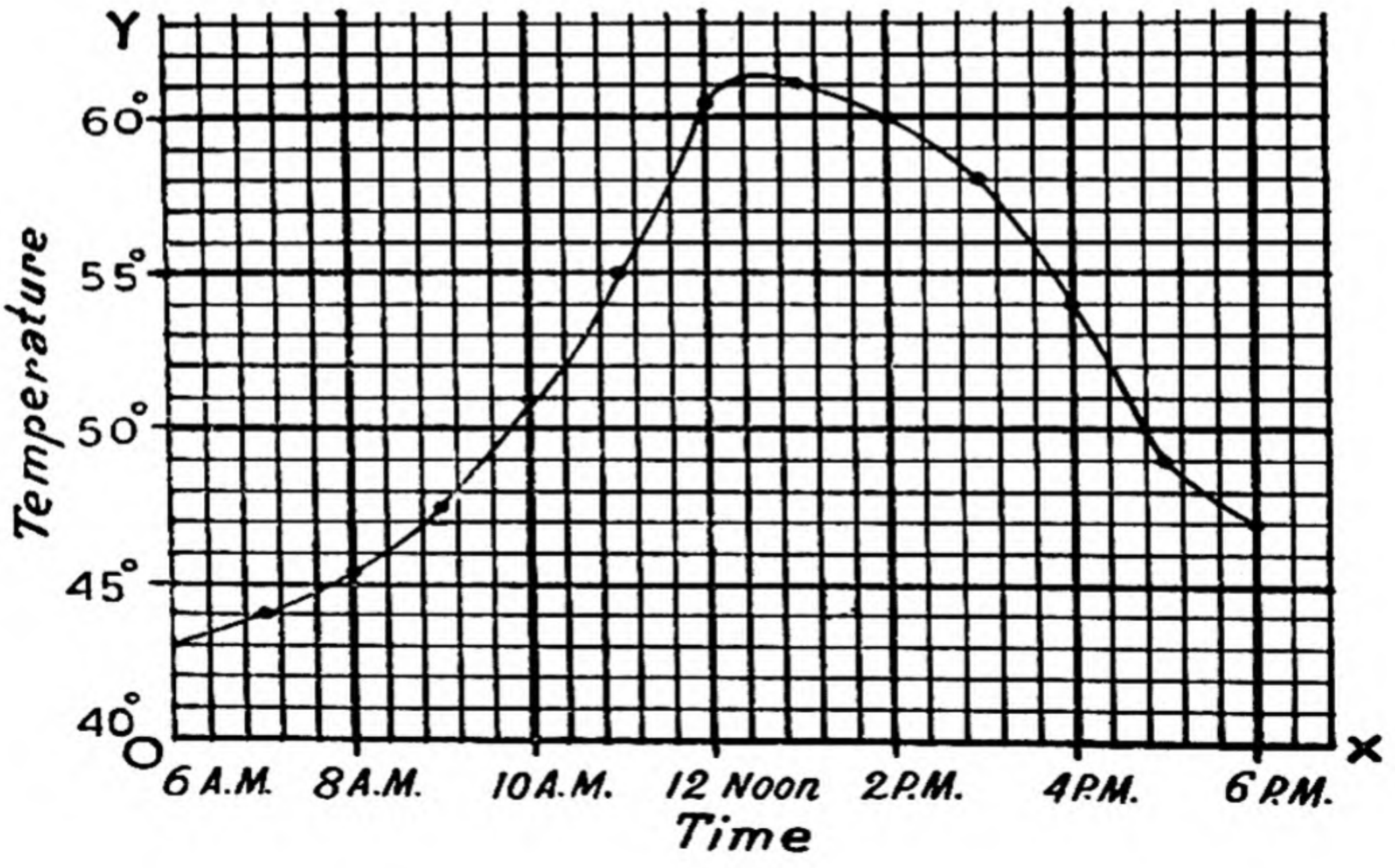
In this case we have taken 6 A.M. and 40° as the *origin*, $2\frac{1}{2}$ divisions for 1 hour along the x-axis and 1 small division for 1° along the y-axis. The successive points are joined by separate st. lines and we get a discontinuous graph.

Now we can read the approximate temperature at any particular time:

e.g., at 10-30 A.M. temp. is 53° (approx.)
and at 2-30 P.M. temp. is 58·8° (approx.)

At 1 P.M. the temperature appears to be the *maximum*, and between 12 noon and 2 P.M. the temperature is fairly steady. Between 10 A.M. and 12 noon there is a rapid increase, and between 3 P.M. and 5 P.M. there is a rapid fall.

In all experimental work, since the observations are made at intervals, the graphs made from the data thus collected would be discontinuous. Really in majority of cases the variation *is not so abrupt but gradual and continuous*; therefore the proper graph in such cases should be continuous, as shown below:



This graph gives us a better representation of the variation of temperature during the day and also gives us more accurate reading, *e.g.*, the maximum temperature is not exactly at 1 P.M. but between 12 noon and 1 P.M.

EXERCISE 101.

1. If rice cost 75 paise per kilogram construct a graph to read off the price of any number of kg. Frame an equation connecting the price of rice in rupees and the number of kilograms.

2. If 40 articles cost Re. 1·25 construct a graph to read off the price of articles correct up to 5 paise.

3. Taking £3 = Rs. 40 construct a conversion graph to read off the number of shillings for a given number of rupees.

4. Taking 1 kilometre = ·62 mile (approx.), draw a graph showing the number of miles in any number of kilometres up to 100. Read off approximately the number of kilometres in 45 miles.

5. Taking 1 cm = ·39″, construct a conversion graph to read off any number of centimetres in inches.

6. The temperature of a patient taken every two hours during a day is recorded as below :

Time	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.	10 p.m.	12 mid-night	2 a.m.	4 a.m.
Temp.	99·2°	100°	100·4°	101°	103°	102·4°	101·6°	100°	99·8°	99·6°	99·4°	99·2°

Draw a continuous graph and read off the temperature at 3 p.m. and the time when the temperature was approximately 100·5°F.

7. The average monthly temperature of a town in Fahrenheit scale is as follows :

Month	Jan.	Feb.	Mar.	Ap.	May	June.	July	Aug.	Sep.	Oct.	Nov.	Dec.
Temp.	63°	72°	88°	95°	102°	106°	104°	100°	96°	90°	75°	65°

Draw a continuous graph and read off approximately the hottest fortnight of the year.

8. The length of the longest day in different latitudes is given below in hours:

Latitude	0°	10°	20°	30°	40°	50°	60°	70°	80	90°
Length of the longest day	12	12·6	13·3	14	14·8	16·1	18·5	24	24	24

Draw a continuous graph. Estimate the length of the longest day in latitudes 16°, 38°, 65°.

9. The following table gives us the heights of places above the sea-level in thousands of metres and the corresponding heights of the barometer in centimetres:

Height above sea-level	0	1·5	3	4.5	6	7·5
Height of barometer in cm	70	62·5	51·5	42·7	35·7	27·5

Draw a continuous graph and read off the height of the barometer at 1,000 m and 3,000 m above the sea-level.

10. The temperature of water in a boiler is 20°C. It is continually heated for half an hour and its temperature is recorded at intervals of 5 minutes and is tabulated below:

Intervals in minutes	5	10	15	20	25	30
Temperature	24°	28·6°	32·8°	36·6°	40°	43°

Show graphically the relation between time and temperature of the water and read off the temperature after 23 minutes.

11. The formula $s = 4{\cdot}90\ t^2$ gives us the relation between the space (s) covered by a body falling freely under gravity in time (t). If in the following table s be given in metres and t in seconds,

t =	1	2	3	4	5	6
s =	4·90	19·6	44·1	78·4	122·25	176·4

draw a continuous graph for the relation between the distance and time, and read off

(i) the height of a tower from the top of which a stone takes 4·5 seconds to fall,

(ii) the depth of water-level of a well which a stone takes 2·25 seconds to reach.

Verify the results by calculation.

12. The average height of boys and girls at different ages is given below in centimetres in two different tables:

Table for boys

Age in years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ht. in centi-metres	68	80	89	93	95	100	107	116	122	124	126	128	132	140	149	156	160	164

Table for girls

Age in years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ht. in centi-metres	68	79	84	88	95	105	111	116	117	120	124	128	140	148	150	154	156	158

Draw *carefully* both these graphs on the same paper with the same axes. Represent one by dotted lines and the other by a continuous line.

Note the following points :

(1) The age up to which both the curves are concurrent.

(2) The ages at which the two curves intersect.

(3) The ages when the springs of boys and girls are almost parallel.

(4) The ages when the plateaus of boys and girls are almost parallel.

(5) The ages when the boys are in springs and the girls in plateaus.

Note that the *plateaus are the periods when the system can be more energetic and the springs are the periods when the system is more liable to fatigue.*

Taking this as a law, point out periods

(i) when the boys can be made to do more work;

(ii) when the girls can be made to do more work;

(iii) when the boys should have a lighter course;

(iv) when the girls should have a lighter course;

(v) at what ages should we hold the public examinations for boys and girls ?

PART II

CHAPTER XII

INDICES

1. The definition of an index is given in Chapter I. According to that definition, in a^5, 5 is the index and it indicates that a is repeated as a factor five times, or $a^5 = a \times a \times a \times a \times a$. It does not cover *fractional* and *negative* indices. So the definition of an index must be extended so that a^{-5}, $a^{\frac{2}{3}}$, etc. may also have some meaning.

We have already established 3 laws, namely:

$$a^m \times a^n = a^{m+n} \quad \ldots \quad \ldots \quad \text{(i)}$$

$$a^m \div a^n = a^{m-n} \quad \ldots \quad \ldots \quad \text{(ii)}$$

and $$(a^m)^n = a^{mn} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Assuming that they are true when m and n are positive integers, we must interpret a^{-m}, $a^{\frac{m}{n}}$, a^o in such a way that these obey the ordinary laws of Algebra.

2. Meaning of $a^{\frac{m}{n}}$.

By the law of indices, $a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}$

$$\therefore \quad \left(a^{\frac{2}{3}}\right)^3 = a^{\frac{2}{3}\times 3} = a^2.$$

Taking the cube root of both sides, we have $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

Similarly, $a^{\frac{4}{5}} = \sqrt[5]{a^4}$ and $a^{\frac{7}{11}} = \sqrt[11]{a^7}$.

For further generalisation, we proceed as below.
By the law of indices,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}\div \ldots \text{to } n \text{ terms}}$$

$$\therefore \quad \left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}\times n} = a^m.$$

Taking the nth root of both sides, we have

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence $a^{\frac{m}{n}}$ is the n^{th} root of a^m.

Cor. $a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a}$.

Thus, $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, and so on.

Since $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots\ldots m$ factors $= a^{\frac{1}{n} \times m} = a^{\frac{m}{n}}$

and $a^{\frac{1}{n}} = \sqrt[n]{a}$, $\left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$, or $a^{\frac{m}{n}} = \left(\sqrt[n]{a^m}\right)$

Thus, $a^{\frac{m}{n}}$ may be interpreted either as the n^{th} root of the mth power of a, or as the mth power of the n^{th} root of a.

3. Meaning of a^0.

By the law of indices, $\because$ $a^m \div a^n = a^{m-n}$,

$$a^4 \div a^2 = a^{4-2} = a^2;$$
$$a^4 \div a^3 = a^{4-3} = a^1;$$
$$a^4 \div a^4 = a^{4-4} = a^0;$$

But any quantity divided by itself gives 1.

i.e. $a^4 \div a^4 = 1$ $\quad\therefore\quad a^0 = 1$.

So quantity raised to the power 0 equals 1.

Again, by the law of indices, $a^0 \times a^m = a^{0+m} = a^m$.

Dividing both sides by a^m, we have

$$\frac{a^0 \times a^m}{a^m} = \frac{a^m}{a^m}$$

$$\therefore \qquad a^0 = 1.$$

Hence for all finite values of a, $a^0 = 1$.

4. Meaning of a^{-m}.

By the law of indices,

$$a^m \times a^{-m} = a^{m-m} = a^0 = 1.$$

Dividing both sides by a^m, we have

$$a^{-m} = \frac{1}{a^m}$$

Also $$a^m = \frac{1}{a^{-m}}.$$

Cor. $\dfrac{a^{-m}}{b^{-n}} = a^{-m} \times \dfrac{1}{b^{-n}} = \dfrac{1}{a^m} \times b^n = \dfrac{b^n}{a^m}$.

Hence any quantity with a *negative* index denotes the reciprocal of the same with the same *positive* index and *vice-versa*.

Example 1: Find the value of $27^{\frac{4}{3}}$.

Sol. $27^{\frac{4}{3}} = \left(\sqrt[3]{27}\right) = 3^4 = 81$.

Example 2: Find the value of $9^{-\frac{3}{2}}$.

Sol. $9^{-\frac{3}{2}} = \dfrac{1}{9^{\frac{3}{2}}} = \dfrac{1}{(\sqrt{9})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$.

Example 3: Multiply together $\sqrt[4]{a^3}$, $a^{\frac{3}{2}}$, $\dfrac{1}{a^{-4}}$ and $a^{-\frac{1}{4}}$.

Sol. The required product $= a^{\frac{3}{4}} \times a^{\frac{3}{2}} \times a^4 \times a^{-\frac{1}{4}}$

$= a^{\frac{3}{4}+\frac{3}{2}+4-\frac{1}{4}}$

$= a^6$.

EXERCISE 102.

Express the following without fractional and negative indices:

1. $a^{\frac{3}{5}}$. **2.** $x^{-\frac{3}{5}}$. **3.** $x^{-\frac{2}{3}} \times 6a^{-\frac{1}{2}}$.

4. $9p^{-2} \times p^{-\frac{3}{4}}$. **5.** $m^{-\frac{2}{3}} \div a^{-\frac{3}{2}}$. **6.** $\dfrac{4}{x^{-\frac{2}{3}}}$.

7. $\sqrt{m^3} \div \sqrt{x^{-2}}$. **8.** $\sqrt{x^{-7}} \times \sqrt{a^5}$.

9. $\sqrt{x^{12}} \div \sqrt{x^{-5}}$.

Express the following without radical signs and negative indices:

10. $(\sqrt{x})^7$. **11.** $(\sqrt{m})^{-9}$. **12.** $\dfrac{1}{\sqrt{m^{-2}}}$. **13.** $\dfrac{1}{(\sqrt[3]{a})^{-4}}$.

14. $\sqrt[4]{x^6} \div (\sqrt{x})^{-1}$. **15.** $\sqrt{m^{-5}} \div (\sqrt{m})^{-6}$.

Find the value of:

16. $4^{-\frac{1}{2}}$. **17.** $8^{\frac{2}{3}}$. **18.** $16^{\frac{3}{4}}$. **19.** $64^{\frac{2}{3}}$.

20. $32^{-\frac{2}{5}}$. **21.** $27^{-\frac{2}{3}}$. **22.** $125^{\frac{2}{3}}$. **23.** $81^{-\frac{3}{4}}$.

24. $128^{-\frac{3}{7}}$. **25.** $343^{-\frac{2}{3}}$. **26.** $\sqrt[5]{243^2}$ **27.** $\frac{1}{9^{-2}}$.

28. $\left(\frac{1}{216}\right)^{-\frac{2}{3}}$. **29.** $\frac{1}{\sqrt{16^{\frac{3}{2}}}}$. **30.** $\sqrt[3]{(16^{-\frac{3}{4}})(125^{-2})}$

Simplify:

31. $16^{\frac{1}{2}}+49^{\frac{1}{2}}$. **32.** $\frac{1}{8^{-\frac{1}{3}}}+\frac{1}{27^{-\frac{1}{3}}}$ **33.** $\frac{(2\cdot5)^3}{10^{-2}}$

34. $\frac{x+x^{-1}}{x-1}$. **35.** $\frac{a^{2m+n}\times a^{3m-6n}}{a^{4m-3n}}$.

5. To prove that $(a^m)^n = a^{mn}$ for all values of m and n.

Case I. When n is a positive integer,

$$(a^m)^n = a^m \times a^m \times a^m \ldots\ldots \text{to } n \text{ factors}$$
$$= a^{m+m+m+\ldots \text{ to } n \text{ terms}} = a^{mn}.$$

Case II. When n is a positive fraction:

Let $n = \frac{p}{q}$, where p and q are positive integers.

Now, $$(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$$
$$= \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}$$
$$= a^{m\frac{p}{q}} = a^{mn}.$$

Case III. When n is any negative quantity:

Let $n = -p$, then p is a positive quantity.

Now $$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p}$$
$$= \frac{1}{a^{mp}} = a^{-mp}$$
$$= a^{mp(-p)} = a^{mn}.$$

Example 4: $(9^{\frac{2}{3}})^{\frac{3}{4}} = 9^{\frac{2}{3}\times\frac{3}{4}} = 9^{\frac{1}{2}} = \sqrt{9} = 3.$

Example 5: $(27^{-\frac{2}{3}})^{\frac{1}{2}} = 27^{-\frac{2}{3}\times\frac{1}{2}} = 27^{-\frac{1}{3}} = \frac{1}{27^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}.$

6. To prove that $a^n b^n = (ab)^n$ for all values of n.

Case I. When n is a positive integer.

$$\begin{aligned} a^n b^n &= (a\times a\times a.....\text{to } n \text{ factors}) \\ &\quad \times (b\times b\times b.....\text{to } n \text{ factors}) \\ &= (ab\times ab\times ab.....\text{to } n \text{ factors}) \\ &= (ab)^n. \end{aligned}$$

Case II. When n is a positive fraction.

Let $n = \frac{p}{q}$, where p and q are both positive integers, so that $p = nq$.

$$\begin{aligned} \text{Now} \quad (ab)^n &= (ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p} \\ &= \sqrt[q]{a^p b^p} = \sqrt[q]{a^{nq} b^{nq}} \\ &= \sqrt[q]{(a^n)^q (b^n)^q} = \sqrt[q]{(a^n b^n)^q} \\ &= (a^n b^n)^{\frac{q}{q}} = a^n b^n. \end{aligned}$$

Case III. When n is any negative quantity.

Let $n = -p$, then p is a positive quantity.

$$\begin{aligned} \text{Now} \quad (ab)^n &= (ab)^{-p} = \frac{1}{(ab)^p} \\ &= \frac{1}{a^p b^p} = a^{-p} b^{-p} = a^n b^n. \end{aligned}$$

Hence generally $(abc.....)^n = a^n \times b^n \times c^n.....$ for all values of n.

Cor. $\frac{a^n}{b^n} = a^n b^{-n} = a^n (b^{-1})^n = (ab^{-1})^n = \left(\frac{a}{b}\right)^n.$

Example 6: $(8^{-2}\times 9^{\frac{3}{4}})^{-\frac{2}{3}} = (8^{-2})^{-\frac{2}{3}} \times (9^{\frac{3}{4}})^{-\frac{2}{3}}$

$$\begin{aligned} &= 8^{\frac{4}{3}} \times 9^{-\frac{1}{2}} = (\sqrt[3]{8})^4 \times \frac{1}{\sqrt{9}} \\ &= 2^4 \times \frac{1}{3} = \frac{16}{3}. \end{aligned}$$

Example 7: $\sqrt[3]{a^{-4}b^2} \times \sqrt{a^3b} \times \sqrt[6]{ab^{-3}}$

$$= a^{-\frac{4}{3}}b^{\frac{2}{3}} \times a^{\frac{3}{2}}b^{\frac{1}{2}} \times a^{\frac{1}{6}}b^{-\frac{1}{2}}$$

$$= a^{-\frac{4}{3}+\frac{3}{2}+\frac{1}{6}} \times b^{\frac{2}{3}+\frac{1}{2}-\frac{1}{2}}$$

$$= a^{\frac{1}{3}} \times b^{\frac{2}{3}}$$

$$= \sqrt[3]{ab^2}.$$

EXERCISE 103.

Simplify :

1. $\left(x^{-\frac{2}{3}}\right)^6$. **2.** $\left(x^{-\frac{3}{4}}y^{\frac{5}{8}}\right)^{\frac{3}{4}}$. **3.** $\left(x^{-\frac{1}{3}}b^{-2}\right)^{-3}$.

4. $\left(x^6y^{\frac{5}{4}}\right)^{-\frac{4}{3}}$. **5.** $\left(\sqrt[4]{m^3n^4}\right)^4$. **6.** $\left(\sqrt[8]{a^{12}b^{-6}}\right)^{-4}$

7. $\sqrt{m^6 \quad \sqrt[4]{m^{-3}}}$ **8.** $\sqrt{x^{-6}y^8} \times \sqrt[4]{x^4y^{-12}}$.

9. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a^{-4}}}}$. **10.** $\sqrt{a^{-4}\sqrt{b^5}} \times \sqrt{a\sqrt[4]{b^3}}$.

11. $(27a^3 \div 64b^{-3})$ **12.** $(125a^3 \div 27b^{-3})^{-\frac{2}{3}}$.

13. $\{(a^{\frac{1}{3}}b^{-\frac{2}{3}})^{-2} \times (a^{-\frac{1}{3}}b^{\frac{2}{3}})^{-7}\}^{\frac{1}{4}}$

14. $\sqrt{xy^{-2}z^3} \div \sqrt[3]{(x^3y^2z^{-3})^{-1}}$

15. $\left\{(x^{a+b+c} \times x^{a-b+c})^b\right\}^c$.

16. $\left(\frac{x^m}{x^n}\right)^{m+n} \times \left(\frac{x^n}{x^l}\right)^{n+l} \times \left(\frac{x^l}{x^m}\right)^{l+m}$

17. Which is greater $(2^3)^2$ or 2^{3^2}, and by how much ?

7. Algebraical operations involving fractional and negative indices.

As in the processes of multiplication, division, factorisation and square root, it is necessary to arrange the terms in ascending or descending powers of some common letter. The student should have sufficient practice in such arrangements while dealing with expressions involving fractional and negative indices.

Example 9: Arrange the following expressions in ascending powers of x:

(i) $x+x^0+x^{-1}+x^{-2}+x^2$.

(ii) $3x^{\frac{1}{2}}+x^{-1}+4x^{-\frac{1}{2}}+x^{\frac{1}{4}}+x^0$.

(iii) $x^{\frac{1}{8}}-2x^{-\frac{3}{4}}+3x^{-\frac{1}{4}}+1-7x^2+5x$.

Sol. (i) Since $0>-1$ and $-1>-2$, the proper arrangement of the terms in the first expression would be

$$x^{-2}+x^{-1}+x^0+x+x^2.$$

(ii) Since $0>-\frac{1}{2}$ and $-\frac{1}{2}>-1$, the proper arrangement of the terms in the second expression would be

$$x^{-1}+4x^{-\frac{1}{2}}+x^0+x^{\frac{1}{4}}+3x^{\frac{1}{2}}.$$

(iii) Since $1=x^0$ and $0>-\frac{1}{4}$ and $-\frac{1}{4}>-\frac{3}{4}$, the proper arrangement of the terms in the third expression would be

$$-2x^{-\frac{3}{4}}+3x^{-\frac{1}{4}}+1+x^{\frac{1}{8}}+5x-7x^2.$$

Example 9: Multiply $ab^{-1}+1+a^2b^{-2}$ by $a^2b^{-2}+1-ab^{-1}$.

Sol. Arranging both the expressions in ascending powers of a and proceeding as usual, we have

$$\begin{array}{l}
1+ab^{-1}+a^2b^{-2} \\
1-ab^{-1}+a^2b^{-2} \\
\hline
1+ab^{-1}+a^2b^{-2} \\
\quad -ab^{-1}-a^2b^{-2}-a^3b^{-3} \\
\qquad\quad +a^2b^{-2}+a^3b^{-3}+a^4b^{-4} \\
\hline
1\qquad\quad +a^2b^{-2}\qquad\quad +a^4b^{-4}.
\end{array}$$

Example 10: Divide $11x^{\frac{1}{5}}-6+x^{\frac{3}{5}}-6x^{\frac{2}{5}}$ by $x^{\frac{1}{5}}-3$.

Sol. Arranging the dividend and divisor according to descending powers of x and proceeding as usual, we have

$$\begin{array}{l}
x^{\frac{1}{5}}-3)x^{\frac{3}{5}}-6x^{\frac{2}{5}}+11x^{\frac{1}{5}}-6(x^{\frac{2}{5}}-3x^{\frac{1}{5}}+2 \\
\quad\; x^{\frac{3}{5}}-3x^{\frac{2}{5}} \\
\quad\; \underline{\qquad\qquad} \\
\qquad\quad -3x^{\frac{2}{5}}+11x^{\frac{1}{5}} \\
\qquad\quad -3x^{\frac{2}{5}}+9x^{\frac{1}{5}} \\
\qquad\quad \underline{\qquad\qquad\qquad} \\
\qquad\qquad\qquad\quad 2x^{\frac{1}{5}}-6 \\
\qquad\qquad\qquad\quad 2x^{\frac{1}{5}}-6 \\
\qquad\qquad\qquad\quad \underline{\qquad\qquad}
\end{array}$$

EXERCISE 104.

1. Arrange the following expressions according to ascending powers of x:

(i) $4x^2+1+3x^{-1}+5x^{-2}-7x$.

(ii) $3x+6x^{-3}+5x^3+4x^{-2}+7x^{-1}$.

(iii) $x+4+6x^{-\frac{1}{2n}}+5x^{\frac{1}{n}}+3x^{\frac{1}{2n}}-x^{-\frac{1}{n}}$, where n is a positive integer.

2. Arrange the following expressions according to descending powers of y:

(i) $2+x^{-2}y^2+x^2y^{-2}$.

(ii) $x^{\frac{2}{3}}y^{-\frac{2}{3}}-1+x^{-\frac{2}{3}}y^{\frac{2}{3}}$.

(iii) $3y^{-3}-xy^{-2}+1-5x^2y^{-1}+7x^3$.

Multiply:

3. $a^{\frac{2}{3}}-1+a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}+a^{-\frac{1}{3}}$.

4. $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

5. $a+3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$ by $a-3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$.

6. $a^{\frac{2}{3}}+2a^{\frac{1}{2}}+3a^{\frac{1}{3}}+2a^{\frac{1}{6}}+1$ by $1+a^{\frac{1}{3}}-2a^{\frac{1}{6}}$.

7. $x^{-1}+y^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ by $x^{-1}-x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

8. $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

Divide:

9. $a^{\frac{2}{3}}b^{-\frac{2}{3}}+1-a^{\frac{2}{3}}-b^{-\frac{2}{3}}$ by $1+b^{-\frac{1}{3}}+a^{\frac{1}{3}}+a^{\frac{1}{3}}b^{-\frac{1}{3}}$.

10. $x^{\frac{4}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}$ by $x^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$.

11. $x^2y^{-2}+x^{-2}y^2+2$ by $x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}-1$.

12. $1-x^{-\frac{1}{2}}+2x^{\frac{3}{4}}+x^{\frac{3}{2}}$ by $1+x^{-\frac{1}{4}}+x^{\frac{3}{4}}$.

Example 11: Find the square of $a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}}$.

Sol. $(a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}})^2=(a^{\frac{1}{5}})^2+(-b^{\frac{1}{7}})^2+(c^{\frac{1}{3}})^2+2(a^{\frac{1}{5}})(-b^{\frac{1}{7}})+2(a^{\frac{1}{5}})(c^{\frac{1}{3}})+2(-b^{\frac{1}{7}})(c^{\frac{1}{3}})=a^{\frac{2}{5}}+b^{\frac{2}{7}}+c^{\frac{2}{3}}-2a^{\frac{1}{5}}b^{\frac{1}{7}}+2a^{\frac{1}{5}}c^{\frac{1}{3}}-2b^{\frac{1}{7}}c^{\frac{1}{3}}$

Example 12: Resolve into factors $a^{\frac{3}{2}}-b^{\frac{3}{2}}$.

Sol. $a^{\frac{3}{2}}-b^{\frac{3}{2}}=(a^{\frac{1}{2}})^3-(b^{\frac{1}{2}})^3=(a^{\frac{1}{2}}-b^{\frac{1}{2}})\{(a^{\frac{1}{2}})^2+(a^{\frac{1}{2}})(b^{\frac{1}{2}})+(b^{\frac{1}{2}})^2\}$
$=(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b)$.

Example 13: Simplify (i) $(x^{2^{n-1}})^2$, (ii) $(x^{3^{n-1}})^9$.

Sol. (i) $(x^{2^{n-1}})^2=[x^{(2^{n-1})}]^2=x^{2\cdot 2^{n-1}}=x^{2^{n-1+1}}=x^{2^n}$.

(ii) $(x^{3^{n-1}})^9=[x^{(3^{n-1})}]^9=x^{9\cdot 3^{n-1}}=x^{3^2\cdot 3^{n-1}}$
$=x^{3^{n1-+2}}=x^{3^{n+1}}$.

EXERCISE 105.

Find the square of:

1. $a^{\frac{1}{3}}+b^{\frac{1}{3}}$. **2.** $a+a^{-1}$. **3.** $a+a^{\frac{1}{2}}-a^{\frac{1}{3}}$.

4. $a^{\frac{1}{3}}-2a^{\frac{1}{2}}+a^{\frac{5}{6}}$. **5.** $1+a^{-\frac{1}{3}}+a^{\frac{1}{3}}$. **6.** $a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{-\frac{1}{4}}+b^{\frac{1}{2}}$.

Multiply by formulae:

7. $a^{\frac{2}{3}}-1+a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}+a^{-\frac{1}{3}}$.

8. $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

9. $x^{-1}+y^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ by $x^{-1}-x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

Divide by formulae:

10. $x^{-1}+y^{-1}$ by $x^{-\frac{1}{3}}+y^{-\frac{1}{3}}$.

11. $8a^{\frac{3}{2}}+b^{-\frac{3}{2}}-c+6a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{3}}$ by $2a^{\frac{1}{2}}+b^{-\frac{1}{2}}-c^{\frac{1}{3}}$.

12. Multiply $a^{2n}+a^nb^{-n}+b^{-2n}$ by a^n-b^{-n}, using a formula.

13. Divide $a^{\frac{3n}{2}}-b^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}}-b^{\frac{n}{2}}$ by factors.

14. Find the L.C.M. of $a^{\frac{3}{2}}+b^{\frac{3}{2}}$, $a^{\frac{3}{2}}-b^{\frac{3}{2}}$, $a^{\frac{1}{2}}+b^{\frac{1}{2}}$, $a-b$ by factors.

Simplify :

15. $\left(x^{2^{n+1}}\right)^4$.

16. $\left(x^{3^{n+1}}\right)^9$.

17. $27^n \div 3^{3n+1}$.

18. $\left[(a^m)m - \frac{1}{m}\right]^{\frac{1}{m+1}}$.

19. $\left(x^{2^{n-1}} + a^{2^{n-1}}\right)\left(x^{2^{n-1}} + a^{2^{n-1}}\right)$.

20. Compare $\left[(2^2)^2\right]^2$ and $2^{2^{2^2}}$.

8. Simplification of expressions with any Indices.

Example 14: Simplify $\dfrac{2^{m+1}.3^{2m-n}.5^{m+n}.6^n.}{6^m.10^{n+2}.15^m}$

Sol. The expression $= \dfrac{2^{m+1}.3^{2m-n}.5^{m+n}.(3^n.2^n)}{(3^m.2^m).(5^{n+2}.2^{n+2}).5^m 3^m}$

$= \dfrac{2^{m+n+1}.3^{2m+n-n}.5^{m+n}}{2^{m+n+2}.3^{m+m}.5^{m+n+2}}$

$= \left(\dfrac{2^{m+n+1}}{2^{m+n+2}}\right) \cdot \left(\dfrac{3^{2m}}{3^{2m}}\right) \cdot \left(\dfrac{5^{m+n}}{5^{m+n+2}}\right)$

$= \dfrac{1}{2}.1.\dfrac{1}{5^2} = \dfrac{1}{2} \cdot \dfrac{1}{25} = \dfrac{1}{50}$.

Example 15: $\left(\dfrac{a-b}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} - \dfrac{a^{\frac{3}{2}} - b^{\frac{3}{2}}}{a-b}\right) - 1$.

Sol. The expression $= \left[\dfrac{(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} - \dfrac{(a^{\frac{1}{2}} - b^{\frac{1}{2}})(a + a^{\frac{1}{2}} b^{\frac{1}{2}} + b)}{(a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}})}\right]^{-1}$

$= \left[(a^{\frac{1}{2}} + b^{\frac{1}{2}}) - \dfrac{a + a^{\frac{1}{2}} b^{\frac{1}{2}} + b)}{a^{\frac{1}{2}} + b^{\frac{1}{2}}}\right]^{-1}$

$= \left[\dfrac{(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - (a + a^{\frac{1}{2}} b^{\frac{1}{2}} + b)}{a^{\frac{1}{2}} + b^{\frac{1}{2}}}\right]^{-1}$

$$= \left[\frac{a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b-a-a^{\frac{1}{2}}b^{\frac{1}{2}}-b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$$

$$= \left[\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1} = \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}} + \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= \frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}.$$

EXERCISE 106.

Simplify :

1. $\dfrac{2^n.6^{m+1}.10^{m-n}.15^{m+n-2}}{4^m.3^{2m+n}.25^{m-1}}$.

2. $\dfrac{3^{n+4}-6.3^{n+1}}{3^{n+2}}$.

3. $\dfrac{2^n.(2^{n-1})^n}{2^{n+1}.2^{n-1}} \times \dfrac{2.2^n}{(2^n)^n}$.

4. $\dfrac{3{\cdot}4 \times 10^4 \times 5{\cdot}7 \times 10^{-6} \times 4{\cdot}4 \times 10^{-3}}{1{\cdot}87 \times 10^{-4} \times 9{\cdot}5 \times 10^3 \times 3 \times 10^{-4}}$.

5. $\dfrac{6^{-n} \times 36^{2n-2}}{6^{3n-2} \times 12^{-1}}$.

6. $\dfrac{3^{n+1}}{(3^n)^{n+1}} \div \dfrac{9^{n+1}}{(3^{n+1})^{n-1}}$.

Prove that

7. $\dfrac{2^n+2^{n-1}}{2^{n+1}-2^n} = \dfrac{3}{2}$.

8. $\dfrac{5^3.2^{\frac{1}{4}}.10^{-\frac{1}{4}}}{15^{\frac{3}{4}}.6^{-\frac{3}{4}}.4^{\frac{3}{8}}} = 25.$

Simplify :

9. $\dfrac{\left(p+\dfrac{1}{q}\right)^m\left(p-\dfrac{1}{q}\right)^n}{\left(q+\dfrac{1}{p}\right)^m\left(q-\dfrac{1}{p}\right)^n}$.

10. $\dfrac{x-x^{-1}}{x-1}$.

11. $\dfrac{x^3-y^{-3}}{x-y^{-1}}$.

12. $\dfrac{x^2}{(1-x)^n} + \dfrac{2}{(1-x)^{n-1}} - \dfrac{1}{(1-x)^{n-2}}$.

13. $\left(\frac{x^a}{x^b}\right)^{a+b} \cdot \left(\frac{x^b}{x^c}\right)^{b+c} \cdot \left(\frac{x^c}{x^a}\right)^{c+a}$.

14. $(2x^{\frac{1}{2}}+x^{-\frac{1}{2}})^2-(2x^{\frac{1}{2}}-x^{-\frac{1}{2}})^2$.

15. $\frac{x^2+y^2-x^{-2}-y^{-2}}{x^2y^2-x^{-2}y^{-2}}+\frac{(x-x^{-1})(y-y^{-1})}{xy+x^{-1}y^{-1}}$.

16. $a^{-2}(bc^{-1}-b^{-1}c)+b^{-2}(ca^{-1}-c^{-1}a)+c^{-2}(ab^{-1}-a^{-1}b)$.

Prove that

17. $\frac{y^{-1}}{x^{-1}+y^{-1}}+\frac{y^{-1}}{x^{-1}-y^{-1}}=\frac{2xy}{y^2-x^2}$.

18. $\left(1-\frac{1}{n}\right)^{-n}=\left(1+\frac{1}{n-1}\right)^n$.

19. Simplify $\frac{1}{1+a^{y-x}+a^{z-x}}+\frac{1}{1+a^{x-y}+a^{z-y}}+\frac{1}{1+a^{y-z}+a^{x-z}}$.

[*Hint*: $\frac{1}{1+a^{y-x}+a^{z-x}}=\frac{a^x}{a^x+a^y+a^z}$.]

20. If $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, prove that $2x^3+6x=3$.

[*Hint*: *Cube both sides* of $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, *and put* x for $(2^{\frac{1}{3}}-2^{-\frac{1}{3}})$.]

21. If $a^l=b^m=c^n$, and $b^2=ac$, then $\frac{1}{l}+\frac{1}{n}=\frac{2}{m}$.

22. If $a^x=b^y$ and $b^x=a^y$, then $a=b$.

23. If $a=b$, $b^y=c$, $c^z=a$, then $xyz=1$.

24. If $x=y^x$, prove that $\left(\frac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$

[*Hint. Proceed analytically*: $\left(\frac{x}{y}\right)^{\frac{x}{y}}$ will be $=x^{\frac{x}{y}-1}$, *if* $\frac{(x^{\frac{x}{y}})}{(y^{\frac{x}{y}})}=\frac{(x^{\frac{x}{y}})}{x}$ or *if* $y^{\frac{x}{y}}=x$ or *if* $y^x=x^y$.]

SELECTED QUESTIONS—G

1. Simplify $a^{\frac{2}{3}} \times a^{-\frac{3}{4}} \times (a^2)^{-\frac{1}{6}} \times \frac{1}{\left(a^{-\frac{1}{12}}\right)^5}$.

2. Find the value of $\frac{(27)^{\frac{2n}{3}} \times 8^{\frac{-n}{6}}}{(18)^{\frac{-n}{2}}}$.

3. Express in the simplest form

$$\frac{\left(9^n \times 3^2 \times \frac{1}{3^{-n}}\right) - (27)^n}{3^{3n} \times 9}.$$

Simplify

4. $\sqrt{(x^{-2}y^3)^{-5}} \div \frac{(x^{\frac{1}{3}}y^2)^{-1}}{xy}$.

5. $\left(\frac{x^p}{x^q}\right)^{p+q} \div \left(\frac{x^{p+q}}{x^{p-q}}\right)^{\frac{p^2}{q}}$.

6. $a^{\frac{1}{4}} b^{\frac{2}{3}} \left(\frac{4a^{\frac{1}{4}}}{9b^{-1}}\right)^{\frac{1}{2}} \div \frac{a^{\frac{5}{2}\frac{5}{4}}}{b^{\frac{5}{6}}}$.

7. $\left(\frac{x^{a+b}}{x^{2b}}\right) \times \left(\frac{x^{b+c}}{x^{2c}}\right) \times \left(\frac{x^{c+a}}{x^{2a}}\right)$.

8. $\left(\frac{a^x}{a^y}\right)^{x+y} \times \left(\frac{a^y}{a^z}\right)^{y+z} \div \left(a^x \times a^z\right)^{x-z}$.

9. $\left(\frac{x^m}{x^n}\right)^{m+n-l} \times \left(\frac{x^n}{x^l}\right)^{n+l-m} \times \left(\frac{x^l}{x^m}\right)^{l+m-n}$.

10. $\left(\frac{x^{a^2+b^2}}{x^{ab}}\right)^{a+b} \times \left(\frac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \times \left(\frac{x^{c^2+a^2}}{x^{ca}}\right)^{c+a}$.

11. $\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$.

12. $\left(\frac{x^{ab}}{x^{a^2+b^2}}\right)^{a+b} \times \left(\frac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \times \left(\frac{x^{ca}}{x^{c^2+a^2}}\right)^{c+a}$.

CHAPTER XIII

RATIO

1. If A and B be two quantities of the *same kind* which when measured with one and the same unit are represented by the numbers a and b, the **ratio** of A to B is the quotient of a by b.

This ratio is written as a/b, $a \div b$ or $a : b$ and is read as 'the ratio of a to b.'

Thus, for example, if A and B be lengths of $3\frac{1}{4}$ metres and $5\frac{1}{2}$ metres respectively, the ratio of A to B is $3\frac{1}{4}/5\frac{1}{2}$ which when simplified becomes $\frac{13}{22}$ or 13 : 22.

Similarly, the ratio of Rs. 3·25 to Rs. 5·50 is 13 : 22.

In the first example the quantities compared are *two lengths* and in the second, *two sums of money*; the ratio in both cases is the same.

Thus, a ratio is simply an abstract number and is independent of the concrete units employed in the quantities compared, provided they are of the same kind.

In the ratio $a : b$, a and b are its **terms**. The first term a is called the **antecedent** and the second term b is called the **consequent.**

The ratio $b : a$ is known as the **inverse ratio** of $a : b$.

As a ratio is represented by a vulgar fraction, *its value remains unaltered if its terms are multiplied or divided by the same number*; for instance, $\frac{a}{b} = \frac{ma}{mb}$ and $\frac{na}{nb} = \frac{a}{b}$.

A ratio is said to be a ratio of *greater inequality*, of *equality* or of *less inequality*, according as the antecedent is greater than, equal to, or less than the consequent.

Thus 4 : 3 or $\frac{4}{3}$ is a ratio of greater inequality
3 : 3 or $\frac{3}{3}$,, ,, ,, equality,
3 : 4 or $\frac{3}{4}$,, ,, ,, less inequality.

Two or more ratios are said to be **compounded,** when they are multiplied together. Thus the ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

When the ratio $a : b$ is compounded with itself, the resulting ratio $a^2 : b^2$ is called the **duplicate** ratio of $a : b$.

Similarly, $a^3 : b^3$ is called the **triplicate** ratio of $a : b$. When $a : b$ is the original ratio, $\sqrt{a} : \sqrt{b}$ is called the **sub-duplicate** ratio and $\sqrt[3]{a} : \sqrt[3]{b}$ is called the **sub-triplicate** ratio of $a : b$.

2. Theorem: *A ratio of greater inequality is diminished and a ratio of less inequality is increased, by adding the same positive quantity to both the terms.*

Let $\frac{a}{b}$ be a ratio, a and b being positive.

Let x, a positive quantity, be added to both its terms.

The new ratio thus formed $= \frac{a+x}{b+x}$.

$\frac{a}{b} > = < \frac{a+x}{b+x}$ according as $\left(\frac{a}{b} - \frac{a+x}{b+x}\right)$ is positive, zero or negative.

Now $\frac{a}{b} - \frac{a+x}{b+x} = \frac{ax-bx}{b(b+x)} = \frac{x(a-b)}{b(b+x)}$.

Since x, b, $(b+x)$ are all positive, the above is positive, zero or negative according as $a-b$ is positive, zero or negative; or according as $a > = < b$.

$\therefore \quad \frac{a}{b} > = < \frac{a+x}{b+x}$ according as $a > = < b$.

It is left to the student to prove the corresponding theorem:

A ratio of greater inequality is increased and a ratio of less inequality is diminished, by subtracting the same positive quantity from both the terms.

Example 1: Find the ratio compounded of the three ratios $4a : 5b$, $b^2 : 2ac$ and $3c^2 : 2ab$.

Sol. The given ratios are $\frac{4a}{5b}$, $\frac{b^2}{2ac}$, $\frac{3c^2}{2ab}$.

The required ratio $= \frac{4a \times b^2 \times 3c^2}{5b \times 2ac \times 2ab} = 3c : 5a$.

Example 2: If $4+x : 5+x$ be the duplicate ratio of 2 : 3, find the value of x.

Sol.
$$\frac{4+x}{5+x} = \frac{2\times 2}{3\times 3} = \frac{4}{9}$$
$$\therefore \quad 36+9x = 20+4x$$
$$\therefore \quad 5x = -16$$
$$\therefore \quad x = -3\tfrac{1}{5}.$$

Example 3: What must be added to the terms of the ratio 4 : 7 to make it equal to 5 : 6?

Sol. Let x be the required number.

Then by the question, $\frac{4+x}{7+x} = \frac{5}{6}$

or $6(4+x) = 5(7+x)$

$\therefore \quad 24+6x = 35+5x$

$\therefore \quad x = 11.$

$\therefore$ 11 should be added.

Example 4: If $x : y = 4 : 7$, find the value of $\frac{2x+y}{x+3y}$

Sol. Dividing the numerator and the denominator by y we have
$$\frac{2x+y}{x+3y} = \frac{2\left(\frac{x}{y}\right)+1}{\frac{x}{y}+3}$$

Substituting the value of $\frac{x}{y}$, we get
$$\frac{2x+y}{x+3y} = \frac{2\times\frac{4}{7}+1}{\frac{4}{7}+3} = \frac{1\frac{8}{7}}{3\frac{4}{7}} = \frac{15}{25} = \frac{3}{5}$$

Example 5: If $3x+4y = 5x-2y$, find the ratio $x : y$.

Sol. Since $3x+4y = 5x-2y$

$\therefore \quad 2x = 6y$

or $x = 3y$

$\therefore \quad \frac{x}{y} = 3 = \frac{3}{1}$

$\therefore \quad x : y = 3 : 1.$

EXERCISE 107

1. Write down the ratio compounded of the ratios of:
(i) $3 : 4$, $5 : 6$, $2 : 3$;
(ii) $a : b$, $b : c$, $c : d$;
(iii) $(x+1) : (x-1)$, $(x^2-x+1) : (x^2+x+1)$.

2. Write down the duplicate ratio of:
(i) $3 : 7$, (ii) $ab : cd$, (iii) $(a+x) : (a-x)$,
(iv) $(x+1) : (x-1)$.

3. Write down the triplicate ratio of:
(i) $4 : 5$, (ii) $mn : pq$, (iii) $(a-b) : (a+b)$, (iv) $x^2 : 2^x$.

4. Write down the sub-duplicate ratio of:
(i) $16 : 25$, (ii) $x^6 : y^8$, (iii) $x^2+4ax+4a^2 : x^2-4ax+4a^2$.

5. Write down the sub-triplicate ratio of:
(i) $64 : 27$, (ii) $x^9 : y^6$, (iii) $125a^6b^{12} : 343x^3y^6$.

6. Compare the ratios:
(i) $2 : 3$ and $5 : 6$, (ii) $14 : 11$ and $8 : 5$.

7. If $4x+3 : 7x-1$ be the duplicate ratio of $3 : 4$, find the value of x.

8. If $m : n$ be the duplicate ratio of $2m-x : n-2x$, shew that $x^2 = mn$.

9. If $(a+x) : (a-x)$ be the duplicate ratio of $(a+b) : (a-b)$, then $(x-b) : (a-x) = b(a+b) : a(a-b)$.

10. If $p : q$ be the triplicate ratio of $p+x : q+x$, shew that $x^3-3pqx-pq(p+q)=0$.

11. What must be added to the terms of $5 : 9$ to make it equal to $7 : 8$?

12. What must be added to the terms of $a : b$ so that the resulting ratio may be in the duplicate ratio of $a : b$?

13. What must be added to the terms of $a : b$ so that the resulting ratio may be in the triplicate ratio of $a : b$?

14. Two numbers are in the ratio of $7 : 8$ and if four be added to each, they will be in the ratio of $8 : 9$. Find the numbers.

[*Hint*: Let $7x$ and $8x$ be the required numbers.]

15. Two numbers are in the ratio of 3 : 4 and if five be subtracted from each, they will be in the ratio of 2 : 3. Find the numbers.

16. Find two numbers in the ratio of 9 : 7 whose difference is 11.

17. Two numbers are in the ratio of 3 : 5 and the sum of their squares is 1666; find them.

18. If x be added to the antecedent of the ratio $a : b$, what quantity must be added to the consequent so that the value of the ratio may remain unaltered?

19. If $x : y = 5 : 6$, find the value of $\frac{5x-3y}{6x+3y}$.

20. If $x : y = 3 : 2$, find the value of $\frac{4x^2-xy+2y^2}{3x^2+xy-y^2}$.

21. If $\frac{7x-4y}{3x+y} = \frac{5}{13}$, find the value of $\frac{x}{y}$.

22. If $\frac{2x^2-3y^2}{x^2+y^2} = \frac{2}{41}$, find $\frac{x}{y}$.

23. If $(8x-2y)^2 = (3x+4y)^2$, find $x : y$.

24. Which is greater, $\frac{y+x}{y-x}$ or $\frac{y^2+x^2}{y^2-x^2}$, when x and y are both positive and $y>x$?

25. Which is greater, $\frac{x^3+y^3}{x^2+y^2}$ or $\frac{x^2+y^2}{x+y}$ when x and y are both positive ?

26. Find the least integer which must be added to each term of the ratio 3 : 5 so that the resulting ratio may be greater than 2 : 3.

27. Find two numbers whose difference is 9 and whose sub-duplicate ratio is 1 : 2.

28. The ages of two persons are in the ratio of 5 : 4 ; 9 years ago they were in the ratio 4 : 3. Find their ages.

29. The bases of two triangles are in the ratio of 3 : 4 and their heights in the ratio of 6 : 5. Find the ratio of their areas.

30. The soldiers in two armies when they met in a battle were in the ratio of 10 : 3. Their respective losses were as 20 : 3 and the survivors as 40 : 13 ; if the number of survivors in the larger army be 24,000, find the original number of soldiers in each army.

3. **Meaning of** $\frac{0}{n}, \quad \frac{m}{\infty}, \quad \frac{m}{0}, \quad \frac{0}{0}$.

(i) The fraction $\frac{m}{n}$ stands for the share of each person when a quantity m is divided equally among n persons.

Similarly, $\frac{0}{n}$ stands for the share of each person when 0 is divided among n persons. As in this case the quantity to be divided is zero, the share of each must be zero.

Or $\qquad \frac{0}{n} = 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad$ (i)

Cor. When $n = 1$, $\frac{0}{1} = 0$.

(ii) If the numerator of a fraction is constant (say m), its value decreases as its denominator increases, *e.g.*, $\frac{m}{3} < \frac{m}{2}, \frac{m}{4} < \frac{m}{3}, \frac{m}{5} < \frac{m}{4}$ ...and when the denominator is *very very large* the value of the fraction is *very very small*, and when the denominator is *infinitely large* its value is *infinitely small*

or $\qquad \frac{m}{\infty} = 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad$ (ii)

Cor. When $m = 1$, $\frac{1}{\infty} = 0$.

(iii) If the numerator of a fraction is constant (say m), its value increases as its denominator decreases, *e.g.*, $\frac{m}{\frac{1}{3}} > \frac{m}{\frac{1}{2}}$ and $\frac{m}{\frac{1}{4}} > \frac{m}{\frac{1}{3}}$ ...and when the denominator is *very very small*, the value of the fraction is *very very large*, and when the denominator is *infinitely small*, its value is *infinitely large*, or

$$\frac{m}{\textit{very very small quantity}} = \infty \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Cor. When $m = 1$, $\frac{1}{0} = \infty$.

(iv) $\frac{x^2-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^2-1}{x-1}=x+1=1+1=2$ when $x=1$ (By identity).

$\therefore \quad \frac{0}{0}=2.$

Likewise, by choosing expressions like $\frac{x^3-1}{x-1}$, $\frac{x^2-1}{x^3-1}$ and putting $x=1$, we find that $\frac{0}{0}=2, \frac{2}{3}$ and so on ; *i.e.* $\frac{0}{0}$ can have any value ; it is **indeterminate.**

Example 6: Find the value of $\frac{x^2+ax-2a^2}{x^2+4ax-5a^2}$ when $x=a$.

Sol. If we substitute a for x in the given expression, it is reduced to the indeterminate form $\frac{0}{0}$.

In order to avoid the indeterminate form, we proceed as follows :

The expn. $=\frac{(x+2a)(x-a)}{(x+5a)(x-a)}\equiv\frac{x+2a}{x+5a}=\frac{a+2a}{a+5a}=\frac{3a}{6a}=\frac{1}{2}$.

EXERCISE 108.

Find the value of:

1. $\frac{2x^2+5x-7}{3x^2-8x+5}$ when $x=1$.

2. $\frac{x^2+3x-40}{x^2+4x-32}$ when $x=-8$.

3. $\frac{x^3+4x^2-8x+3}{2x^3-5x^2+2x+1}$ when $x=1$.

4. $\frac{2x^3-7x^2a+6a^2x-a^3}{3x^4-3x^3a+4a^3x-4a^4}$ when $x=a$.

5. $\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-\ 4a^2x-48a^3}$ when $x=2a$.

6. $\frac{x^3-5ax^2+3a^2x+9a^3}{x^3-7ax^2+15a^2x-9a^3}$ when $x=3a$.

CHAPTER XIV

PROPORTION

1. If $\frac{a}{b} = \frac{c}{d}$, the numbers a, b, c, d are said to be **in proportion** (or **proportionals**) and it is read as 'a to b equals c to d' or 'a is to b as c is to d'.

Of this proportion a, d are called the **extremes,** b, c the **means** and d the **fourth proportional** to a, b and c.

Two quantities are said to be *proportional* to two others when the ratio of the first two is equal to the ratio of the second two, and two quantities are said to be *inversely proportional* to two others when the *inverse* ratio of the first two is equal to the ratio of the other two.

If a, b, c, d, e...be such that $a : b = b : c = c : d = d : e = \ldots$ a, b, c, d, e are said to be in **continued proportion.**

When a, b c, are in continued proportion, b is said to be the **mean proportional** between a and c, and c is said to be the **third proportional** to a and b.

2. If $a : b = c : d$, then $ad = bc$.

Since $\frac{a}{b} = \frac{c}{d}$, multiplying both sides by bd, we get

$$ad = bc.$$

Thus, if *four quantities are proportional, the product of the extremes is equal to the product of the means.*

Conversely, if $ad = bc$, we can easily prove by dividing both sides by bd that $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$.

Cor. If $a : b = b : c$, then $ac = b^2$, *i.e., if three quantities are in continued proportion, the product of the extremes is equal to the square of the middle one.*

3. If $a : b = b : c$, then $a : c = a^2 : b^2$.

For, $\frac{a}{b} = \frac{b}{c}$. $\therefore$ $\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$.

$\therefore$ $\frac{a}{c} = \frac{a^2}{b^2}$, or $a : c = a^2 : b^2$.

Thus, if *three quantities are in continued proportion the first is to the third in the duplicate ratio of the first to the second.*

4. If $a : b = b : c = c : d$, then $a : d = a^3 : b^3$.

For, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$. $\therefore \quad \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$.

$\therefore \quad \frac{a}{d} = \frac{a^3}{b^3}$, or $a : d = a^3 : b^3$.

Thus, if *four quantities are in continued proportion the first is to the fourth in the triplicate ratio of the first to the second.*

Example 1: Find the fourth proportional to 2, 3, 4.

Sol. Let x be the fourth proportional required.

Then $\frac{2}{3} = \frac{4}{x}$ or $2x = 12$ *i.e.* $x = 6$

$\therefore$ the 4th proportional is 6.

Example 2: Find the mean proportional to 9, 25.

Sol. Let x be the mean proportional required.

Then $\frac{9}{x} = \frac{x}{25}$ $\therefore \quad x^2 = 9 \times 25,$ $\therefore \quad x = 15.$

$\therefore$ the mean proportional is 15.

Example 3: Find the third proportional to 2, 8.

Sol. Let x be the third proportional required.

Then $\frac{2}{8} = \frac{8}{x},$ $\therefore \quad 2x = 64,$ $\therefore \quad x = 32.$

$\therefore$ the 3rd proportional is 32.

EXERCISE 109.

Find the fourth proportional to:

1. 8, 12, 14. **2.** 9, 12, 15. **3.** ·0024, 2·4, ·04. **4.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

5. $3a$, $6a^2$, $2ab^2$.

Find the mean proportional to:

6. 9, 16,. **7.** 49, 25. **8.** 9, 64. **9.** 121, 144. **10.** $4\frac{1}{2}$, 2.

11. x^2, y^2. **12.** $(a+b)\ (a-b)^3$ and $(a+b)^3\ (a-b)$.

Find the third proportional to :

13. 18, 12. **14.** 20, 16. **15.** 15, 6. **16.** x^2-y^2, $(x+y)^2$.

17. If $1+x$, $3+x$, $6+x$ are in continued proportion, find x.

Example 4: What must be added to each of the numbers 7, 11 and 19 so that the resulting numbers may be in continued proportion ?

Sol. Let x be the required number.

Then $$\frac{7+x}{11+x} = \frac{11+x}{19+x}.$$

Multiplying cross-wise, we have

$$(7+x)(19+x) = (11+x)^2$$
$$\therefore \quad 133+26x+x^2 = 121+22x+x^2$$
$$\therefore \quad 4x = -12, \text{ or } x = -3.$$
$$\therefore \quad \text{The number to be added is } -3.$$

Example 5: A cask is filled with wine and water in the ratio of 4 : 3. If 12 litres of the mixture be drawn off and the cask be filled with water, their ratio becomes 3 : 4. How many litres can the cask hold ?

Sol. Let x be the required number of litres.

When 12 litres are drawn off, the number of litres in it $= x-12$. Out of every 7 litres of mixture, 4 are wine and 3 are water, or $\frac{4}{7}$ of the mixture is wine and $\frac{3}{7}$ is water.

Hence $\frac{4}{7}(x-12)$ litres of wine are left and $\frac{3}{7}(x-12)$ litres are water.

When the cask is again filled up with water, the quantity of water $= \frac{3}{7}(x-12)+12$.

By the question, we have

$$\frac{\frac{4}{7}(x-12)}{\frac{3}{7}(x-12)+12} = \frac{3}{4}.$$

Multiplying cross-wise, we have

$$16(x-12) = 9(x-12)+252$$
$$\therefore \quad 7x = 336$$
$$\therefore \quad x = 48.$$
$$\therefore \quad \text{the cask can hold 48 litres.}$$

EXERCISE 110.

1. What must be added to each of the numbers 3, 7, 13 and 22 so that the results may be in proportion ?

2. What must be subtracted from each of the numbers 27, 41, 30 and 46 so that the results may be in proportion ?

3. Shew that the fourth proportional to $x^2+7x+12$, $5(x+3)$ and $6(x+4)$ is independent of x.

4. Find two numbers such that their mean proportional is 12 and the third proportional is 324.

5. The mean proportional between 45 and a certain number is three times the mean proportional between 5 and 22. Find the number.

6. If y is the mean proportional between x and z, shew that $xy+yz$ is the mean proportional between x^2+y^2 and y^2+z^2.

7. Two numbers consisting of the same two digits are in the ratio of 4 : 7. Find the numbers.

8. A certain kind of brass is made up of copper, zinc, lead and tin. The ratio of copper to zinc is 1 : 2, that of zinc to lead is 3 : 4 and that of lead to tin is 4 : 5. Find the quantity of each metal in 540 kg of brass.

9. Two vessels contain a mixture of wine and water in the ratios of 8 : 5 and 4 : 1. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 3 : 1 ?

10. Two vessels contain mixtures of wine and water in the ratios of 7 : 3 and 3 : 2. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 9 : 5 ?

5. Certain Useful Propositions.

When $a : b = c : d$, the following relations always hold good and may be used when occasions demand.

Given $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$.

(i) $\because \quad \frac{a}{b} = \frac{c}{d}$, their reciprocals are equal.

$\therefore \quad \frac{b}{a} = \frac{d}{c} \ldots$ This is known as **Invertendo.**

(ii) $\because \quad \frac{a}{b} = \frac{c}{d}, \; ad = bc$

Dividing by cd, $\frac{a}{c} = \frac{b}{d} \ldots$ This is known as **Alternando.**

(iii) $\because \quad \frac{a}{b} = \frac{c}{d}, \; \frac{a}{b} + 1 = \frac{c}{d} + 1.$

i.e. $\frac{a+b}{b} = \frac{c+d}{d} \ldots$ This is known as **Componendo.**

(iv) $\because \quad \frac{a}{b} = \frac{c}{d}, \; \frac{a}{b} - 1 = \frac{c}{d} - 1$

i.e. $\frac{a-b}{b} = \frac{c-d}{d} \ldots$ This is known as **Dividendo.**

(v) Dividing (iii) by (iv), we get

$\frac{a+b}{a-b} = \frac{c+d}{c-d} \ldots$ This is known as **Componendo-Dividendo.**

Example 6: If $\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$, then $\frac{a}{b} = \frac{c}{d}$.

Sol. By componendo-dividendo, we have

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)} = \frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\therefore \quad \frac{8a}{10b} = \frac{8c}{10d}.$$

Multiplying both sides by $\frac{10}{8}$, we have

$$\frac{a}{b} = \frac{c}{d}$$

or $$a : b = c : d.$$

Example 7: If $a : b = c : d$, then $\frac{2a+3b}{2a-3b} = \frac{2c+3d}{2c-3d}$.

First Method.

Let $\frac{a}{b} = \frac{c}{d} = k$; then $a = bk$ and $c = dk$.

$$\frac{2a+3b}{2a-3b} = \frac{2bk+3b}{2bk-3b} = \frac{b(2k+3)}{b(2k-3)} = \frac{2k+3}{2k-3}.$$

and $$\frac{2c+3d}{2c-3d} = \frac{2dk+3d}{2dk-3d} = \frac{d(2k+3)}{d(2k-3)} = \frac{2k+3}{2k-3}$$

$$\therefore \quad \frac{2a+3b}{2a-3b} = \frac{2c+3d}{2c-3d}$$

Second Method.

Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{c} = \frac{b}{d}$ (Alternando).

or $$\frac{2a}{2c} = \frac{3b}{3d}$$

$\therefore \quad \frac{2a}{3b} = \frac{2c}{3d}$ (Alternando).

$\therefore \quad \frac{2a+3b}{2a-3b} = \frac{2c+3d}{2c-3d}$ (Componendo-dividendo.)

(The second method is more elegant, but the first is surer.)

Example 8: If $3a+4b+3c+4d : 2a+3b+2c+3d = 3a+4b-6c-8d : 2a+3b-4c-6d$, then $a : b = c : d$.

Sol. Let $\frac{3a+4b+3c+4d}{2a+3b+2c+3d} = \frac{3a+4b-6c-8d}{2a+3b-4c-6d} = k$.

Then $3a+4b+3c+4d = (2a+3b+2c+3d)k$ ... (i)

and $3a+4b-6c-8d = (2a+3b-4c-6d)k$ ... (ii)

From (i) and (ii), we have

$a(3-2k)+b(4-3k)+c(3-2k)+d(4-3k) = 0$ ... (iii)

$a(3-2k)+b(4-3k)-2c(3-2k)-2d(4-3k) = 0$ ... (iv)

From (iii) and (iv), by factors and transposition, we have

$(a+c)(3-2k) = (b+d)(3k-4)$ ... ... (v)

$(a-2c)(3-2k) = (b-2d)(3k-4)$ ... ... (vi)

From (v) and (vi), we have

$\frac{a+c}{b+d} = \frac{3k-4}{3-2k}$ ... ... (vii)

$\frac{a-2c}{b-2d} = \frac{3k-4}{3-2k}$ ... ... (viii)

From (vii) and (viii), we have

$$\frac{a+c}{b+d} = \frac{a-2c}{b-2d}$$

Multiplying cross-wise, we have

$$ab-2ad+bc-2cd = ab+ad-2bc-2cd$$

$$\therefore \quad -3ad = -3bc$$

$$\text{or} \quad ad = bc$$

$$\text{or} \quad \frac{a}{b} = \frac{c}{d}$$

$$\text{or} \quad a:b = c:d.$$

EXERCISE 111.

If $a:b = c:d$, prove that

1. $a+3c : b+3d = 3a+c : 3b+d.$

2. $3a-2c : 3b-2d = 5a-4c : 5b-4d.$

3. $ma+nc : mb+nd = pq+ac : pb+qd.$

4. $ma-nc : mb-nd = pa-qc : pb-qd.$

5. $a+b : ma-nb = c+d : mc-nd.$

6. $a^2+b^2 : a^2-b^2 = c^2+d^2 : c^2-d^2.$

7. $\frac{ma+nb}{mc+nd} = \frac{b^2c}{d^2a}.$

8. $pa^2+qc^2 : pb^2+qd^2 = ma^2-nc^2 : mb^2-nd^2.$

9. $a^2+ab+b^2 : a^2-ab+b^2 = c^2+cd+d^2 : c^2-cd+d^2.$

10. $a^3+c^3 : b^3+d^3 = \left(\frac{a+c}{b+d}\right)^3.$

11. $\frac{la^2+mab+nb^2}{pa^2+qab+rb^2} = \frac{lc^2+mcd+nd^2}{pc^2+qcd+rd^2}.$

12. $\frac{(a-b)\ (a-c)}{a} = (a+d)-(b+c).$

[*Hint*: Simplify the left-hand side and substitute ad for bc.]

13. $4(a+b)\ (c+d) = bd\ \left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2.$

[*Hint*: Apply componendo, then $\frac{a+b}{b}+\frac{c+d}{d} = \frac{2(a+b)}{b}$ $= \frac{2(c+d)}{d}$; hence $\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2 = \frac{2(a+b)}{b} \times \frac{2(c+d)}{d}.$]

14. $(5a+6c)(5b-6d) = (5a-6c)(5b+6d).$

15. If $\frac{2a+3b}{2a-3b} = \frac{2c+3d}{2c-3d}$, then $a:b = c:d$.

16. If $\frac{a^2+c^2}{b^2+d^2} = \frac{ac}{bd}$, then $a:b = c:d$.

17. If $\left(\frac{a-c}{b-d}\right)^2 = \frac{a^2}{b^2}$, then $a:b = c:d$.

18. If $\left(\frac{a+c}{b+d}\right)^3 = \frac{a(a+c)^2}{b(b+d)^2}$, then $a:b = c:d$.

19. If $\frac{ac+bd}{ac-bd} = \frac{a^2+b^2}{a^2-b^2}$, then $a:b = c:d$.

20. If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, then $a:b = c:d$.

21. If $(2a+3b+5c+4d)(2a-3b-5c+4d) = (2a+3b-5c-4d)(2a-3b+5c-4d)$, then $ad:bc = 15:8$.

22. If $\frac{3a+b-9c-3d}{a+2b-3c-6d} = \frac{3a+b+6c+2d}{a+2b+2c+4d}$, then $a:b = c:d$.

23. If $(a^2+b^2)(x^2+y^2) = (ax+by)^2$, then $a:x = b:y$.

24. If $2b = a+c$ and $\frac{2}{c} = \frac{1}{b}+\frac{1}{d}$, show that $a:b = c:d$.

Example 9: If $x = \frac{2ab}{a+b}$, find the value of $\frac{x+a}{x-a}+\frac{x+b}{x-b}$.

Sol. Since $x = \frac{2ab}{a+b}$, $\therefore \quad \frac{x}{a} = \frac{2b}{a+b}$ ... ... (i)

and $\frac{x}{b} = \frac{2a}{a+b}$ ... ... (ii)

Applying componendo and dividendo in (i) and (ii), we get $\frac{x+a}{x-a} = \frac{a+3b}{-a+b}$ ... ... (iii),

and $\frac{x+b}{x-b} = \frac{3a+b}{a-b}$ ... ... (iv)

Adding (iii) and (iv), we get

$$\frac{x+a}{x-a}+\frac{x+b}{x-b} = \frac{a+3b}{-a+b}+\frac{3a+b}{a-b}$$

$$= \frac{-a-3b+3a+b}{a-b}$$

$$= \frac{2(a-b)}{a-b} = 2.$$

Example 10: If $\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$, then $\frac{x+y}{x-y}=\frac{p+q}{p-q}$.

Sol. Since $\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$,

$\therefore \quad \frac{x^2+y^2}{2xy}=\frac{p^2+q^2}{2pq}$ [Dividing by 2.]

$\therefore \quad \frac{x^2+y^2+2xy}{x^2+y^2-2xy}=\frac{p^2+q^2+2pq}{p^2+q^2-2pq}$. [Componendo-dividendo.]

$\therefore \quad \frac{(x+y)^2}{(x-y)^2}=\frac{(p+q)^2}{(p-q)^2}$

$\therefore \quad \frac{x+y}{x-y}=\frac{p+q}{p-q}$. [Sq. root of both sides.]

EXERCISE 112.

1. If $x=\frac{4ab}{a+b}$, find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$.

2. If $x=\frac{6ab}{a+b}$, find the value of $\frac{x+3a}{x-3a}+\frac{x+3b}{x-3b}$.

3. If $\frac{x^2+y^2}{p^2+q^2}=\frac{xy}{pq}$, then $\frac{(x^2+y^2)^2}{(x^2-y^2)^2}=\frac{(p^2+q^2)^2}{(p^2-q^2)^2}$.

4. If $\frac{x+y}{p+q}=\frac{y}{q}$, then $\frac{x(x+2y)}{y^2}=\frac{p(p+2q)}{q^2}$.

5. If $\frac{x+y+p+q}{x-y+p-q}=\frac{x+y-p-q}{x-y-p+q}$, then $\frac{x+p}{y+q}=\frac{x-p}{y-q}$.

6. If $\frac{x+y-p+q}{x-y+p+q}=\frac{y+p+x+q}{y-p-x+q}$, then $\frac{x+q}{y-p}=\frac{y+q}{p+x}$.

7. If $\frac{x^3+3xy^2}{3x^2y+y^3}=\frac{p^3+3pq^2}{3p^2q+p^3}$, then $\frac{x+y}{x-y}=\frac{p+q}{p-q}$.

Example 11: If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a^3+c^3}{b^3+d^3}=\frac{e^3}{f^3}$.

Sol. Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$, then $a=bk$, $c=dk$ & $e=fk$.

$\therefore \quad \frac{a^3+c^3}{b^3+d^3}=\frac{b^3k^3+d^3k^3}{b^3+d^3}=\frac{k^3(b^3+d^3)}{b^3+d^3}=k^3$.

But $\frac{e}{f}=k \qquad \therefore \quad \frac{e^3}{f^3}=k^3$

$\frac{a^3+c^3}{b^3+d^3}=\frac{e^3}{f^3}$, for each is equal to the same quantity.

Example 12: If $a:b=b:c$, then $\frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a^2}{b^2}$.

Sol. Since $\frac{a}{b}=\frac{b}{c} \qquad \therefore \quad b^2=ac.$

$$\therefore \quad \frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a^2+ab+ac}{ac+bc+c^2}=\frac{a(a+b+c)}{c(a+b+c)}=\frac{a}{c}$$

$$=\frac{a}{b}\times\frac{a}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

EXERCISE 113.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ prove that

1. $\frac{a+c}{b+d}=\frac{c-e}{d-f}$

2. $\frac{b-3f}{a-3e}=\frac{d-b}{c-a}$.

3. $\frac{c^3+e^3}{d^3+f^3}=\frac{a^3-a^2c}{b^3-b^2d}$.

4. $\frac{bd-f^2}{b^2+d^2}=\frac{ac-e^2}{a^2+c^2}$.

5. $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$.

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that

6. $\frac{a^2+b^2+c^2}{b^2+c^2+d^2}=\frac{ab+bc+c^2}{bc+cd+d^2}$.

7. $\frac{a^2+b^2+c^2}{b^2+c^2+d^2}=\frac{a^2+bc}{b^2+cd}$.

8. $\frac{a^3+b^3+c^3}{b^3+c^3+d^3}=\frac{a}{d}$.

9. $\left(\frac{a-b}{b-c}\right)^3=\frac{a}{d}$.

10. $a-2b+c=\frac{(a-b)^2}{a}=\frac{(b-c)^2}{c}$.

11. $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{a+b+c}{a-b+c}$.

12. $(b-c)^2=(a-b)(c-d)$.

13. $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

14. $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$.

15. $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

16. a^2+b^2, b^2+c^2 and c^2+d^2 are in continued proportion.

17. If a, b, c are in continued proportion, find the simplest value of $\dfrac{abc(a+b+c)^3}{(ab+ac+bc)^3}$.

18. If a, b, c are in continued proportion and if $a(b-c) = 2b$, prove that $a-c = 2\left(\dfrac{a+b}{a}\right)$.

6. Theorem. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \ldots$ then each of the ratios $= \dfrac{pa+qc+re+}{pb+qd+rf+} \ldots$, where $p,q,r \ldots$ are any quantities whatever.

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \ldots = k.$

Then $a = bk,\ c = dk,\ c = fk \ldots$

$\therefore\quad pa = pbk,\ qc = qdk,\ re = rfk \ldots$

By addition we have

$$pq+qc+re+ = k(pb+qd+rf+\ldots)$$

$$\therefore\ \frac{pa+qc+re+}{pb+qd+rf+} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

Cor. 1. Putting $p = q = r = \ldots = 1$, we have

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots = \frac{a+c+e+\ldots}{b+d+f+\ldots}.$$

Thus, *when several fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

Cor. 2. Putting $p = 1$ and $q = -1$, we have

$$\frac{a}{b} = \frac{c}{d} = \ldots = \frac{a-c}{b-d}.$$

Thus, *when two fractions are equal, each is equal to the first numerator minus the second numerator divided by the first denominator minus the second denominator.*

EXERCISE 114.

If $a : b = c : d = e : f$, then

1. $\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2} = \dfrac{a^2}{b^2}$.

2. $\dfrac{a^3-2c^3+3e^3}{b^3-2d^3+3f^3} = \dfrac{a^3}{b^3}$.

3. $\dfrac{3a^2-4c^2+5e^2}{3b^2-4d^2+5f^2}=\dfrac{ac}{bd}$ **4.** $\dfrac{3a+4c+2e}{3b+4d+2f}=\dfrac{2a-3c-4e}{2b-3d-4f}$.

5. $\dfrac{a^3+c^3+e^3}{b^3+d^3+f^3}=\dfrac{a^2c+c^2e+e^2a}{b^2d+d^2f+f^2b}$.

6. $(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2$.

7. If $\dfrac{4x+5}{6x+7}=\dfrac{4x+6}{6x+4}$, then each $=-\dfrac{1}{3}$.

8. If $\dfrac{1-4x}{1+6x}=\dfrac{3+2x}{2-3x}$, then each $=\dfrac{7}{5}$.

9. If $\dfrac{x^2+3x+2}{x^2+7x+12}=\dfrac{x+3}{x+7}$, then each $=\dfrac{1}{6}$.

10. If $\left(\dfrac{2x+3}{2x+5}\right)^2=\dfrac{x+3}{x+5}$, then $\dfrac{9}{x+3}=\dfrac{25}{x+5}$.

Fill up the blanks in the following:

11. $\dfrac{x}{8}=\dfrac{y}{3}=\dfrac{x+y}{?}=\dfrac{x-y}{?}=\dfrac{15x-12y}{?}$.

12. $\dfrac{x}{7}=\dfrac{y}{3}=\dfrac{?}{10}=\dfrac{?}{4}=\dfrac{?}{36}$.

13. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, shew that $\left(\dfrac{ax+by+cz}{a^2+b^2+c^2}\right)^3=\dfrac{xyz}{abc}$.

14. If a, b, c, d are in proportion, prove that $ab+cd$ is the mean proportional between a^2+c^2 and b^2+d^2; and if a, b, c, d are in continued proportion, prove that $b+c$ is the mean proportional between $a+b$ and $c+d$.

7. Harder Examples.

Example 13: If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, find the value of $(b-c)x+(c-a)y+(a-b)z$.

Sol. Let $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}=k$.

Then
$$x=(b+c-a)k$$
$$y=(c+a-b)k$$
$$z=(a+b-c)k$$

$\therefore$ the given expression
$= (b-c)(b+c-a)k+(c-a)(c+a-b)k+(a-b)(a+b-c)k$
$= k\{\, b^2-c^2-a(b-c)+c^2-a^2-b(c-a)+a^2-b^2-c(a-b)\,\}$
$= k\times 0 = 0.$

Example 14: If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$, then each $=\frac{1}{2}$ or -1.

Sol. Each ratio $=\dfrac{\text{sum of the numerators}}{\text{sum of the denominators}}$

$=\dfrac{a+b+c}{2a+2b+2c}$

$=\dfrac{a+b+c}{2(a+b+c)}$

$=\frac{1}{2}$, if $a+b+c$ *is not equal to* O.

When $a+b+c=0$, then $b+c=-a$

$\therefore$ each ratio $=\dfrac{a}{b+c}=\dfrac{a}{-a}=-1.$

Example 15: If $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$,

then $\dfrac{a}{y+z-x}=\dfrac{b}{z+x-y}=\dfrac{c}{x+y-z}.$

Sol. Let $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}=k.$

Then $x=(b+c)k, y=(c+a)k$ and $z=(a+b)k.$

$\therefore\quad y+z-x=(c+a)k+(a+b)k-(b+c)k$
$=k(c+a+a+b-b-c)$
$=2ka$

$\therefore\quad \dfrac{a}{y+z-x}=\dfrac{1}{2k}.$

Similarly, $\dfrac{b}{z+x-y}=\dfrac{1}{2k}$ and $\dfrac{c}{x+y-z}=\dfrac{1}{2k}$

$\therefore\quad \dfrac{a}{y+z-x}=\dfrac{b}{z+x-y}=\dfrac{c}{x+y-z}.$

Example 16: If $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}.$

show that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}.$

Sol. Let each of the given ratios $= k$.

Then $ay-bx = ck$, $cx - az = bk$, $bz-cy = ak$;

$\therefore$ $c(ay-bx) = c^2k$, $b(cx-az) = b^2k$, $a(bz-cy) = a^2k$.

Adding these results, we get

$$k(a^2+b^2+c^2) = abz-acy+bcx-abz+acy-bcx = 0$$

$\therefore$ $k = 0$

$\therefore$ $ay-bx = 0$ or $ay = bx$, $\therefore$ $\frac{x}{a} = \frac{y}{b}$,

also $cx-az = 0$ or $cx = az$, $\therefore$ $\frac{x}{a} = \frac{z}{c}$.

$\therefore$ $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

EXERCISE 115.

1. If $\frac{x}{a-b} = \frac{y}{b-c} = \frac{z}{c-a}$, prove that $x+y+z = 0$ and $cx+ay+bz = 0$.

2. If $\frac{x}{bc(b-c)} = \frac{y}{ca(c-a)} = \frac{z}{ab(a-b)}$, prove that $a(b+c)x+b(c+a)y+c(a+b)z = 0$.

3. If $\frac{a}{b+c-a} = \frac{b}{c+a-b} = \frac{c}{a+b-c}$, prove that each ratio $= 1$.

4. If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$ and $a+b+c$ be not equal to zero. then $a = b = c$.

5. Assuming that $\frac{a+b-c}{a+b} = \frac{b+c-a}{b+c} = \frac{c+a-b}{c+a}$ and $a+b+c$ is not zero, show that $a = b = c$.

6. If $\frac{a+b}{b+c} = \frac{c+d}{d+a}$, prove that either $a = c$ or $a+b+c+d = 0$.

7. If $\frac{x}{a+2b+2c} = \frac{y}{b+2c+2a} = \frac{z}{c+2a+2b}$, show that each of these ratios $= \frac{x+y-z}{a+b+3c}$.

8. If $\frac{x}{ax+by+cz} = \frac{y}{bx+cy+az} = \frac{z}{cx+ay+bz}$,
show that each of these ratios $= \frac{1}{a+b+c}$, provided $x+y+z$ is not equal to zero.

9. If $(a+b+c)x = (b+c-a)y = (c+a-b)z = (a+b-c)w$
then $\frac{1}{y}+\frac{1}{z}+\frac{1}{w} = \frac{1}{x}$.

10. If $a+c = 2b$ and $\frac{1}{b}+\frac{1}{d} = \frac{2}{c}$, show that $a : b = c : d$.

7. Theorem. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$

then each of the ratios $= \left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}}$.

where $p, q, r \ldots\ldots n$ are any quantities whatever.

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots = k$.

Then $a = bk, \quad c = dk, \quad e = fk \ \ldots\ldots$

$$pa^n = pb^n k^n$$
$$qc^n = qd^n k^n$$
$$re^n = rf^n k^n \text{ etc.}$$

Hence $pa^n + qc^n + re^n + \ldots\ldots = (pb^n + qd^n + rf^n + \ldots)k^n$

$$\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots} = k^n.$$

Taking n^{th} root of both sides, we have

$$\left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}} = (k^n)^{\frac{1}{n}}$$

$$= k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

Example 17: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each of these ratios is equal to $\sqrt{\left(\frac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right)^{-1}}$

Sol. Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k.$

$\therefore \quad a = bk,\ c = dk,\ e = fk.$

$\therefore \quad 3a^{-2} = 3b^{-2}k^{-2}.$

$-5c^{-2} = -5d^{-2}k^{-2}$

$7e^{-2} = 7f^{-2}k^{-2}.$

$\therefore\ 3a^{-2} - 5c^{-2} + 7e^{-2} = k^{-2}(3b^{-2} - 5d^{-2} + 7f^{-2})$

or $\left(\frac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right) = k^{-2}$

or $\left(\frac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right)^{-1} = (k^{-2})^{-1} = k^2.$

Taking the sq. root of both sides, we have

$$\sqrt{\left(\frac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right)^{-1}} = \sqrt{k^2} = k.$$

$$\therefore \quad \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \sqrt{\left(\frac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right)^{-1}}.$$

EXERCISE 116

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each of these ratios is equal to

1. $\sqrt{\frac{a^2 - c^2 + e^2}{b^2 - d^2 + f^2}}$.

2. $\left(\frac{a^3 + 2c^3 - 3e^3}{b^3 + 2d^3 - 3f^3}\right)^{\frac{1}{3}}$.

3. $\sqrt{\frac{5a^2 - 7c^2 + 6e^2}{5b^2 - 7d^2 + 6f^2}}$.

4. $\left(\frac{3a^3 - 4c^3 + 5e^3}{3b^3 - 4d^3 + 5f^3}\right)^{\frac{1}{3}}$.

5. $\sqrt[4]{\left(\frac{2a^{-4} + 5c^{-4} + 7e^{-4}}{2b^{-4} + 5d^{-4} + 7f^{-4}}\right)^{-1}}$.

6. $\left(\frac{la^7 - mc^7 + ne^7}{lb^7 - md^7 + nf^7}\right)^{\frac{1}{7}}$.

If $\frac{a}{b} = \frac{c}{d}$, prove that

7. $\sqrt{\frac{a^2 + c^2}{b^2 + d^2}} = \frac{ma + nc}{mb + nd}$.

8. $\frac{(\sqrt{a} + \sqrt{b})^2}{(\sqrt{c} + \sqrt{d})^2} = \frac{a - b}{c - d}$.

9. $\frac{\sqrt{3a^2 + 4c^2}}{\sqrt[3]{5a^3 - 6c^3}} = \frac{\sqrt{3b^2 + 4d^2}}{\sqrt[3]{5b^3 - 6d^3}}$.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that

10. $\sqrt{\frac{pa^2+qc^2+re^2}{pb^2+qd^2+rf^2}} \quad \sqrt[3]{\frac{ace}{bdf}}$.

11. $\sqrt[3]{\frac{a^2c-c^2e-e^2a}{b^2d-d^2f-f^2b}} = \sqrt{\frac{a^2-c^2-e^2}{b^2-d^2-f^2}}$.

SELECTED QUESTIONS—H

1. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, show that

(*i*) $\frac{x^3}{a^3} - \frac{y^3}{b^3} + \frac{z^3}{c^3} = \frac{xyz}{abc}$.

(*ii*) $\left(\frac{a^2x^2+b^2y^2+c^2z^2}{a^3x+b^3y+c^3z}\right)^{\frac{3}{2}} = \sqrt{\frac{xyz}{abc}}$

2. If $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$, prove that $\frac{x^3+y^3+z^3}{a^3+b^3+c^3} = \frac{xyz}{abc}$.

3. $ax = by = cz$, show that

$$\frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} = \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}.$$

4. If a, b, c are in continued proportion, prove that $a^2+ab : b^2 :: b^2+bc : c^2$.

5. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that

$$\sqrt{(a+c+e)(b+d+f)} = (ab)^{\frac{1}{2}} + (cd)^{\frac{1}{2}} + (ef)^{\frac{1}{2}}.$$

6. If $a : b = b : c = c : d$, show that

$$(a+d)(b+c) - (a+c)(b+d) = (b-c)^2$$

7. If a, b, c, d are in continued proportion, show that

(i) $(a+c)^4 : ac = (b+d)^4 : d^2$

(ii) $a : d = a^3+b^3+c^3 : b^3+c^3+d^3$.

(iii) $(a^2+c^2)(b^2+d^2) = (ab+cd)^2$

8. If $a : b :: x : y$, prove that

$$ab : xy :: a^2+b^2 : x^2+y^2.$$

9. If $a:b::c:d$, prove that
$$a^2+c^2 : b^2+d^2 :: ac : bd.$$

10. If $x:y=a:a+b$, prove that
$$x^2-xy+y^2 : a^2+ab+b^2 = x^2 : a^2.$$

11. If $\frac{x}{b^2-c^2}=\frac{y}{c^2-a^2}=\frac{z}{a^2-b^2}$, show that
$$\frac{x}{b+c}+\frac{y}{c+a}+\frac{z}{a+b}=0.$$

12. If $\frac{b+c}{a}=\frac{c+a}{b}=\frac{a+b}{c}$, prove that either $a+b+c=0$ or $a=b=c$.

13. If $\frac{y+z-x}{a}=\frac{z+x-y}{b}=\frac{x+y-z}{c}$, prove that
$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}.$$

14. If $(a+b+c)x=(-a+b+c)y=(a-b+c)z=(a+b-c)w$, show that $y^{-1}+z^{-1}+w^{-1}=x^{-1}$.

15. If $(b-c)x+(c-a)y+(a-b)z=0$, show that
$$\frac{y-z}{b-c}=\frac{z-x}{c-a}=\frac{x-y}{a-b}.$$

16. If $\frac{x}{p+2q+r}=\frac{y}{p-r}=\frac{z}{p-2q+r}$, show that
$$\frac{x+2y+z}{p}=\frac{x-z}{q}=\frac{x-2y+z}{r}.$$

17. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, show that $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$.

CHAPTER XV

FRACTIONAL EQUATIONS AND WITH LITERAL CO-EFFICIENTS EQUATIONS

1. Equations involving fractions which contain the unknown in the denominators.

Example 1: Solve $\frac{1}{x}-\frac{3}{2x}+\frac{5}{3x}-\frac{7}{4x}=\frac{5}{6x}-\frac{1}{24x}-1$.

Sol. Multiplying both sides by $24x$, the L.C.M. of the denominators, we have

$$24-36+40-42 = 20-1-24x.$$

By transposition, we get

$$24x = 20-1-24+36-40+42$$

or $24x = 33$

$$\therefore \quad x = 1\frac{3}{8}.$$

Example 2: Solve $\frac{2}{3x-4}-\frac{3}{4x-5}=0$.

Sol. Multiplying both sides by $(3x-4)(4x-5)$, the L.C.M. of the denominators, we have

$$2(4x-5)-3(3x-4) = 0$$

$$8x-10-9x+12 = 0$$

or $$-x = -2$$

or $$x = 2.$$

EXERCISE 117.

Solve the equations:

1. $\frac{1}{2x}+\frac{1}{3x}-2=\frac{4}{3x}-3$.

2. $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}+\frac{1}{4x}=6$.

3. $\frac{\cdot 4}{x}=\cdot 25-\frac{\cdot 65}{x}$.

4. $\frac{3}{2x}+\frac{4}{5x}+11=\frac{1}{x}+\frac{5}{3x}$.

5. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$.

6. $\frac{1}{5x-11}=\frac{2}{3x-1}$.

7. $\frac{5}{3x+4}=\frac{4}{5(x-3)}$.

8. $\frac{3}{2x+3}=\frac{4}{3x+2}$.

9. $\frac{5}{3-4x} = \frac{6}{6-5x}$.

10. $\frac{2}{3x-1} + \frac{3}{2x-1} = 0$.

11. $\frac{3}{8x+5} - \frac{5}{6x-47} = 0$.

12. $\frac{a}{a-x} + \frac{b}{b-x} = 0$.

13. $\frac{a}{bx+a} + \frac{b}{ax+b} = 0$.

14. $\frac{3}{6x^2+2x+5} = \frac{4}{8x^2+3x+2}$.

15. $\frac{a}{ax^2+bx+1} = \frac{b}{bx^2+ax+1}$

16. $\frac{2}{4x^3+2x^2+x+1} = \frac{3}{6x^3+3x^2+2x+1}$.

Example 3: Solve $\frac{1}{(x-1)(x-2)} + \frac{1}{(x-1)(x-3)} = \frac{1}{(x-2)(x-3)} + \frac{1}{(x-3)(x-4)}$.

Sol. By transposition, we have

$$\frac{1}{(x-1)(x-2)} - \frac{1}{(x-2)(x-3)} = \frac{1}{(x-3)(x-4)} - \frac{1}{(x-1)(x-3)}$$

$$\therefore \quad \frac{x-3-x+1}{(x-1)(x-2)(x-3)} = \frac{x-1-x+4}{(x-1)(x-3)(x-4)}$$

$$\therefore \quad \frac{-2}{(x-1)(x-2)(x-3)} = \frac{3}{(x-1)(x-3)(x-4)}$$

Multiplying both sides by $(x-1)(x-3)$, we have

$$\frac{-2}{x-2} = \frac{3}{x-4}.$$

Multiplying cross-wise, we have

$$-2x+8 = 3x-6$$

$$\therefore \quad 5x = 14$$

or $$x = 2\tfrac{4}{5}.$$

NOTE: In the above process, the first step of transposition is taken to remove x from the numerators on both sides, when they are simplified.

Example 4: Solve $\frac{10}{2x+1} + \frac{3}{3x-2} = \frac{2}{2x-3} + \frac{15}{3x+2}$

Sol. Here, we arrange the terms in such a way as to remove x from the numerator on both sides, when they are simplified.

By transposition, we have

$$\frac{10}{2x+1}-\frac{15}{3x+2}=\frac{2}{2x-3}-\frac{3}{3x-2}$$

$$\therefore \quad \frac{30x+20-30x-15}{(2x+1)(3x+2)}=\frac{6x-4-6x+9}{(2x-3)(3x-2)}$$

$$\frac{5}{(2x+1)(3x+2)}=\frac{5}{(2x-3)(3x-2)}$$

or

$$\frac{1}{(2x+1)(3x+2)}=\frac{1}{(2x-3)(3x-2)}.$$

Multiplying cross-wise, we have

$$(2x-3)(3x-2)=(2x+1)(3x+2)$$

$$6x^2-13x+6=6x^2+7x+2$$

$$\therefore \quad 20x=4$$

$$\therefore \quad x=\tfrac{1}{5}.$$

EXERCISE 118.

Solve the equations:

1. $\frac{1}{x+1}-\frac{1}{x+3}=\frac{1}{x+2}-\frac{1}{x+4}$.

2. $\frac{1}{x+2}+\frac{1}{x+10}=\frac{1}{x+4}+\frac{1}{x+8}$.

3. $\frac{1}{x}-\frac{1}{x-1}+\frac{1}{x+3}-\frac{1}{x+4}=0$.

4. $\frac{1}{x-6}-\frac{1}{x-3}=\frac{1}{x-5}-\frac{1}{x-2}$.

5. $\frac{1}{x-10}-\frac{1}{x-5}=\frac{1}{x-7}-\frac{1}{x-2}$.

6. $\frac{1}{x^2+7x+12}+\frac{1}{x^2+8x+15}=\frac{1}{x^2+9x+18}+\frac{1}{x^2+9x+20}$.

7. $\frac{1}{(x+a)^2-b^2}+\frac{1}{(x+b)^2-a^2}=\frac{1}{x^2-(a+b)^2}+\frac{1}{x^2-(a-b)^2}$.

8. $\frac{2}{2x-1}+\frac{3}{3x+1}=\frac{2}{2x+1}+\frac{3}{3x-2}$.

9. $\frac{9}{x-4}+\frac{3}{x-8}=\frac{4}{x-9}+\frac{8}{x-3}$.

10. $\frac{3}{x-3}-\frac{4}{x+9}-\frac{5}{x-27}+\frac{6}{x-15}=0$.

11. $\frac{6}{5x+7}-\frac{4}{5x+13}=\frac{9}{5x+13}-\frac{7}{5x+19}$.

12. $\frac{1}{2x-1}+\frac{8}{4x-1}-\frac{6}{3x-1}-\frac{3}{6x-1}=0$.

Example 5: Solve $\frac{1}{x+5}+\frac{10}{2x-5}=\frac{18}{3x-5}$.

Method. Decompose $\frac{18}{3x-5}$ into two such fractions that when one of them is combined with $\frac{1}{x+5}$ and the other with $\frac{10}{2x-5}$ and then simplified, x disappears from the numerators on both sides.

Thus, $\frac{1}{x+5}+\frac{10}{2x-5}=\frac{3+15}{3x-5}=\frac{3}{3x-5}+\frac{15}{3x-5}$.

Combining $\frac{3}{3x-5}$ with $\frac{1}{x+5}$ and $\frac{15}{3x-5}$ with $\frac{10}{2x-5}$,

we have $\frac{1}{x+5}-\frac{3}{3x-5}=\frac{15}{3x-5}-\frac{10}{2x-5}$

$$\frac{3x-5-3x-15}{(x+5)(3x-5)}=\frac{30x-75-30x+50}{(3x-5)(2x-5)}$$

$$\frac{-20}{(x+5)(3x-5)}=\frac{-25}{(3x-5)(2x-5)}.$$

Multiplying both sides by $\frac{3x-5}{-5}$ we have

$$\frac{4}{x+5}=\frac{5}{2x-5}.$$

Multiplying cross-wise we have

$$8x-20 = 5x+25$$
$$\therefore \quad 3x = 45$$
$$\therefore \quad x = 15.$$

Example 6: Solve $\frac{6}{2x+3} - \frac{5}{5x+2} = \frac{2}{x+6}$.

This can be solved by decomposing $\frac{2}{x+6}$ into $\frac{3}{x+6}$ and $-\frac{1}{x+6}$ and proceeding as in the previous example.

It can also be solved directly thus:

$$\frac{6(5x+2)-5(2x+3)}{(2x+3)(5x+2)} = \frac{2}{x+6}.$$

or

$$\frac{30x+12-10x-15}{10x^2+19x+6} = \frac{2}{x+6}.$$

Multiplying cross-wise,

$$(20x-3)(x+6) = 2(10x^2+19x+6)$$
$$20x^2+117x-18 = 20x^2+38x+12$$
$$79x = 30$$
$$x = \frac{30}{79}.$$

EXERCISE 119.

Solve the equations:

1. $\frac{1}{x+1} + \frac{2}{x+2} = \frac{3}{x+3}$.

2. $\frac{3}{x+1} + \frac{4}{x+2} = \frac{7}{x+3}$.

3. $\frac{2}{2x-5} + \frac{1}{x-3} = \frac{6}{3x-1}$.

4. $\frac{7}{x+2} + \frac{3}{x+4} = \frac{10}{x-5}$.

5. $\frac{1}{x+a} + \frac{1}{x+b} = \frac{2}{x}$.

6. $\frac{a}{x-1} + \frac{b}{x-2} = \frac{a+b}{x}$.

7. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{a+b}{x-c}$.

8. $\frac{a}{x+a} - \frac{b}{x+b} = \frac{a-b}{x+c}$.

9. $\frac{3}{4x+1} + \frac{4}{4x+5} = \frac{7}{4x+3}$.

10. $\frac{6}{3x+15} - \frac{1}{x-5} = \frac{2}{2x-5}$.

11. $\frac{4}{5x+1}+\frac{7}{10x+1}=\frac{3}{2x+1}$. **12.** $\frac{6}{2x-3}+\frac{2}{1-2x}=\frac{2}{x-1}$.

13. $\frac{10}{5x-9}+\frac{14}{2x-9}=\frac{9}{x-8}$. **14.** $\frac{7}{7x+1}+\frac{6}{4x+1}=\frac{15}{6x+5}$.

15. $\frac{b-c}{x+a}+\frac{a-b}{x+b}=\frac{a-c}{x+c}$. **16.** $\frac{1}{x+2}+\frac{3}{x+3}+\frac{5}{x+5}=\frac{9}{x+4}$.

(*Hint*: *Decompose* $\frac{9}{x+4}$ *into three parts with numerators* 1, 3, 5, *and then associate these parts with the fractions on the left-hand side so that x may disappear from the numerators after simplification*)

17. $\frac{1}{x-6a}+\frac{2}{x+3a}+\frac{3}{x-2a}=\frac{6}{x-a}$.

18. $\frac{3}{3x+1}+\frac{4}{4x+1}+\frac{48}{8x+1}=\frac{48}{6x+1}$.

2. Equations involving fractions which contain the unknown both in the numerators and the denominators.

Example 7: Solve $\frac{4x+1}{5x+2}=\frac{2}{3}$.

Sol. Multiplying cross-wise, we have

$$3(4x+1) = 2(5x+2)$$
$$\therefore \quad 12x+3 = 10x+4$$
$$\therefore \quad 2x = 1 \text{ and } x = \tfrac{1}{2}.$$

Example 8: Solve $\frac{4x+5}{6x+7}=\frac{2x+3}{3x+2}$.

Sol. Multiplying cross-wise, we have

$$(4x+5)(3x+2) = (2x+3)(6x+7)$$
$$\therefore \quad 12x^2+23x+10 = 12x^2+32x+21$$
$$\therefore \quad -9x = 11$$
$$\therefore \quad x = -1\tfrac{2}{9}.$$

Example 9: Solve $\frac{x^2+2x-3}{x^2-9}=3$.

Sol. The left-hand side $=\frac{(x+3)(x-1)}{(x+3)(x-3)}=\frac{x-1}{x-3}$

$$\therefore \quad \frac{x-1}{x-3}=3.$$

Multiplying cross-wise, we have

$$3x-9 = x-1$$
$$\therefore \quad x = 4.$$

EXERCISE 120.

Solve the equations:

1. $\frac{5x+2}{3x-4} = 3\frac{2}{3}$.

2. $\frac{7x-3}{2x+5} = 1\frac{2}{3}$.

3. $\frac{4x+5}{6x-7} = 1\frac{6}{11}$.

4. $\frac{2-3x}{1-4x} + \frac{1}{2} = 0$.

5. $\frac{11-5x}{7-4x} = 2$.

6. $\frac{ax+b}{bx+a} = \frac{c}{d}$.

7. $\frac{2x+1}{3x+2} = \frac{2x+5}{3x-4}$.

8. $\frac{3x+2}{3x+1} = \frac{4x-3}{4x+1}$.

9. $\frac{3-5x}{7-3x} = \frac{5x+2}{3x+4}$.

10. $\frac{7-12x}{5-2x} = \frac{6x-5}{x-3}$.

11. $\frac{x^2+x+1}{x^2+3x+3} = \frac{x+1}{x+3}$.

12. $\frac{ax+b^2}{ax+c^2} = \frac{x+a}{x+b}$.

Solve the equations:

13. $\frac{x^2+x-6}{x^2-4} = 1\frac{1}{5}$.

14. $\frac{x^2+x-12}{x^2-16} = 2$.

15. $\frac{x^2-7x+12}{x^2-9} = \frac{2}{9}$.

16. $\frac{x^2-x-20}{x^2+x-12} = \frac{1}{2}$.

17. $\frac{x^2+4x-21}{x^2+3x-28} = 1\frac{1}{4}$.

18. $\frac{x^2+x-30}{x^2+3x-18} = \frac{1}{2}$.

Example 10: Solve $\frac{4x+3}{9} + \frac{29-7x}{12-5x} = \frac{8x+19}{18}$.

Sol. Transposing the terms with numerical denominators to one side, we have

$$\frac{29-7x}{12-5x} = \frac{8x+19}{18} - \frac{4x+3}{9}$$
$$= \frac{8x+19-8x-6}{18}$$
$$= \frac{13}{18}.$$

Multiplying cross-wise, we have

$$18(29-7x) = 13(12-5x)$$

$$\therefore \quad 522-126x = 156-65x$$

$$61x = 366 \text{ and } x = 6.$$

Example 11: Solve $\frac{4x-13}{2x-5}+\frac{9x-17}{3x-7}=5.$

Sol. Decomposing the terms on the left-hand side into integral and fractional parts, we have

$$\frac{4x-10-3}{2x-5}+\frac{9x-21+4}{3x-7}=5$$

or $$-\frac{2(2x-5)-3}{2x-5}+\frac{3(3x-7)+4}{3x-7}=5$$

or $$2-\frac{3}{2x-5}+3+\frac{4}{3x-7}=5$$

$$\therefore \quad -\frac{3}{2x-5}+\frac{4}{3x-7}=0$$

By transposition, we have

$$\frac{4}{3x-7}=\frac{3}{2x-5}.$$

16458.

Multiplying cross-wise, we have

$$\therefore \quad 8x-20 = 9x-21$$

$$x = 1.$$

Example 12: Solve $\frac{6x-1}{3x+4}+\frac{25x-40}{5x-6}=\frac{7x+9}{x+2}.$

Sol. Decomposing all the terms into integral and fractional parts, we have

$$\frac{2(3x+4)-9}{3x+4}+\frac{5(5x-6)-10}{5x-6}=\frac{7(x+2)-5}{x+2}$$

or $$2-\frac{9}{3x+4}+5-\frac{10}{5x-6}=7-\frac{5}{x+2}$$

$$\frac{9}{3x+4}+\frac{10}{5x-6}=\frac{5}{x+2}=\frac{3+2}{x+2}$$

$$=\frac{3}{x+2}+\frac{2}{x+2}.$$

By transposition, we have

$$\frac{9}{3x+4}-\frac{3}{x+2}=\frac{2}{x+2}-\frac{10}{5x-6}$$

$$\therefore \quad \frac{9x+18-9x-12}{(3x+4)(x+2)}=\frac{10x-12-10x-20}{(x+2)(5x-6)}$$

$$\therefore \quad \frac{6}{(3x+4)(x+2)}=\frac{-32}{(x+2)(5x-6)}.$$

Multiplying both sides by $\frac{x+2}{2}$, we have

$$\frac{3}{3x+4}=\frac{-16}{5x-6}.$$

Multiplying cross-wise, we have

$$15x-18=-48x-64$$

$$\therefore \quad 63x=-46 \qquad \therefore \quad x=-\frac{46}{63}.$$

EXERCISE 121.

Solve the equations:

1. $\frac{x-3}{4}+\frac{x+1}{12x+7}=\frac{3x-11}{12}$.

2. $\frac{2x-1}{5}+\frac{3x+1}{15x+8}=\frac{8x-7}{20}$.

3. $\frac{x+4}{3}+\frac{3x+2}{2x+5}=\frac{3x+16}{9}$

4. $\frac{x-a}{b}+\frac{2x+a}{x+b}=\frac{ax+b^2}{ab}$.

5. $\frac{x}{2}+\frac{8x-12}{10x-4}=\frac{18x+20}{36}$.

6. $\frac{91x-21}{56}+\frac{24x-93}{35x-138}=\frac{13x+9}{8}$.

Solve the equations:

7. $\frac{3x+1}{x-1}+\frac{2x+3}{x+1}=5$.

8. $\frac{8x-7}{2x+1}+\frac{6x+1}{3x-2}=6$.

9. $\frac{5-2x}{4-x}-\frac{3-x}{1-x}=1$.

10. $\frac{5-4x}{3-2x}+\frac{2-3x}{1-x}=5$.

11. $\frac{x^2+5x+6}{x+1}+\frac{x^2+3x+5}{x+2}=2x+5$.

[*Hint*: *Decompose these fractions by division.*]

12. $\frac{2x^2-9x-8}{2x+1}+\frac{x^2-7x+15}{x-3}=2x-9.$

13. $\frac{4x+3}{4x-3}+\frac{15x-7}{5x-4}=\frac{8x+1}{2x-1}.$

14. $\frac{x+2}{x+1}+\frac{x+3}{x+2}=\frac{2x+8}{x+3}.$

15. $\frac{6x+1}{2x-1}+\frac{6x+7}{3x-1}=\frac{25x+20}{5x-1}.$

16. $\frac{15x+11}{3x+4}+\frac{12x+5}{4x+3}=\frac{8x+44}{x+6}.$

17. $\frac{6x-5}{2x-3}-\frac{6x-14}{3x-1}=\frac{4x+23}{4x-1}.$

18. $\frac{42x-37}{6x-1}+\frac{20x+13}{5x+12}=\frac{11x+76}{x+8}.$

19. $\frac{2x+7}{x+2}+\frac{3x+13}{x+3}=\frac{5x+27}{x+4}.$

20. $\frac{4x-7}{4x+5}+\frac{15x+11}{5x+7}=\frac{12x+1}{3x+4}.$

21. $\frac{x^2+4}{x-1}+\frac{x^2-x+2}{x-2}=\frac{2x^2-4x+3}{x-3}.$

22. $\frac{2x^2+x+5}{2x-1}+\frac{3x^2-2x+20}{3x+1}=\frac{2x^2+6x+10}{x+3}.$

23. $\frac{x}{2x-a}+\frac{x}{2x-b}=1.$ [*Hint: Multiply both sides by 2 and proceed as before.*]

24. $\frac{3x+5}{x+1}=\frac{4x+8}{3x+3}+\frac{10x+1}{6x+3}.$

Example 13: Solve $\frac{x-8}{x-10}+\frac{x-5}{x-7}=\frac{x-7}{x-9}+\frac{x-4}{x-6}.$

Sol. Decomposing each fraction into integral and fractional parts, we have

$$\frac{(x-10)+2}{x-10}+\frac{(x-7)+2}{x-5}=\frac{(x-9)+2}{x-9}+\frac{(x-6)+2}{x-6}$$

$$\therefore \quad \left(1+\frac{2}{x-10}\right)+\left(1+\frac{2}{x-5}\right)=\left(1+\frac{2}{x-9}\right)+\left(1+\frac{2}{x-6}\right)$$

or $$\frac{2}{x-10}+\frac{2}{x-5}=\frac{2}{x-9}+\frac{2}{x-6}$$

or $$\frac{1}{x-10}+\frac{1}{x-5}=\frac{1}{x-9}+\frac{1}{x-6}.$$

By transposition, we have

$$\frac{1}{x-10}-\frac{1}{x-6}=\frac{1}{x-9}-\frac{1}{x-5}.$$

or $$\frac{x-6-x+10}{(x-10)(x-6)}=\frac{x-5-x+9}{(x-9)(x-5)}$$

or $$\frac{4}{(x-10)(x-6)}=\frac{4}{(x-9)(x-5)}$$

or $$(x-9)(x-5)=(x-10)(x-6)$$

or $$x^2-14x+45=x^2-16x+60$$

or $$16x-14x=60-45$$

or $$2x=15 \qquad \therefore \quad x=7\tfrac{1}{2}$$

EXERCISE 122.

Solve the equations:

1. $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+4}{x+5}-\frac{x+5}{x+6}.$

2. $\frac{2x-11}{x-2}+\frac{x+4}{x-3}=\frac{x-5}{x+2}+\frac{2x+9}{x+1}.$

3. $\frac{7x-55}{x-8}+\frac{2x-17}{x-9}=\frac{6x-71}{x-12}+\frac{3x-14}{x-5}.$

4. $\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}.$

5. $\frac{2x-3}{x-4}-\frac{2x+1}{x-2}=5\left\{\frac{x-2}{x-1}-\frac{x-4}{x-3}\right\}.$

6. $\frac{4(x+1)}{2x-3}+\frac{2x+1}{x+3}=\frac{4(x+2)}{2x-1}+\frac{2x+3}{x+4}.$

7. $\frac{x}{x-2}+\frac{9-x}{7-x}=\frac{x+1}{x-1}+\frac{8-x}{6-x}.$

8. $\frac{2x-3}{x-2}+\frac{3x-20}{x-7}=\frac{x-3}{x-4}+\frac{4x-19}{x-5}.$

9. $\dfrac{2x+11}{x+5}-\dfrac{9x-9}{3x-4}=\dfrac{4x+13}{x+3}-\dfrac{15x-47}{3x-10}$.

10. $\dfrac{x^2+4x+5}{x+2}+\dfrac{x^2+10x+26}{x+5}=\dfrac{x^2+10x+25}{x+6}+\dfrac{x^2+4x+4}{x+1}$.

11. $\dfrac{x^2-4x-20}{x-7}+\dfrac{x^2+7x+11}{x+5}=\dfrac{x^2+3x+3}{x+2}+\dfrac{x^2-15}{x-4}$.

3. Solution of fractional equations by the application of the fundamental principles of proportion.

Example 14: Solve $\dfrac{4x+5}{6x+7}=\dfrac{2x+3}{3x+2}$.

Sol. The equations of this type can be easily solved by multiplying cross-wise, but the method illustrated below is considered to be neater.

Multiplying the numerator and the denominator of the term on the right-hand side by 2 and rewriting the equation.

$$\frac{4x+5}{6x+7}=\frac{4x+6}{6x+4}$$

$$\therefore \quad \text{each} = \frac{(4x+5)-(4x+6)}{(6x+7)-(6x+4)} = -\tfrac{1}{3}$$

$$\therefore \quad \frac{4x+6}{6x+4} \text{ or } \frac{2x+3}{3x+2} = -\tfrac{1}{3}$$

$$\text{or} \quad 6x+9 = -3x-2$$

$$\text{or} \quad 9x = -11 \qquad \therefore \quad x = -1\tfrac{2}{9}.$$

Example 15: Solve $\dfrac{1-4x}{1+6x}=\dfrac{3+2x}{2-3x}$.

Sol. Multiplying the numerator and the denominator of the term on the right-hand side by 2, and rewriting the equation,

$$\frac{1-4x}{1+6x}=\frac{6+4x}{4-6x}$$

$$\therefore \quad \text{each} = \frac{(1-4x)+(6+4x)}{(1+6x)+(4-6x)} = \frac{7}{5}$$

$$\therefore \quad \frac{1-4x}{1+6x} = \frac{7}{5}$$

or $$7+42x = 5-20x$$

$\therefore$ $$62x = -2 \quad \text{or} \quad x = -\frac{1}{31}.$$

EXERCISE 123.

Solve the equations:

1. $\frac{6x+1}{4x+3} = \frac{3x-5}{2x-7}$.

2. $\frac{9x-5}{6x-1} = \frac{3x-2}{2x-3}$.

3. $\frac{8x-7}{12x+5} = \frac{2x-3}{3x+4}$.

4. $\frac{3-4x}{1+6x} = \frac{1+2x}{2-3x}$.

5. $\frac{2+3x}{3-5x} = \frac{1-9x}{4+15x}$.

6. $\frac{1+2x}{2-3x} = \frac{5-6x}{1+9x}$.

7. $\frac{x+p}{x+q} = \frac{px+q^2}{px+r^2}$

8. $\frac{3+bx}{2+abx} = \frac{1-x}{1-ax}$.

Example 16: Solve $\frac{(x+1)(x+2)}{(x+3)(x+4)} = \frac{x+3}{x+7}$

Sol. Removing the brackets, we have

$$\frac{x^2+3x+2}{x^2+7x+12} = \frac{x+3}{x+7}.$$

Multiplying the numerator and the denominator of the term on the right-hand side by x, we have

$$\frac{x^2+3x+2}{x^2+7x+12} = \frac{x^2+3x}{x^2+7x}$$

$$\therefore \quad \text{each} = \frac{(x^2+3x+2)-(x^2+3x)}{(x^2+7x+12)-(x^2+7x)} = \frac{2}{12} = \frac{1}{6}.$$

$\therefore$ $$\frac{x+3}{x+7} = \frac{1}{6}$$

$\therefore$ $$6x+18 = x+7$$

$\therefore$ $$5x = -11 \qquad \therefore \quad x = -2\tfrac{1}{5}.$$

Otherwise

By *alternando*, we have

$$\frac{x^2+3x+2}{x+3} = \frac{x^2+7x+12}{x+7}.$$

Decomposing each fraction into integral and fractional parts, we have

$$x+\frac{2}{x+3}=x+\frac{12}{x+7}$$

$$\therefore \quad \frac{2}{x+3}=\frac{12}{x+7}$$

$$\frac{1}{x+3}=\frac{6}{x+7}$$

$$\therefore \quad 6x+18=x+7 \qquad \therefore \quad x=-2\tfrac{1}{5}.$$

Example 17: Solve $\left(\frac{2x+3}{2x+5}\right)^2=\frac{x+3}{x+5}$.

Sol. By *alternando*, we have

$$\frac{(2x+3)^2}{x+3}=\frac{(2x+5)^2}{x+5}$$

or $$\frac{4x^2+12x+9}{x+3}=\frac{4x^2+20x+25}{x+5}$$

Decomposing each fraction into integral and fractional parts, we have

$$4x+\frac{9}{x+3}=4x+\frac{25}{x+5}$$

$$\therefore \quad \frac{9}{x+3}=\frac{25}{x+5}.$$

Multiplying cross-wise, we have

$$25x+75=9x+45$$

$$\therefore \quad 16x=-30 \qquad \therefore \quad x=-1\tfrac{7}{8}.$$

EXERCISE 124.

Solve the following equations by both the methods illustrated above:

1. $\frac{(x+2)(x+3)}{(x+4)(x+5)}=\frac{x+5}{x+9}$.

2. $\frac{(x+3)(x+4)}{(x+5)(x+6)}=\frac{x+7}{x+11}$.

3. $\frac{x^2+6x+8}{x^2+8x+15}=\frac{x+6}{x+8}$.

4. $\frac{x^2+7x+10}{x^2+10x+24}=\frac{x+7}{x+10}$.

5. $\dfrac{(x+3)(x+5)}{(x+4)(x+1)} = \dfrac{x+8}{x+5}$.

6. $\dfrac{2-5x-6x^2}{2-2x-3x^2} = \dfrac{6x+5}{3x+2}$.

Solve the equations:

7. $\left(\dfrac{x+2}{x+3}\right)^2 = \dfrac{x+4}{x+6}$.

8. $\left(\dfrac{x+3}{x+5}\right)^2 = \dfrac{x+6}{x+10}$.

9. $\left(\dfrac{x-4}{x-5}\right)^2 = \dfrac{x-8}{x-10}$.

10. $\left(\dfrac{2x-3}{2x-5}\right)^2 = \dfrac{x-3}{x-5}$.

11. $\left(\dfrac{3x-5}{3x-2}\right)^2 = \dfrac{3x-10}{3x-4}$.

12. $\left(\dfrac{x+3}{x+5}\right)^3 = \dfrac{x+1}{x+7}$.

13. $\left(\dfrac{ax-b}{ax-c}\right)^2 = \dfrac{ax-2b}{ax-2c}$.

14. $\left(\dfrac{ax-b}{ax+c}\right)^3 = \dfrac{ax-2b-c.}{ax+2c+b}$

Example 18: Solve $(x^2+3x-7)(x^2+2x-5) = (x^2-3x+7)(x^2-2x+5)$.

Sol. Dividing both sides of the equation by $(x^2-3x+7)(x^2+2x-5)$, we have

$$\frac{x^2+3x-7}{x^2-3x+7} = \frac{x^2-2x+5}{x^2+2x-5}.$$

By componendo-dividendo, we have

$$\frac{(x^2+3x-7)+(x^2-3x+7)}{(x^2+3x-7)-(x^2-3x+7)} = \frac{(x^2-2x+5)+(x^2+2x-5)}{(x^2-2x+5)-(x^2+2x-5)}.$$

or $$\frac{2x^2}{6x-14} = \frac{2x^2}{-4x+10}.$$

Dividing both sides by $2x^2$ and multiplying cross-wise, we have $-4x+10 = 6x-14$

or $-10x = -24$ $\therefore$ $x = \frac{12}{5} = 2\frac{2}{5}$.

Example 19: Solve $16\left(\dfrac{1-x}{1+x}\right)^3 = \dfrac{1+x}{1-x}$.

Sol. Multiplying both sides by $\dfrac{1-x}{1+x}$, we get

$$16\left(\frac{1-x}{1+x}\right)^4 = 1.$$

Dividing both sides by 16, we get

$$\left(\frac{1-x}{1+x}\right)^4 = \frac{1}{16} = \left(\frac{1}{2}\right)^4 \qquad \text{or} \qquad \frac{1-x}{1+x} = \tfrac{1}{2}.$$

From this, we get $x = \frac{1}{3}$.

EXERCISE 125.

Solve the equations:

1. $\dfrac{x^2-2x+3}{x^2+2x-3} = \dfrac{x+5}{x-5}$. **2.** $\dfrac{x^2-3x+4}{3x-4} = \dfrac{x^2-2x+5}{2x-5}$.

3. $\dfrac{x^2+2x-5}{x^2-2x+5} = \dfrac{x^2-x+3}{x^2+x-3}$. **4.** $\dfrac{x^2-2x+5}{3x^2+4x-1} = \dfrac{x^2-2x-5}{3x^2-4x+1}$.

5. $(x^2+5x+9)(x^2+3x-7) = (x^2-3x+7)(x^2-5x-9)$.

6. $(x^2-7x+15)(3x-5) = (x^2-3x+5)(7x-15)$.

7. $81\left(\dfrac{2-x}{2+x}\right)^3 = \left(\dfrac{2+x}{2-x}\right)$. **8.** $\dfrac{x^3+3x}{3x^2+1} = \dfrac{76}{49}$.

4. **Literal Equations.** The methods *generally* employed in the solution of literal equations are similar to those used in ordinary equations. However, a few typical examples are added below.

Example 20: Solve $a(x-a)+b(x-b) = 2ab$.

Sol. Removing the brackets, we get

$$ax-a^2+bx-b^2 = 2ab.$$

By transposition, $ax+bx = a^2+b^2+2ab$

or $x(a+b) = (a+b)^2$

$\therefore \quad x = \dfrac{(a+b)^2}{a+b} \qquad \therefore \quad x = a+b.$

Example 21: Solve $(x+a)^3+(x+b)^3+(x+c)^3 = 3(x+a)(x+b)(x+c)$.

Sol. By transposition,

$$(x+a)^3+(x+b)^3+(x+c)^3-3(x+a)(x+b)(x+c) = 0$$

Factorising the left-hand side, we get

$$\tfrac{1}{2}\{(x+a)+(x+b)+(x+c)\}\,[\{(x+a)-(x+b)\}^2+\{(x+a)-(x+c)\}^2+\{(x+b)-(x+c)\}^2] = 0.$$

On simplification, we get

$$\tfrac{1}{2}(3x+a+b+c)\{(a-b)^2+(a-c)^2+(b-c)^2\} = 0.$$

Either the first factor $= 0$ or the second $= 0$.

but the second being a *constant* cannot be equal to zero.

Hence $$(3x+a+b+c) = 0$$

$$\therefore \qquad x = -\frac{a+b+c}{3}.$$

Example 22: Solve $\dfrac{x+a}{b+c}+\dfrac{x+b}{c+a}+\dfrac{x+c}{a+b}+3 = 0.$

Sol. Decomposing 3 into $1+1+1$ and associating *one* with each fraction, we have

$$\left(\frac{x+a}{b+c}+1\right)+\left(\frac{x+b}{c+a}+1\right)+\left(\frac{x+c}{a+b}+1\right) = 0$$

or $$\frac{x+a+b+c}{b+c}+\frac{x+a+b+c}{c+a}+\frac{x+a+b+c}{a+b} = 0$$

or $$(x+a+b+c)\left\{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right\} = 0.$$

Since the 2nd factor is a constant, $x+(a+b+c) = 0$

$$\therefore \qquad x = -(a+b+c).$$

Example 23: Solve $(x-2a)^3+(x-2b)^3 = 2(x-a-b)^3$.

Sol. Put $x-2a = p$ and $x-2b = q$. Then $x-a-b = \dfrac{p+q}{2}$.

Substituting in the given equation, we have

$$p^3+q^3 = 2\left(\frac{p+q}{2}\right)^3$$

or $$p^3+q^3 = 2\times\frac{p^3+q^3+3pq(p+q)}{8}$$

or $$4p^3+4q^3 = p^3+q^3+3pq(p+q)$$
or $$3p^3+3q^3 = 3pq(p+q)$$
or $$p^3+q^3 = pq(p+q)$$
or $$(p+q)(p^2-pq+q^2) = pq(p+q)$$
or $$(p+q)(p^2-pq+q^2-pq) = 0$$
or $$(p+q)(p-q)^2 = 0.$$

But $p-q = x-2a-x+2b = 2(b-a)$ which is not zero

$\therefore \quad p+q = 0$

$x-2a+x-2b = 0$

$\therefore \quad x = a+b.$

Example 24: Solve $\frac{mx+b}{x+b}+\frac{nx+a}{x+a} = m+n.$

Sol. By transposition, $\left(\frac{mx+b}{x+b}-m\right)=\left(n-\frac{nx+a}{x+a}\right)$

or $\frac{mx+b-mx-mb}{x+b} = \frac{nx+na-nx-a}{x+a}$

or $\frac{b-mb}{x+b} = \frac{na-a}{x+a}.$

Multiplying cross-wise, we have

$(x+a)(b-mb) = (x+b)(na-a)$

or $xb-xmb+ab-abm = xna-xa+abn-ab.$

By transposition,

$xb-xmb-xna+xa = abn-ab-ab+abm$

or $x(b-mb-na+a) = ab(n-2+m)$

$\therefore \quad x = \frac{ab(m+n-2)}{a+b-mb-na}.$

EXERCISE 126.

Solve the equation:

1. $a^2(x-a)+b^2(x-b) = abx.$
2. $b(x-2a)+a(x-2b) = (a-b)^2.$
3. $x(x-a)+x(x-b) = 2(x-a)(x-b).$
4. $(x-a)^2-(x-b)^2 = 0.$
5. $(x-a)^2+(x-b)^2 = 2(x-a-b)^2.$
6. $\frac{a-x}{a}+\frac{2a-x}{2a} = \frac{3a-x}{3a}.$
7. $\frac{a}{bx}+\frac{b}{ax} = a^2+b^2.$
8. $\frac{c-x^2}{bx}-\frac{b-x}{c} = \frac{c-x}{b}-\frac{b-x^2}{cx}.$
9. $\frac{bx}{a}+\frac{ax}{b}+\frac{c^2x}{ab} = a^2+b^2+c^2.$

10. $(x-a)^3+(x-b)^3+(x-c)^3 = 3(x-a)(x-b)(x-c)$.

11. $(x-a)^3+(b-x)^3+(a-b)^3 = 0$.

12. $\dfrac{x-b}{a+1}+\dfrac{x-a}{b+1}+\dfrac{x-1}{a+b}=3$.

13. $\dfrac{x-ab}{b+a}+\dfrac{x-bc}{b+c}+\dfrac{x-ca}{c+a}=a+b+c$.

14. $\dfrac{bc(ax-1)}{b+c}+\dfrac{ca(bx-1)}{c+a}+\dfrac{ab(cx-1)}{a+b}=a+b+c$.

15. $\dfrac{x-a^2}{b+c}+\dfrac{x-ab}{c+a}+\dfrac{x-ac}{a+b}=3a$.

16. $\dfrac{x-a^3}{b^2-bc+c^2}+\dfrac{x-b^3}{c^2-ca+a^2}+\dfrac{x-c^3}{a^2-ab+b^2}=2(a+b+c)$.

[*Hint*: $2(a+b+c) = (a+b)+(b+c)+(c+a)$.]

17. $\dfrac{a+b}{x-a-b}=\dfrac{a}{x-a}+\dfrac{b}{x-b}$.

18. $\dfrac{ax}{ax-1}+\dfrac{bx}{bx-1}=2$.

19. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+b}+\dfrac{1}{x}$.

20. $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{1}{ac-ax}$.

21. $\dfrac{a}{ax+1}+\dfrac{b}{bx+1}=\dfrac{2c}{cx+1}$.

22. $\dfrac{1}{x+a}+\dfrac{1}{x-a}=\dfrac{2}{x^2-a^2}$.

23. $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b$.

24. $\dfrac{x+2a}{x-2a}-\dfrac{x-2b}{x+2b}=\dfrac{4(a+b)}{x}$.

25. $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$.

26. $(x+a+b-c)^3+(x+b+c-a)^3=2(x+b)^3$.

27. $(x-27)^3+(x-23)^3=2(x-25)^3$.

28. $\left(\dfrac{x+a}{x+b}\right)^3=\dfrac{x+2a-b}{x+2b-a}$.

29. $16\left(\dfrac{a-x}{a+x}\right)^3=\dfrac{a+x}{a-x}$.

CHAPTER XVI

SIMULTANEOUS EQUATIONS—THREE UNKNOWNS

1. Simultaneous equations involving three unknown quantities.

Example 1: Solve $3x+y+2z=13$ ... ... (i)

$x+4y+3z=14$ ... ... (ii)

$2x+3y+4z=16$ ... ... (iii)

Sol. Multiplying (i) by 3 and (ii) by 2, we have

$9x+3y+6z=39$ ... ... (iv)

$2x+8y+6z=28$ ... ... (v)

Subtracting (v) from (iv), we have

$7x-5y=11$ ... ... (vi)

Again multiplying (ii) by 4 and (iii) by 3, we have

$4x+16y+12z=56$ ... ... (vii)

$6x+9y+12z=48$ ... ... (viii)

Subtracting (viii) from (vii), we have

$-2x+7y=8$ ... (ix)

Multiplying (vi) by 7 and (ix) by 5, we have

$49x-35y=77$ ... ... (x)

$-10x+35y=40$ ... ... (xi)

Adding (x) and (xi), we have

$39x=117$

or $x=3$.

Substituting this value in (vi), we have

$21-5y=11$, $\therefore \quad y=2$.

Substituting the values of x and y in (i), we have

$9+2+2z=13$, $\therefore \quad z=1$.

Thus $x=3$, $y=2$ and $z=1$.

Example 2: Solve $y+z=17$ ... ... (i)

$z+x=14$ ... ... (ii)

$x+y=11$ ... ... (iii)

Sol. Adding (i), (ii) and (iii), we have

$2(x+y+z)=42$

$x+y+z=21$. ... ... (iv)

Subtracting (i), (ii) and (iii), respectively from (iv), we have

$x=4$, $y=7$, $z=10$.

Example 3: Solve $\frac{xy}{x+y}=\frac{1}{3}$ ... ... (i)

$$\frac{yz}{y+z}=\frac{1}{5} \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\frac{zx}{z+x}=\frac{1}{4} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Sol. Taking the reciprocals of (i), (ii) and (iii), we have

$$\frac{x+y}{xy}=3, \text{ or } \frac{1}{y}+\frac{1}{x}=3 \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$\frac{y+z}{yz}=5, \text{ or } \frac{1}{z}+\frac{1}{y}=5 \quad \ldots \quad \ldots \quad \text{(v)}$$

$$\frac{z+x}{zx}=4, \text{ or } \frac{1}{x}+\frac{1}{z}=4 \quad \ldots \quad \ldots \quad \text{(vi)}$$

Adding (iv), (v) and (vi), we get

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=12 \quad \text{or} \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6 \quad \ldots \quad \text{(vii)}$$

Subtracting (v), (vi) and (iv) respectively from (vii),

we have $\frac{1}{x}=1, \frac{1}{y}=2$ and $\frac{1}{z}=3$

$\therefore \quad x=1, y=\frac{1}{2}$ and $z=\frac{1}{3}$.

Example 4: Solve $x+ay+a^2z=a^3$ ... ... (i)

$x+by+b^2z=b^3$ ... ... (ii)

$x+cy+c^2z=c^3$ ... ... (iii)

Sol. Subtracting (ii) from (i), we have

$$y(a-b)+z(a^2-b^2)=a^3-b^3$$

or $$y+z(a+b)=a^2+ab+b^2 \quad \text{(iv)}$$

Subtracting (iii) from (ii), we have

$$y(b-c)+z(b^2-c^2)=b^3-c^3$$

or $$y+z(b+c)=b^2+bc+c^2 \quad \text{(v)}$$

Subtracting (v) from (iv), we have

$$z(a-c)=(a^2-c^2)+b(a-c)$$

or $$z=a+c+b$$

$$=a+b+c.$$

Subtituting the value of z in (iv), we have

$$y+(a+b+c)(a+b)=a^2+ab+b^2$$

or $$y=-(ab+bc+ca).$$

Substituting the values of y and z in (i), we have

$$x+a(ab+bc+ca)+a^2(a+b+c) = a^3$$

or $$x = abc.$$

EXERCISE 127.

Solve the following equations:

1. $\left.\begin{aligned} x-y+z &= 1 \\ x-2y+4z &= 8 \\ x-3y+9z &= 27. \end{aligned}\right\}$

2. $\left.\begin{aligned} x+2y+3z &= 14 \\ 2x+3y+4z &= 20 \\ 3x+y+6z &= 23. \end{aligned}\right\}$

3. $\left.\begin{aligned} x-3y+2z &= 1 \\ 2x-4y+z &= 4 \\ 3x+y-5z &= -11. \end{aligned}\right\}$

4. $\left.\begin{aligned} x-y-z &= -2 \\ x+y+2z &= 7 \\ 5x+6y+4z &= 32. \end{aligned}\right\}$

5. $\left.\begin{aligned} x+2y+3z &= 22 \\ 2x-3y+z &= 5 \\ 3x+4y-2z &= 7. \end{aligned}\right\}$

6. $\left.\begin{aligned} x+2y+3z &= 20 \\ 2x+3y-5z &= -7 \\ 4x-5y+7z &= 21. \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= 6 \\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z} &= 8 \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z} &= 10 \end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{z} &= 2 \\ \frac{3}{x}+\frac{4}{y}-\frac{5}{z} &= 18 \\ \frac{2}{x}+\frac{3}{y}+\frac{4}{z} &= 38 \end{aligned}\right\}$

9. $x+y = 20,\ y+z = 16,\ z+x = 18.$

10. $x+y = 2a,\ y+z = 2b,\ z+x = 2c.$

11. $\frac{1}{x}+\frac{1}{y} = 1,\ \frac{1}{y}+\frac{1}{z} = 2,\ \frac{1}{z}+\frac{1}{x} = 4.$

12. $\frac{1}{x}+\frac{1}{y} = a,\ \frac{1}{y}+\frac{1}{z} = b,\ \frac{1}{z}+\frac{1}{x} = c.$

13. $\left.\begin{aligned} y+z-x &= 8 \\ z+x-y &= 12 \\ x+y-z &= 16. \end{aligned}\right\}$

14. $\left.\begin{aligned} x-3y+z &= -3 \\ y-3z+x &= 1 \\ z-3x+y &= -7. \end{aligned}\right\}$

15. $ay+bx = c,\ cx+az = b,\ bz+cy = a.$

[*Hint: Multiply first by c, second by b and third by a.*]

16. $\frac{x+y}{xy} = \frac{y+z}{yz} = \frac{z+x}{zx} = \frac{2}{5}.$

17. $\left.\begin{aligned} 2x+3y &= 5xy \\ 3y+4z &= 7yz \\ 4z+5x &= 9zx. \end{aligned}\right\}$

18. $\left.\begin{aligned} 3xy &= 4(x+y) \\ 2xz &= 3(x+z) \\ 5yz &= 12(y+z). \end{aligned}\right\}$

19. $\left.\begin{aligned} x+y &= axy \\ y+z &= byz \\ z+x &= czx \end{aligned}\right\}$

20. $\left.\begin{aligned} x-ay+a^2z &= a^3 \\ x-by+b^2z &= b^3 \\ x-cy+c^2z &= c^3 \end{aligned}\right\}$

21. $\left.\begin{aligned} x+ay+bcz &= a^2 \\ x+by+caz &= b^2 \\ x+cy+abz &= c^2. \end{aligned}\right\}$

22. $xy=20,\ yz=24,\ zx=30.$

[*Hint: Multiply the three equations and take the square root.*]

23. $\left.\begin{aligned} x(x+y+z) &= 60 \\ y(x+y+z) &= 75 \\ z(x+y+z) &= 90. \end{aligned}\right\}$

[*Hint: Add the three equations.*]

24. $\left.\begin{aligned} (x+y)(y+z) &= 96 \\ (y+z)(z+x) &= 80 \\ (z+x)(x+y) &= 120. \end{aligned}\right\}$

2. Problems involving simultaneous equations.

Example 5: A number consists of three digits whose sum is 17 ; the middle digit exceeds the sum of the other two by 1 ; if the digits be reversed, the number is diminished by 396. Find the number.

Sol. Let x, y and z be the digits in hundreds, tens and units place respectively.

Since the sum of the digits $=x+y+z$,

$$\therefore \quad x+y+z=17 \quad \ldots \quad \ldots \quad \text{(i)}$$

Since the middle digit y exceeds the sum of the other digits x and z by 1,

$$\therefore \quad y=x+z+1$$

$$\text{or} \quad x-y+z=-1 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Since the actual number is $100x+10y+z$ and it becomes $100z+10y+x$ when the digits are reversed,

$$\therefore \quad (100x+10y+z)-(100z+10y+x)=396.$$

Simplifying and dividing both sides by 99, we have

$$x-z=4 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Adding (i) and (ii), we have

$$2x+2z=16$$

$$\text{or} \quad x+z=8 \quad \ldots \quad \ldots \quad \text{(iv)}$$

From equations (iii) and (iv), we have

$$x=6 \text{ and } z=2.$$

Substituting the values of x and z in (i), we have

$$6+y+2=17, \qquad \therefore \quad y=9.$$

Hence the required number is 692.

EXERCISE 128.

1. The sum of three numbers is 77. Three times the first added to the third is less than four times the second by 3. The excess of the third over the second is 18. Find the numbers.

2. The sum of two numbers is increased by 10 and the result divided by 8 ; the quotient is less than their difference by 1. Also $\frac{1}{2}$ of the greater is greater than $\frac{1}{4}$ of the less by 6. Find the two numbers.

3. Find three numbers such that the excess of the first over the second is 8, the excess of the second over the third is 4, and the sum of the first and the third is 28.

4. Find three numbers x, y, z such that x increasd by $\frac{1}{2}$ of y, y increased by $\frac{1}{8}$ of z, and z increased by $\frac{1}{4}$ of x may each be 250.

5. A number of two digits when divided by 5 gives a certain quotient and a remainder 4 ; when divided by 8, gives another quotient and a remainder 7. The digit in the tens place is equal to the second quotient diminished by 2 and the other digit is less than the first quotient by 6. Find the number.

6. A certain number of 3 digits exceeds the sum of its digits by 414 and the number formed by reversing the digits exceeds the same sum by 612. If the given number and one formed by reversing the digits be in the ratio of 71 : 104, find the number.

7. A number consists of 3 digits whose sum is 18. The digit in the hundreds place is $\frac{1}{8}$ of the number formed by the remaining two digits, and the digit in the units place is $\frac{1}{8}$ of the number formed by the remaining two digits. Find the number.

8. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two. The number formed by reversing the digits is greater than the original number by 99. Find the number.

3. Cross-multiplication Method.

We have seen that three independent equations are required if there are three unknowns whose values are to be found. If there are two equations with three unknowns, we can only find the ratio of their values.

Theorem: If $ax+by+c=0$ ... ... (i)
and $a'x+b'y+c'z=0$ ... ... (ii)

then $\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$.

Proof: Multiplying (i) by c' and (ii) by c we get

$ac'x+bc'y+cc'z=0$ ... ... (iii)
$a'cx+b'cy+cc'z=0$ ... ... (iv)

Subtraction (iv) from (iii), we have

$(ac'-a'c)x+(bc'-b'c)y=0$
$\therefore \quad (bc'-b'c)y=-(ac'-a'c)x=(ca'-c'a)x.$

Dividing both sides by $(bc'-b'c)(ca'-c'a)$, we have

$$\frac{y}{ca'-c'a}=\frac{x}{bc'-b'c}$$

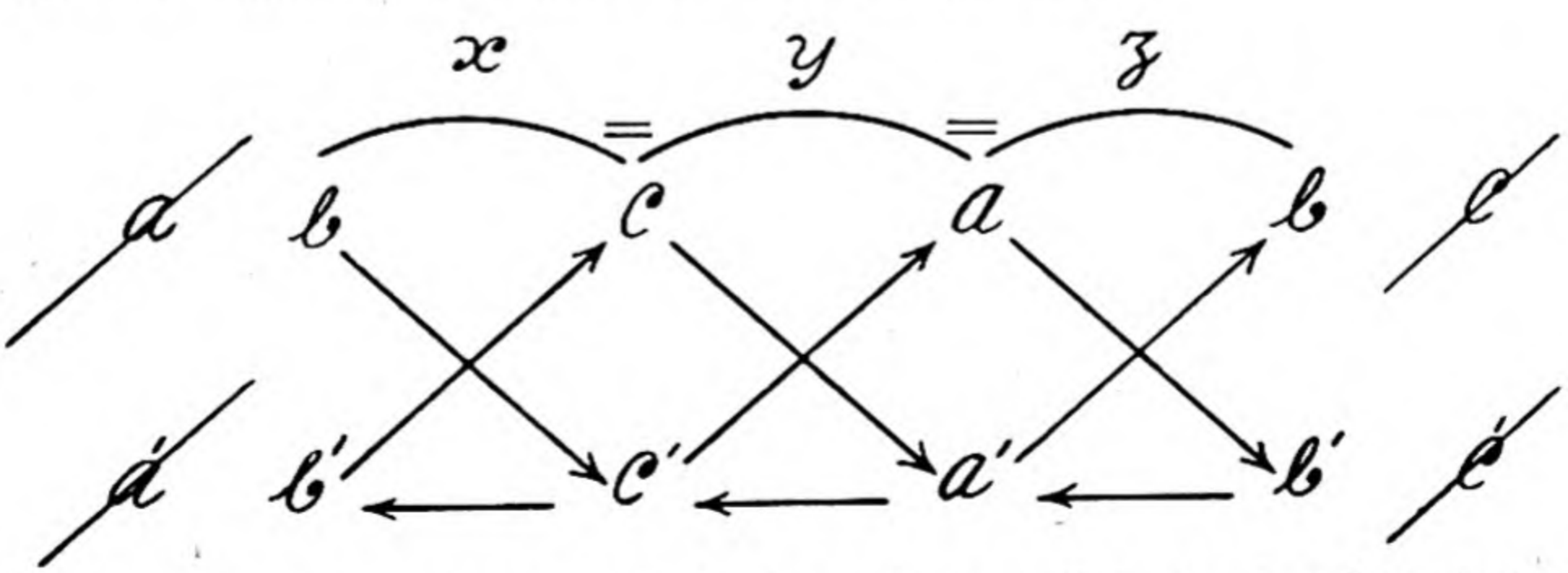
Similarly, $\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$.

$$\therefore \quad \frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \quad \ldots \quad \ldots \quad \text{(v)}$$

This is a very useful theorem and the following mechanical rule gives a convenient method of writing down the result at once.

Rule: (i) *Reduce the original equations to the standard form with 0 on the right hand side.*

(ii) *Write down the co-efficients of x, y, z in two rows and repeat the same so that there are six in each row.*



(iii) *Strike off the extreme co-efficients, leaving four in each row with 3 gaps.*

(iv) *Put down x, y, z over the 3 gaps.*

(v) *Multiply the co-efficients cross-wise as shown by the arrows and put down the denominators $bc'-b'c$, $ca'-c'a$, $ab'-a'b$ under x, y, z respectively.*

This is known as the rule of **Cross-multiplication.**

Cor. If $z = 1$, the equations (i) and (ii) become

$$ax+by+c = 0$$

and $$a'x+b'y+c' = 0$$

and (v) becomes

$$\frac{x}{bc'-b'c} = \frac{y}{ca'-c'a} = \frac{1}{ab'-a'b}.$$

From this we get $x = \dfrac{bc'-b'c}{ab'-a'b}$ and $y = \dfrac{ca'-c'a}{ab'-a'b}$.

NOTE: The rule of cross-multiplication is applicable only when two equations are of the form $ax+by+cz = 0$. The corollary gives us an elegant method of solving simultaneous equations of the first degree involving two variables.

Example 6: Find $x : y : z$, if

$$4x-5y+2z = 0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$2x-7y+4z = 0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. Arranging the co-efficients of x, y, z, we have

$$\overset{x}{\frown} = \overset{y}{\frown} = \overset{z}{\frown}$$

$$\begin{array}{cccccc} +4 & -5 & +2 & +4 & -5 & +2 \\ +2 & -7 & +4 & +2 & -7 & +4 \end{array}$$

By the rule of cross-multiplication,

$$\frac{x}{(-5)(4)-(-7)(2)} = \frac{y}{(2)(2)-(4)(4)} = \frac{z}{(4)(-7)-(2)(-5)}$$

or $$\frac{x}{-20+14} = \frac{y}{4-16} = \frac{z}{-28+10}$$

or $$\frac{x}{-6} = \frac{y}{-12} = \frac{z}{-18} \qquad \text{or} \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

$\therefore$ $x : y : z = 1 : 2 : 3.$

Example 7: Solve the equations

$$2x-3y+3z = 0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$x+2y+3z = 32 \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$x-4y+5x = 0 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Sol. From (i) and (iii), by the rule of cross-multiplication,

$$\frac{x}{(-3)(5)-(-4)(3)} = \frac{y}{(3)(1)-(5)(2)} = \frac{z}{(2)(-4)-(1)(-3)}$$

or $$\frac{x}{-15+12} = \frac{y}{3-10} = \frac{z}{-8+3}$$

or $$\frac{x}{-3} = \frac{y}{-7} = \frac{z}{-5}$$

or $$\frac{x}{3} = \frac{y}{7} = \frac{z}{5}.$$

Let $$\frac{x}{3} = \frac{y}{7} = \frac{z}{5} = k.$$

$\therefore$ $$x = 3k,\ y = 7k,\ z = 5k.$$

Substituting these values of x, y, z in (ii), we get

$$3k+14k+15k = 32$$

or $$32k = 32 \quad \text{or} \quad k = 1$$

$\therefore$ $$x = 3, \quad y = 7, \quad z = 5.$$

EXERCISE 129.

Find $x : y : z$, if

1. $\left.\begin{aligned} x-2y+z &= 0 \\ 9x-8y+3z &= 0. \end{aligned}\right\}$

2. $\left.\begin{aligned} 6x-8y+7z &= 0 \\ 2x-7y+11z &= 0. \end{aligned}\right\}$

3. $\left.\begin{aligned} 4x-13y+8z &= 0 \\ 7x+6y-9z &= 0. \end{aligned}\right\}$

4. $\left.\begin{aligned} ax+by+cz &= 0 \\ bx+cy+az &= 0. \end{aligned}\right\}$

Solve the equations by the method of Cross-multiplication:

5. $\left.\begin{aligned} 4x-2y-z &= 0 \\ 6x-y-4z &= 0 \\ x+y+z &= 25. \end{aligned}\right\}$

6. $\left.\begin{aligned} 5x+6y+8z &= 0 \\ 3x+4y+6z &= 0 \\ x+5y+16z &= 3. \end{aligned}\right\}$

7. $\left.\begin{aligned} x+2y+z &= 0 \\ 2x-5y-4z &= 0 \\ 4x+5y+6z &= 24. \end{aligned}\right\}$

8. $\left.\begin{aligned} 15x &= 10y = 6z \\ 3x+4y+5z &= 172. \end{aligned}\right\}$

9. $\left.\begin{aligned} 4x-13y+8z &= 0 \\ 7x+6y-9z &= 0 \\ \frac{6}{x}+\frac{8}{y}+\frac{20}{z} &= 8. \end{aligned}\right\}$

10. $\left.\begin{aligned} x-2y+z &= 0 \\ 2x-5y+4z &= 0 \\ 3x^2+4y^2+5z^2 &= 48. \end{aligned}\right\}$

11. $\left.\begin{aligned} 7x-5y+z &= 0 \\ 5x+2y-3z &= 0 \\ 2x^3+5y^3-z^3 &= 15. \end{aligned}\right\}$

12. $\left.\begin{aligned} x+y+z &= 0 \\ ax+by+cz &= 0 \\ \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b} &= 3. \end{aligned}\right\}$

13. $x-y=y-z=\dfrac{x+z}{6}=2$. **14.** $\left.\begin{aligned}\dfrac{y+z}{4}=\dfrac{z+x}{3}=\dfrac{x+y}{2}.\\ x+y+z=27.\end{aligned}\right\}$

4. Rule of cross-multiplication applied to Simultaneous Equations in 2 unknowns.

Example 8: Solve the equations

$$4x-5y+8=0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$2x-3y+6=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. From (i) and (ii), by rule of cross-multiplication,

$$\frac{x}{(-5)\times 6-(-3)\times 8}=\frac{y}{8\times 2-6\times 4}=\frac{1}{4\times(-3)-2\times(-5)}$$

or $$\frac{x}{-30+24}=\frac{y}{16-24}=\frac{1}{-12+10}$$

or $$\frac{x}{-6}=\frac{y}{-8}=\frac{1}{-2}$$

or $$\frac{x}{3}=\frac{y}{4}=1$$

$\therefore$ $$x=3,\ y=4.$$

EXERCISE 130.

1. $\left.\begin{aligned}3x+4y&=15\\ 6y-5x&=-6.\end{aligned}\right\}$

2. $\left.\begin{aligned}7y&=9x\\ 3y-4x+1&=0.\end{aligned}\right\}$

3. $\left.\begin{aligned}7x+4y&=5\\ 5x+6y&=2.\end{aligned}\right\}$

4. $\dfrac{x+5}{y+2}=\dfrac{5x+3}{4y+1}=2.$

5. $\left.\begin{aligned}3x+y-2p&=0\\ -5x+py-3&=0.\end{aligned}\right\}$

6. $\left.\begin{aligned}lx+my&=p\\ mx&=ly.\end{aligned}\right\}$

7. $x-ay+b=ax-y+b=1.$

8. If $x+5y=18$ and $3x+2y=41$, find the value of $x-8y$.

9. Find the values of x and y from the equation

$$(3x-2y+8)^2+(5x+13y-3)^2=0.$$

(*Hint*: As the sum of two squares $=0$, each $=0$.)

Example 9: Solve the equations

$$\frac{3}{x}+\frac{4}{y}=18 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$\frac{2}{x}+\frac{5}{y}=19 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. By transposition, we get

$$\frac{3}{x}+\frac{4}{y}-18=0 \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\frac{2}{x}+\frac{5}{y}-19=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

By the cross-multiplication method, we get

$$\frac{\frac{1}{x}}{(-76)-(-90)}=\frac{\frac{1}{y}}{(-36)-(-57)}=\frac{1}{15-8}$$

or $$\frac{\frac{1}{x}}{14}=\frac{\frac{1}{y}}{21}=\frac{1}{7}$$

$\therefore$ $$\frac{1}{x}=\frac{14}{7}=2 \text{ and } \frac{1}{y}=\frac{21}{7}=3$$

$\therefore$ $$x=\frac{1}{2} \text{ and } y=\frac{1}{3}.$$

Example 10: Solve the equations

$$5x+4y=9xy \quad \ldots \quad \ldots \quad \text{(i)}$$

$$3x+2y=5xy \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. Dividing (i) and (ii) by xy, we get

$$\left.\begin{aligned}\frac{5}{y}+\frac{4}{x}&=9\\ \frac{3}{y}+\frac{2}{x}&=5\end{aligned}\right\}$$

or $$\frac{4}{x}+\frac{5}{y}-9=0 \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\frac{2}{x}+\frac{3}{y}-5=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

By the cross-multiplication method, we get

$$\frac{\frac{1}{x}}{-25+27}=\frac{\frac{1}{y}}{-18+20}=\frac{1}{12-10}$$

or $$\frac{\frac{1}{x}}{2}=\frac{\frac{1}{y}}{2}=\frac{1}{2}$$

$\therefore \quad \frac{1}{x} = \frac{1}{y} = 1$

$\therefore \quad x = 1$ and $y = 1$.

EXERCISE 131.

Solve the equations by the method of cross-multiplication:

1. $\left.\begin{aligned} \frac{3}{x}+\frac{4}{y} &= 18 \\ \frac{2}{x}+\frac{5}{y} &= 19. \end{aligned}\right\}$

2. $\left.\begin{aligned} \frac{3}{x}-\frac{2}{y} &= -4 \\ \frac{2}{x}+\frac{3}{y} &= 19. \end{aligned}\right\}$

3. $\left.\begin{aligned} \frac{3}{x}-\frac{4}{y} &= 5 \\ \frac{4}{x}-\frac{5}{y} &= 6. \end{aligned}\right\}$

4. $\left.\begin{aligned} \frac{3}{x}+\frac{5}{y} &= 11 \\ \frac{2}{x}-\frac{3}{y} &= -5\tfrac{1}{8}. \end{aligned}\right\}$

5. $\left.\begin{aligned} \frac{1}{3x}+\frac{1}{5y} &= 2 \\ \frac{1}{4x}+\frac{1}{6y} &= \frac{19}{12} \end{aligned}\right\}$

6. $\left.\begin{aligned} 2x+\frac{3}{y} &= 5 \\ 5x-\frac{2}{y} &= 3. \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{a}{x}+\frac{b}{y} &= m \\ \frac{b}{x}+\frac{a}{y} &= n. \end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{2}{ax}+\frac{3}{by} &= 5 \\ \frac{5}{ax}-\frac{2}{by} &= 3. \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{8}{x+y}+\frac{6}{x-y} &= 25 \\ \frac{4}{x+y}-\frac{3}{x-y} &= 10. \end{aligned}\right\}$

10. $\left.\begin{aligned} 10y+14x &= 9xy \\ 4x-6y &= 7xy. \end{aligned}\right\}$

11. $\left.\begin{aligned} 2y-x &= 4xy \\ \frac{4}{y}-\frac{3}{x} &= 9. \end{aligned}\right\}$

12. $\frac{2+5x}{3x} = \frac{y+4}{2} = \frac{2xy+9x+2}{6x}$.

5. Other Simultaneous Equations adaptable to Rule of Cross-multiplication.

Example 11: Solve the equations

$$4x+2y-3z=2 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$3x+4y-2z=10 \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$2x-5y+4z=5 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Sol. Multiplying (i) by 5 and then subtracting (ii) from it,

$$17x+6y-13z=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

Multiplying (iii) by 2 and then subtracting it from (ii),

$$-x+14y-10z=0 \quad \ldots \quad \ldots \quad \text{(v)}$$

By the cross-multiplication method from (iv) and (v), we get

$$\frac{x}{-60+182}=\frac{y}{13+170}=\frac{z}{238+6}$$

or $$\frac{x}{122}=\frac{y}{183}=\frac{z}{244}$$

or $$\frac{x}{2}=\frac{y}{3}=\frac{z}{4}=k \text{ (Let each be equal to } k.)$$

Then $$x=2k,\ y=3k \text{ and } z=4k.$$

Substituting these values in (i), we get

$$8k+6k-12k=2 \qquad \text{or} \qquad k=1$$

$$x=2,\quad y=3,\quad z=4.$$

EXERCISE 132.

Solve the equations:

1. $\left.\begin{aligned} x+y+z&=1\\ 2x+3y+z&=4\\ 4x+9y+z&=16. \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x-3y+2z&=8\\ 3x-4y+5z&=6\\ -6x+5y+7z&=-1. \end{aligned}\right\}$

3. $\left.\begin{aligned} 3x-2y+4z&=5\\ 5x-7y+8z&=8\\ 11x-10y+16z&=21. \end{aligned}\right\}$

4. $\left.\begin{aligned} x+5y-4z&=5\\ 3x-2y+2z&=14\\ -10x+8y+z&=6. \end{aligned}\right\}$

5. $x+2y+3z=3x+y+2z=2x+3y+z=6.$

6. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=6\\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=8\\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=10. \end{aligned}\right\}$

7. $\left.\begin{aligned} 3x+\frac{4}{y}+\frac{5}{z}&=-13\\ 4x+\frac{5}{y}+\frac{3}{z}&=7.\\ 5x+\frac{3}{y}+\frac{4}{z}&=6. \end{aligned}\right\}$

SECTIONAL REVISION IV

TEST PAPERS

PAPER 1

1. Explain the terms:

(i) duplicate ratio and sub-duplicate ratio,
(ii) triplicate ratio and sub-triplicate ratio.

2. (i) Write in shorter form
(a) 2,70,000 (b) $2{\cdot}3 \times 2{\cdot}3 \times 2{\cdot}3 \times 1{,}000{,}000$.
(ii) Simplify (a) $x^0 \times 1$ (b) $x^0 \times x$.

3. Prove that a ratio of greater inequality is increased and a ratio of less inequality is diminished by subtracting the same positive quantity from both the terms.

4. If $\frac{6a-5b}{6a+5b} = \frac{6c-5d}{6c+5d}$, show that $\frac{a}{b} = \frac{c}{d}$.

5. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that each is equal to $\frac{a+c+e}{b+d+f}$

6. Solve the equations:

(i) $\frac{a}{ax+1} + \frac{b}{bx+1} = \frac{2c}{cx+1}$. (ii) $9\left(\frac{x-a}{x+a}\right)^3 = \frac{x-a}{x+a}$.

7. Solve graphically the equations: $\left.\begin{matrix} x+y=1 \\ x-y=5. \end{matrix}\right\}$

PAPER 2

1. Find the compounded ratio of the following three ratios: $3a : 4b$, $b^2 : 5ac$ and $5c^2 : 6ab$.

2. If $6+x : 8+x$ be the sub-duplicate ratio of $4 : 9$, find x.

3. If $a : b = c : d$, then prove $\frac{4a+3b}{4a-3b} = \frac{4c+3d}{4c-3d}$.

4. If $\frac{x}{a+b-c} = \frac{y}{b+c-a} = \frac{z}{c+a-b}$, show that each of these fractions $= \frac{x+y+z}{a+b+c}$.

5. Solve the equations

(i) $(x^2+5x-4)(x^2+3x-2) = (x^2-5x+4)(x^2-3x+2)$.

(ii) $\frac{6x+5}{8x+3} = \frac{3x+2.}{4x+1}$

6. Simplify : $\frac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$.

7. If $\frac{3}{y-5}+\frac{4}{2-x} = \frac{14}{(x-2)(y-5)}$, find the value of $\frac{x}{y}$.

PAPER 3

1. What must be added to the terms of 5 : 8 to make it equal to 9 : 10 ?

2. If $x : y = 5 : 6$, find the value of $\frac{3x+4y}{4x+3y}$.

3. Two vessels contain a mixture of wine and water in the ratio of 5 : 3 and 3 : 1. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 2 : 1 ?

4. If $x = \frac{ab}{a+b}$, find the value of $\frac{x+a}{x-a}+\frac{x+b}{x-b}$.

5. Solve the equations :

(i) $\frac{5-6x}{3+4x} = \frac{2+3x}{5-2x}$.

(ii) $\frac{(x+2)(x+3)}{(x+4)(x+5)} = \frac{x+5}{x+9}$.

6. The equation of a line is $5x+4y = 3$. Put it in the form $y = mx+c$ and state its slope and the intercept on y-axis.

7. Solve the following equations by the method of cross-multiplication :

$$\frac{3}{x}+\frac{2}{y}-2 = 0 \qquad \frac{4}{x}+\frac{3}{y}-2\tfrac{5}{6} = 0.$$

PAPER 4

1. (i) One factor of $a+b$ is $a^{\frac{1}{3}}+b^{\frac{1}{3}}$, write down the other.

(ii) Find the continued product of

$$x^{\frac{1}{4}}-y^{\frac{1}{4}}, \quad x^{\frac{1}{4}}+y^{\frac{1}{4}}, \quad x^{\frac{1}{2}}+y^{\frac{1}{2}}, \quad x+y.$$

2. If $7x-6y = 5x+2y$, find the value of $x : y$.

3. The bases of two triangles are in the ratio of 5 : 6 and their altitudes are in the ratio of 8 : 9. Find the ratio between their areas.

4. If l, m, n are in continued proportion and if $l(m-n) = 2m$, prove that $l-n = 2\left(\frac{l+m}{l}\right)$.

5. Solve the equations :

(i) $\left(\frac{2x-3}{2x-5}\right)^2 = \frac{x-3}{x-5}$.

(ii) $(x-3)(x+4)(x+7) = (x+2)(x+1)(x+5)$.

6. Draw the graph of the function $\frac{2x-3}{6}$.

7. Solve the equations :
$$\left.\begin{aligned} \frac{3}{x}+\frac{2}{y} &= \frac{17}{12} \\ \frac{2}{z}-\frac{1}{y} &= \frac{2}{3} \\ \frac{1}{x}+\frac{1}{z} &= \frac{3}{4}. \end{aligned}\right\}$$

PAPER 5

1. Prove that $\frac{0}{n} = 0$, $\frac{m}{\infty} = 0$ and $\frac{m}{0} = \infty$, where m and n are finite quantities.

2. What must be added to each of the numbers 12,16, 21 so that the resulting numbers may be in continued proportion ?

3. If $\frac{2a^2+3ab+4b^2}{5a^2+6ab+7b^2} = \frac{2c^2+3cd+4d^2}{5c^2+6cd+7d^2}$, prove that $a : b = c : d$.

4. If $\frac{a}{y+z-x} = \frac{b}{z+x-y} = \frac{c}{x+y-z}$, show that
$$\frac{x}{b+c} = \frac{y}{c+a} = \frac{z}{a+b}.$$

5. Solve the equations:

(i) $81\left(\frac{a-x}{a+x}\right)^3 = \frac{a-x}{a+x}$. (ii) $\frac{x^2-5x+7}{5x-7} = \frac{x^2-3x+5}{3x-5}$.

6. Simplify $\frac{\left\{(x^m)^{\frac{1}{r}} . (x^n)^{-\frac{1}{p}}\right\}^{pr}}{\left\{\sqrt[n]{x^p} . \sqrt[m]{x^r}\right\}^{mn}}$

PAPER 6

1. Write down the values of:

(i) $(81)^{\frac{3}{4}}$, (ii) $(81)^{1\cdot25}$, (iii) $\sqrt[5]{(32)^4}$ (iv) $(100)^{-1\cdot5}$.

2. Find two numbers of the same two digits which are in the ratio of 8 : 3.

3. If $\frac{pa+qb}{pc+qd} = \frac{b^2c}{d^2a}$, prove that $a : b = c : d$.

4. If $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$, shew that $\frac{abc}{xyz} = \frac{a^3+b^3+c^3}{x^3+y^3+z^3}$.

5. Solve the equations:

(i) $\frac{x^2-3x+4}{2x^2-5x+1} = \frac{x^2+3x-4}{2x^2+5x-1}$.

(ii) $\frac{x^3+3x}{3x^2+1} = \frac{91}{37}$.

6. Solve graphically the equations $\left.\begin{aligned} y &= 2x-1 \\ 6y &= 2x-3. \end{aligned}\right\}$

7. The average weights of infants for the first 12 months are given below in kilograms:

Months	1	2	3	4	5	6	7	8	9	10	11	12
Weights	3·6	4·3	5	5·7	6·3	7	7·5	8·1	8·55	9	9·45	9·9

Draw the graph and compare the weight of any infant you know.

PAPER 7

1. Find the value of $\frac{x^2+2x-35}{x^2+4x-21}$ when $x=-7$.

2. If $1+x$, $2+x$, $4+x$ are in continued proportion, find x.

3. If $a:b=c:d$, prove that $(6a-5c)(6b+5d) = (6a+5c)(6b-5d)$.

4. If $\frac{x}{a}=\frac{y}{b}$, prove that $(x^2+y^2)(a^2+b^2)=(ax+by)^2$.

5. Solve the equations:

(i) $\frac{x^2+x-30}{x^2-25}=2$.

(ii) $\frac{3x-2}{x-1}+\frac{6x-13}{2x-5}=\frac{3x-5}{x-2}+\frac{6x-7}{2x-3}$.

6. Solve graphically and verify the solution of
$$\left.\begin{aligned}\frac{x+y}{3}&=5\\ 2x-3y&=5.\end{aligned}\right\}$$

PAPER 8

1. (i) Multiply $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

(ii) Divide $x-y$ by $x^{\frac{1}{4}}-y^{\frac{1}{4}}$.

2. The soldiers in two armies when met in a battle were in the ratio of 5 : 3; their respective losses were in the ratio of 3 : 2 and their survivors as 69 : 41. If the number of survivors in the smaller army be 16,400, find the original number of soldiers.

3. Find (i) the mean proportional between 16, 49;
(ii) the fourth proportional to 12, 15, 18;
and (iii) the third proportional to 24, 30.

4. If a, b, c, d are in continued proportion, show that $ab+bc+cd$ is the mean proportional between $a^2+b^2+c^2$ and $b^2+c^2+d^2$.

5. Solve the equations:

(i) $\frac{2x+7}{x+2}+\frac{3x+13}{x+3}=\frac{5x+27}{x+4}$.

(ii) $\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}$.

6. Find the speed of a train if 90 minutes are saved in 150 km when the speed is increased by 5 km an hour.

7. Find the vertices of the triangle formed by the straight lines

$$\left.\begin{aligned} 8x+5y+7&=0\\ 3x+7y-23&=0\\ 5x-2y-11&=0. \end{aligned}\right\}$$

OBJECTIVE TYPE ITEMS—IV

1. Rewrite the following, filling up the blanks:

(i) The abscissa of every point on y-axis is.....

(ii) The equation of the first degree in x and y always represents a.....

(iii) A ratio compounded by itself is called a.....

(iv) The square of the mean proportional is equal to ..... of the other two quantities.

(v) A ratio of less inequality is.....by adding the same number to both its terms.

(vi) The rule of cross-multiplication can be used in solving simultaneous equations of the first degree involving three variables, provided .....

(vii) $(2,-1)$ is a point on the graph of $3x-7y=$ .....

2. For each question write the capital letter which stands by the correct answer in the given answer box.

(i) The mean proportional between $\frac{3p}{5q}$ and $\frac{5p}{3q}$ is

A. $\frac{3p}{5q}+\frac{5q}{3q}$. B. $\frac{3p}{5q}-\frac{5p}{3q}$.

C. $\frac{3p}{5q}\div\frac{5p}{3q}$. D. $\frac{3p}{5q}\times\frac{5p}{3q}$.

E. $p\div q$.

(ii) If $\frac{a^2+b^2}{a^2-b^2}=\frac{41}{9}$, then $\frac{a}{b}$ is equal to

A. $\frac{\sqrt{41}}{3}$. B. $\frac{50}{32}$. C. $\frac{5}{4}$. D. $\frac{16}{25}$.

(iii) The ratio $p : q$ is the same as

A. $ap : bq$ B. $p+k : q+k$.
C. $p-k : q-k$. D. $kp : kq$.
E. $p^2 : q^2$.

(iv) The slope of the straight line represented by $7x-3y=8$ is

A. 7. B. -3. C. $\frac{7}{3}$. D. $\frac{3}{7}$.

(v) Any number, other than zero, raised to a power of zero is equal to

A. zero. B. infinite.
C. number itself D. one.

3. Rewrite the following such that there is no x in the numerators of the fractions, keeping the denominators same.

$$\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}.$$

4. Which of the following points lie on the graph of the equation $7x-4y=2$?

(i) (7, -4) (ii) (-4, 7).
(iii) (2, 3). (iv) (2, 11).

5. In the space provided against each item in column I, write the corresponding letter of the equal quantity from column II.

Column I		*Column II*
..... (i)	$a^m \div a^n$	A. $a^{m \div n}$
		B. $a^{m \times n}$
..... (ii)	$a^m \times a^n$	C. a^{m+n}
		D. a^{m-n}
.....(iii)	$(a^m)^n$	E. a^{m^n}
.....(iv)	$\sqrt[n]{a^m}$	F. a^{n^m}

CHAPTER XVII

INVOLUTION AND EVOLUTION

1. Involution.

It is the process of finding the expansion of expressions when raised to different powers.

First let us study the continued product of certain expressions from which expansion of powers can be derived.

By actual multiplication,

$(x+a)\ (x+b) = x^2+(a+b)\ x+ab$ and

$(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2+(ab+bc+ca)x+abc.$

Here we observe that

(i) the degree of the product is the same as the number of factors ;

(ii) the first term is the product of the first terms of the factors ;

(iii) the co-efficient of the second term is the sum of the second terms of the factors ;

(iv) the co-efficient of the third term is the sum of the products of the second terms of the factors, taken two at a time ;

(v) the last term is the product of the second terms of the factors.

Putting Σ for 'the sum of like terms', we have

$(x+a)\ (x+b)\ (x+c) = x^3+x^2\Sigma a+x\Sigma ab+abc.$

Similarly,

$(x+a)(x+b)(x+c)(x+d) = x^4+x^3\Sigma a+x^2\Sigma ab+x\Sigma abc+abcd.$

Example 1: Find the continued product of

$(x+1)(x+2)(x+3)(x-4).$

Sol. Putting a for 1, b for 2, c for 3 and d for -4. we get

the product $= x^4+x^3\Sigma a+x^2\Sigma ab+x\Sigma abc+abcd.$

$\Sigma a = 1+2+3-4 = 2,$

$\Sigma ab = 2+3-4+6-8-12 = -13,$

$\Sigma abc = 6-8-24-12 = -38,$

$abcd = -24,$

$\therefore$ the product $= x^4+2x^3-13x^2-38x-24.$

Example 2: Expand $(x+a)^4$.

$$(x+a)^4 = (x+a)(x+a)(x+a)(x+a)$$

Sol. In this case the second term of each factor is a

The co-efficient of x^3 in the product $= a+a+a+a = 4a$,

the co-efficient of x^2 in the product

$= a.a+a.a+a.a+a.a+a.a+a.a = 6a^2$,

the co-efficient of x in the product

$= a.a.a+a.a.a+a.a.a+a.a.a = 4a^3$,

the last term $= a.a.a.a = a^4$.

$\therefore \quad (x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4.$

EXERCISE 138.

Find the continued product of:

1. $(x-1)(x-2)(x+4)$. **2.** $(x-2)(x+3)(x+4)$,

3. $(x+2)(x+3)(x+4)$. **4.** $(x-2)(x-3)(x-4)$.

5. $(x-a)(x-b)(x-c)$. **6.** $(x+1)(x+2)(x-4)(x-5)$.

7. $(x-2)(x-3)(x+4)(x+5)$.

8. $(x-a)(x-b)(x-c)(x-d)$.

Expand

9. $(x+a)^5$. **10.** $(x+a)^6$.

11. $(x-a)^4$. **12.** $(x-a)^5$.

2. Expansion of powers of Binomials.

We already know that

$$\left.\begin{aligned}(x+a)^3 &= x^3+3x^2a+3xa^2+a^3\\ (x-a)^3 &= x^3-3x^2a+3xa^2-a^3\end{aligned}\right\}$$

$$\left.\begin{aligned}(x+a)^4 &= x^4+4x^3a+6x^2a^2+4xa^3+a^4\\ (x-a)^4 &= x^4-4x^3a+6x^2a^2-4xa^3+a^4\end{aligned}\right\}$$

$$\left.\begin{aligned}(x+a)^5 &= x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5\\ (x-a)^5 &= x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5.\end{aligned}\right\}$$

From the above cases, we observe that:

(i) The number of terms in the expansion is greater than the index of the binomial by one.

(ii) The first and the last terms in the expansion are respectively x and a, each raised to the same power as the binomial.

(iii) In each successive term, the index of x decreases and that of a increases by one.

(iv) The sum of the indices of x and a in any term is equal to the index of the binomial.

(v) The co-efficient of the first term is one, and that of each succeeding term is obtained by multiplying the co-efficient of the preceding term by the index of x in that term and dividing the product by the *number* of terms already obtained. The co-efficient of the last term is one.

(vi) The co-efficients of the terms, equi-distant from the middle, are equal.

(vii) If the sign between the two terms of the binomial is positive, all the terms in the expansion are positive, and if negative, then the odd terms are positive and the even terms are negative.

Example 3: Expand $(x+a)^8$.

Sol. The total number of terms in the expansion $= 9$.

the first term $= x^8$,

the second term $= \frac{1.8}{1}x^7a = 8x^7a$,

the third term $= \frac{8.7}{2}x^6a^2 = 28x^6a^2$,

the fourth term $= \frac{28.6}{3}x^5a^3 = 56x^5a^3$,

the fifth term $= \frac{56.5}{4}x^4a^4 = 70x^4a^4$.

The numerical co-efficients of the sixth, seventh, eighth and ninth terms are respectively 56, 28, 8 and 1.

Hence, we have

$$(x+a)^8 = x^8+8x^7a+28x^6a^2+56x^5a^3+70x^4a^4+56x^3a^5+28x^2a^6+8xa^7+a^8.$$

Example 4: Expand $(2a-3b)^5$.

Sol. The total number of terms in the expansion $= 6$,

the first term $= (2a)^5$,

the second term $= \frac{1.5}{1}(2a)^4(3b) = 5(2a)^4(3b)$,

the third term $= \frac{5.4}{2}(2a)^3(3b)^2 = 10(2a)^3(3b)^2$.

The numerical co-efficients of the fourth, fifth and sixth terms are respectively 10, 5 and 1.

Hence, we have

$$(2a-3b)^5 = (2a)^5-5(2a)^4(3b)+10(2a)^3(3b)^2-10(2a)^2(3b)^3 +5(2a)(3b)^4-(3b)^5$$
$$= 32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4-243b^5.$$

EXERCISE 139.

Expand :

1. $(x+1)^5$. **2.** $(a-b)^6$. **3.** $(p+q)^7$.
4. $(1-x)^6$. **5.** $(2+m)^5$. **6.** $(3-a)^4$.
7. $(2x-1)^5$. **8.** $(1-3x)^6$. **9.** $(1-x)^{10}$.
10. $(2+3x)^6$. **11.** $(3a-2b)^5$. **12.** $(a+\frac{1}{2}b)^4$.
13. $\left(a+\frac{1}{a}\right)$. **14.** $\left(x-\frac{1}{x}\right)^7$.

Simplify :

15. $(x+1)^4-(x-1)^4$. **16.** $(x-1)^5+(x+1)^5$.
17. $(x+a)^6-(x-a)^6$. **18.** $(3p+4q)^4-(3p-4q)^4$.

3. Coefficients of Particular Term in the Expansion.

Example 5: Find the co-efficient of x^2 in the expansion of $(x+2)^3\ (x-1)^4$.

Sol. The given expression
$$= (x^3+6x^2+12x+8)(x^4-4x^3+6x^2-4x+1).$$

The terms containing x^2 can be obtained by multiplying $(+6x^2)$ by $(+1)$, $(+12x)$ by $(-4x)$ and $(+8)$ by $(+6x^2)$.

Hence, the required co-efficient
$$= (+6)(+1)+(+12)(-4)+(+8)(+6)$$
$$= 6-48+48 = 6.$$

Example 6: Find an approximate value of $(1002)^5$.

Sol. We have $(1002)^5 = (1000+2)^5 = (1000)^5\left(1+\frac{2}{1000}\right)^5$.

$$\text{Now}\left(1+\frac{2}{1000}\right)^5 = 1+5\left(\frac{2}{1000}\right)+\frac{5.4}{2}\left(\frac{2}{1000}\right)^2+\ldots\ldots$$
$$= 1+\frac{1}{100}+\frac{4}{100{,}000}+\ldots\ldots$$
$$= 1{\cdot}01004\ldots\ldots\ldots\text{(approximately.)}$$

(*In the expansion of* $\left(1+\frac{2}{1000}\right)^5$, *we neglect the terms beyond the third, because their value is very small as compared with 1.*)

Hence $(1002)^5 = (1000)^5\ (1{\cdot}01004)$
$= 10^{15}\ (1{\cdot}01004)$
$= 1{,}010{,}040{,}000{,}000{,}000$ (approximately).

EXERCISE 140.

Find the co-efficient of:

1. x^2 in the expansion of $(2x-5)^2(3x+1)^4$.

2. x^3 in the expansion of $(x+1)^5(x-2)^3$.

3. x^4 in the expansion of $(1-x)^6\left(2x+\frac{3}{x}\right)^2$.

4. x^5 in the expansion of $(1+3x+3x^2+x^3)^3$.

Find an approximate value of:

5. $(1{\cdot}005)^5$. **6.** $(1{\cdot}02)^{10}$. **7.** $(1{\cdot}0003)^{25}$.

8. Re. 1 at 4% per annum compound interest after 10 years amounts to Rs. $(1+{\cdot}04)^{10}$. Calculate the value of this amount to 3 decimal places, by the above method.

4. Evolution.

Evolution is the process of determining that expression which has been expanded into the given form. It is thus the *reverse* of 'Involution'. In other words, it is the process of finding the square-root, cube-root, fourth root, etc. of given expressions.

We know that $\sqrt{16} = 4$ or -4 and that the product of two positive or two negative quantities can never be negative.

Consider the following:

$\because\ (\pm 4a^2x^3)^2 = +16a^4x^6,\quad \therefore\ \sqrt{+16a^4x^6} = \pm 4a^2x^3$

$\because\ (-2ax^2)^3 = -8a^3x^6,\quad \therefore\ \sqrt[3]{-8a^3x^6} = -2ax^2$

$\because\ (\pm 2a^2)^4 = +16a^8,\quad \therefore\ \sqrt[4]{+16a^8} = \pm 2a^2$

$\because\ (-2a^2)^5 = -32a^{10},\quad \therefore\ \sqrt[5]{-32a^{10}} = -2a^2$

$\because\ (+2a^2)^5 = +32a^{10},\quad \therefore\ \sqrt[5]{+32a^{10}} = +2a^2.$

It must be clear that

(i) an *even* root of a positive quantity may be either positive or negative :

(ii) an *odd* root of a quantity has the same sign as that of the quantity itself.

(iii) a negative quantity cannot have a real square root.

EXERCISE 141.

Find the square root of :

1. $a^4b^{16}c^2$. **2.** $64a^5b^8c^{10}$. **3.** $\frac{81a^6b^2c^4}{x^{10}y^{12}z^{16}}$.

4. $1{\cdot}21.\frac{a^{14}b^{18}}{c^6d^{10}}$. **5.** $\frac{1{\cdot}44a^6(x-y)^4}{{\cdot}36\ b^8(x+y)^6}$.

Find the cube root of :

6. $125a^6b^3c^9$. **7.** $-64x^3b^6a^{15}$.

8. $343.\frac{a^3b^6}{x^9y^{12}}$. **9.** $-\frac{27}{64}\cdot\frac{a^6b^9}{x^{12}y^{18}}$.

Evaluate :

10. $\sqrt[4]{16x^4y^8z^{12}}$. **11.** $\sqrt[7]{-128a^{14}b^{21}}$.

12. $\sqrt[5]{\frac{32}{x^5y^{10}}}$. **13.** $\sqrt[5]{-\frac{x^{10}y^{25}}{243}}$.

5. Square Root.

Since $a^2+2ab+b^2=(a+b)^2$

and $a^2-2ab+b^2=(a-b)^2$

$\therefore\quad \sqrt{(a^2+2ab+b^2)}=\sqrt{(a+b)^2}=a+b$

and $\sqrt{(a^2-2ab+b^2)}=\sqrt{(a-b)^2}=a-b$.

Thus, we can write down the square root of a trinomial, when it can be put in the form of

$a^2+2ab+b^2$ or $a^2-2ab+b^2$.

Example 7: Find the square root of $25x^{12}-30xy+9y^2$.

Sol. Reducing the expression to the standard form, we have

$$\sqrt{(25x^2-30xy+9y^2)}=\sqrt{(5x)^2-2(5x)(3y)+(3y)^2}$$
$$=\sqrt{(5x-3y)^2}=\pm(5x-3y).$$

Example 8: Find the square root of $\frac{x^4}{4}-3x^2y+9y^2$.

Sol. Reducing the expression to the standard form, we have

$$\sqrt{\left(\frac{x^4}{4}-3x^2y+9y^2\right)}=\sqrt{\left(\frac{x^2}{2}\right)^2-2.\left(\frac{x^2}{2}\right)(3y)+(3y)^2}$$

$$=\sqrt{\left(\frac{x^2}{2}-3y\right)^2}=\pm\left(\frac{x^2}{2}-3y\right).$$

EXERCISE 142.

Find the square root of the following:

1. $4a^2-12ab+9b^2$.
2. $16x^4+40x^2+25$.
3. $9x^4+30x^2y^2+25y^4$.
4. $81a^2-18ab^2+b^4$.
5. a^6-2a^3+1.
6. $1-12x+36x^2$.
7. $x^2-x+\frac{1}{4}$.
8. $x^4y^2+x^2y+\frac{1}{4}$.
9. $m^4-\frac{2}{3}m^2+\frac{1}{9}$.
10. $\frac{a^2}{4b}-2+\frac{4b^2}{a^2}$.
11. $\frac{x^4}{y^4}-1+\frac{y^4}{4x^4}$.
12. $x^4-\frac{1}{2}x^2y^2+\frac{y^4}{16}$.
13. $\frac{1}{9}a^2-\frac{1}{6}ab+\frac{1}{16}b^2$.
14. $\frac{1}{4x^2}+\frac{1}{3y}+\frac{x^2}{9y^2}$.
15. $a^4-a^3b+\frac{1}{4}a^2b^2$.

6. Square Root of Reciprocal Expressions.

Example 9: Find the square root of

$$x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38.$$

Sol. The expression

$$=\left(x^2+2+\frac{1}{x^2}\right)+2.6\left(x+\frac{1}{x}\right)+36 \quad [\because\ 38=2+36.]$$

$$=\left(x+\frac{1}{x}\right)^2+2.6\left(x+\frac{1}{x}\right)+6^2=\left\{\left(x+\frac{1}{x}\right)+6\right\}^2$$

$\therefore$ the square root $=\pm\left(x+6+\frac{1}{x}\right)$.

Example 10: Find the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-6\left(x+\frac{1}{x}\right)^2+21.$$

Sol. The expression $=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}+2\right)+21$

$$=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}\right)-12+21$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}\right)+9$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-2.3\left(x^2+\frac{1}{x^2}\right)+3^2$$

$$=\left\{\left(x^2+\frac{1}{x^2}\right)-3\right\}^2$$

$\therefore$ the sq. root $=\pm\left(x^2-3+\frac{1}{x^2}\right)$.

EXERCISE 143.

Find the square root of:

1. $x^2+\frac{1}{x^2}+8\left(x+\frac{1}{x}\right)+18.$

2. $x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)+6.$

3. $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3.$

4. $x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1.$

5. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right).$

6. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12.$

7. $\left(x-\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)+8.$

8. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x-\frac{1}{x}\right)^2-4.$

7. Square Root of Miscellaneous Expressions.

Example 11: Find the square root of

$$4(a+b)^2-4(a^2-b^2)+(a-b)^2.$$

Sol. Reducing the expression to the standard form, we have

$$\{2(a+b)\}^2-2\{2(a+b)(a-b)\}+(a-b)^2.$$

$\therefore$ the square root $= \pm\{2(a+b)-(a-b)\}$

$= \pm (a+3b).$

Example 12: Find the square root of

$$\frac{y^2}{4}-xy+x^2-4x+4+2y.$$

Sol. Arranging the terms according to descending powers of x we have

$$x^2-4x-xy+4+2y+\frac{y^2}{4}$$

$$=x^2-2x\left(2+\frac{y}{2}\right)+\left(2+\frac{y}{2}\right)^2=\left\{x-\left(2+\frac{y}{2}\right)\right\}^2$$

$\therefore$ the required square root $=\pm\left(x-2-\frac{y}{2}\right)$

Example 13: Find the square root of

$$x^4-6x^3+19x^2-30x+25.$$

Sol. Here, break up $19x^2$ into two parts, such that one of them and the first two terms may be a perfect square.

Thus, the expression

$= x^4-6x^3+9x^2+10x^2-30x+25$

$=(x^2-3x)^2+2.5(x^2-3x)+5^2=\{(x^2-3x)+5\}^2$

$\therefore$ the required square root $=\pm(x^2-3x+5).$

EXERCISE 144.

Find the square root of:

1. $4(x^2+1)^2+4(x^2+1)+1.$

2. $(3a^2+1)^2+14(3a^2+1)+49.$

3. $\left(\frac{x}{y}-2\right)^2+4\left(\frac{x}{y}-2\right)+4.$

4. $4(a+b)^2-12(a^2-b^2)+9(a-b)^2.$

5. $(a^4+1)^2-(a^4-1)^2$ **6.** $\left(\frac{a+b}{a-b}\right)^2-2+\left(\frac{a-b}{a+b}\right)$

7. $9(a+b)^2+6(a+b)(x+y)+x^2+2xy+y^2.$

Find the square root of:

8. $x^4-4x^3+6x^2-4x+1.$

9. $4x^4+12x^3-11x^2-30x+25.$

10. $x^4+2x^3y+3x^2y^2+2xy^3+y^4.$

11. $9x^4-30x^3y+49x^2y^2-40xy^3+16y^4.$

12. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+3+\frac{2x}{y}+\frac{2y}{x}.$

13. $x^2+\frac{y^2}{16}+z^2+\frac{xy}{2}-2xz-\frac{yz}{2}.$

14. $\frac{x^2}{y^2}+\frac{2x}{z}+\frac{y^2}{z^2}+\frac{2y}{x}+\frac{z^2}{x^2}+\frac{2z}{y}.$

15. $(ab+bc+ca)^2-4abc(a+c).$

[*Hint*: $4abc(a+c)=4ac(ab+bc)$. *Put* x *for* $(ab+bc)$].

16. $(2x^2+5xy+2y^2)(6x^2+5xy+y^2)(3x^2+7xy+2y^2).$

17. $\left(\frac{y+z}{6}\right)^2+\frac{x(x-y-z)}{9}.$

[*Hint*: *From the second fraction separate* $\frac{x^2}{9}$].

18. $x^2+2x+1+\frac{2}{x-1}+\frac{1}{x^4-2x^2+1}.$

19. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4).$

Find the fourth root of:

20. $x^4-4x^3+6x^2-4x+1.$

21. Show that $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square.

22. Show that $(x+3)(x+4)(x+5)(x+6)+1$ is a perfect square.

Example 14: Find the square root of $4^m+9^n-2^{m+1}.3^n$.

Sol. The expression $=(2^2)^m+(3^2)^n-2.2^m.3^n$
$=(2^m)^2+(3^n)^2-2.2^m.3^n$
$=(2^m-3^n)^2$

$\therefore$ the sq. root $=\pm(2^m-3^n)$.

EXERCISE 145.

Find the square root of:

1. $x^{2m}-4x^{m+n}+4x^{2n}$. **2.** $16^m+9^n+2^{2m+1}.3^n$

3. $25x^{2m}-20x^{m+n}+4x^{2n}$.

8. General Method for finding the Square Root.

It is not always possible to find the square root of expressions by factorisation. So we require a method which is applicable to all cases. This is the method of division and it is based on the formulae:

$(a+b)^2-a^2 \equiv (2a+b)\times b$ ... ... (i)
$(a+b+c)^2-(a+b)^2 \equiv \{2(a+b)+c\}\times c$ ... (ii)
$(a+b+c+d)^2-(a+b+c)^2 \equiv \{2(a+b+c)+d\}\times d$ ... (iii)

and the like.

Suppose the first one or more terms of the square root have been obtained by trial division. Then the remainder at that stage must be equated to (twice the portion of the square root so obtained +a new term) $\times$ that new term.

This principle governs the extraction of the square root by the division method.

For example, let us obtain the square-root of

$$9x^2-24xy+16y^2.$$

The first term of the root is obviously $3x$.

The remainder $=-24xy+16y^2$.

Twice the portion of the root found out $=2\times 3x=6x$.

To this add a new term, say $(-4y)$. It becomes $6x-4y$.

Then the remainder must be equated to $(6x-4y)\times 4y$ Then subtract. If there is a further remainder, repeat the same process.

$$\begin{array}{r|l} & 3x-4y \\ \hline 3x & 9x^2-24xy+16y^2 \\ & 9x^2 \\ \hline & -24xy+16y^2 \\ 6x-4y & -24xy+16y^2 \\ \hline & \times \quad \times \\ \hline \end{array}$$

$\therefore$ The required square root is $\pm(3x-4y)$.

Example 15: Find the square root of $x^4-6x^3+4-12x+13x^2$.

Sol. The expression when arranged in descending powers of x is $x^4-6x^3+13x^2-12x+4$.

$$
\begin{array}{r|l}
x^2 & x^4-6x^3+13x^2-12x+4 \;\big(\; x^2-3x+2 \\
 & x^4 \\
\hline
2x^2-3x & -6x^3+13x^2-12x+4 \\
 & -6x^3+9x^2 \\
\hline
2x^2-6x+2 & +4x^2-12x+4 \\
 & +4x^2-12x+4 \\
\hline
 & \times \quad \times \quad \times \\
\hline
\end{array}
$$

(i) Here the 1st term $=\sqrt{x^4}=x^2$.

(ii) When $(x^2)^2$ or x^4 is subtracted from the expression, the first remainder $=-6x^3+13x^2-12x+4$.

(iii) $-6x^3 \div$ *twice* $x^2=-3x$, which is the 2nd term of the square root.

(iv) Multiply (*twice* x^2-3x) by $-3x$ and subtract the product from the first remainder; the result $+4x^2-12x+4$ is the second remainder.

(v) $+4x^2 \div$ *twice* $x^2=2$, which is the 3rd term of the square root.

(vi) Multiply [*twice* x^2+ *twice* $(-3x)+2$] by 2; the product is equal to the second remainder.

$\therefore \quad \pm(x^2-3x+2)$ is the required sq. root.

EXERCISE 146.

Find the square root of:

1. $16x^2-56xy+49y^2$.

2. $x^4+6x^3+15x^2+18x+9$.

3. $16x^4-24x^3+25x^2-12x+4$.

4. $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$.

5. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

6. $a^2x^4+4ax^3+2(3a^2+2)x^2+12ax+9a^2$.

7. $\frac{x^4}{64}+\frac{x^3}{8}-x+1$.

8. $9x^4-18x^3+9x+\frac{9}{4}$.

9. $\frac{4}{9}x^4+x^3+\frac{5}{4}\frac{9}{8}x^2+\frac{3}{4}x+\frac{1}{4}$.

10. $9x^2-24x+19-\frac{4}{x}+\frac{1}{4x^2}$.

11. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$.

12. $\frac{2}{4}\frac{5}{}+16a^2b^2+9a^4b^4-5ab-6a^3b^3$.

9. Square Root of Non-integral Expressions having x in the Denominators.

Example 16: Find the square root of

$$x^4+\frac{4}{x^2}-2+4x-x^3+\frac{x^2}{4}.$$

Sol. In arranging the terms in descending powers of x, -2 must be placed before $\frac{4}{x^2}$, apart from other changes.

Arranging the terms in descending powers of x and proceeding as usual, we have

$$\begin{array}{l|l}
x^2 & x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}\left(x^2-\frac{x}{2}+\frac{2}{x}\right. \\
 & x^4 \\
\hline
2x^2-\frac{x}{2} & -x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2} \\
 & -x^3+\frac{x^2}{4} \\
\hline
2x^2-x+\frac{2}{x} & +4x-2+\frac{4}{x^2} \\
 & +4x-2+\frac{4}{x^2} \\
\hline
 & \times \quad \times \quad \times
\end{array}$$

$\therefore$ the square root $= \pm\left(x^2-\frac{x}{2}+\frac{2}{x}\right)$

EXERCISE 147.

Find the square root of

1. $\frac{a^2}{4b^2}-\frac{a}{b}+\frac{4b^2}{a^2}-1+\frac{4b}{a}$.

2. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{z}-\frac{7}{4}$.

3. $\frac{9x^2}{4y^2}+\frac{5}{2}+\frac{y^2}{4x^2}-\frac{3x}{y}-\frac{y}{x}$.

4. $4x^4+9\left(x^2+\frac{1}{x^2}\right)+12x^2\left(x+\frac{1}{x}\right)+18$.

5. $a^6+\frac{1}{4a^6}+2-2\left(a^3+\frac{1}{2a^3}\right)$.

6. $\frac{x^4}{y^4}+\frac{y^4}{x^4}-4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+6$.

7. $x^2y^{-2}+2xy^{-1}+x^{-2}y^2+3+2x^{-1}y$.

Find the fourth root of:

8. $1-4x+6x^2-4x^3+x^4$.

9. $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

Example 17: Find the square root of $1-4x$ up to 4 terms.

Sol.

$$
\begin{array}{r|l|l}
 & 1-4x & 1-2x-2x^2-4x^3 \\
1 & 1 & \\
\hline
 & -4x & \\
2-2x & -4x+4x^2 & \\
\hline
 & -4x^2 & \\
2-4x-2x^2 & -4x^2+8x^3+4x^4 & \\
\hline
 & -8x^3-4x^4 & \\
2-4x-4x^2-4x^3 & -8x^3+16x^4+16x^5+16x^6. &
\end{array}
$$

$\therefore$ the square root up to 4 terms $=1-2x-2x^2-4x^3$.

Example 18: Find (i) what may be added to, (ii) what may be subtracted from, or (iii) what value may be given to m, so that the expression $16x^4-16x^3-20x^2+12x+m$ may be a perfect square.

Sol.

$$
\begin{array}{r|l|l}
 & 16x^4-16x^3-20x^2+12x+m & 4x^2-2x-3 \\
4x^2 & 16x^4 & \\
\hline
 & -16x^3-20x^2 & \\
8x^2-2x & -16x^3+\ 4x^2 & \\
\hline
 & -24x^2+12x+m & \\
8x^2-4x-3 & -24x^2+12x+9 & \\
\hline
 & m-9 &
\end{array}
$$

Obviously, the given expression will be a perfect square if the remainder is 0; *i.e.*

(i) if $-m+9$ is added to it, or

(ii) if $m-9$ is subtract from it, or

(iii) if $m-9=0$, or $m=9$.

EXERCISE 148.

1. Find the square root of $4x^4+8x^3+8x^2+4x+1$, and *hence* extract the square root of 48841.

2. Find the square root of $x^4+2x^3+7x^2+6x+9$, and *hence* extract the square root of 12769.

Find up to 4 terms the square root of:

3. a^2+x^2. **4.** $1+x-x^2$.

5. What may be added to, or subtracted from $9x^4+12x^3+7x^2+2x+a$, so that it may be a perfect square ?

6. For what value of x will $x^4-12x^3+38x^2-9x-5$ be a perfect square ?

7. Find the condition so that x^2+px+q may be a perfect square.

8. Find the values of a and b for which $x^4+4x^3+10x^2+ax+b$ is a perfect square.

SELECTED QUESTIONS—I

Find the square root of:

1. $a^2+\frac{1}{4a^2}+2a-\frac{1}{a}$.

2. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$.

3. $x^4+\frac{1}{x^4}+4\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)+2$.

4. $x^4+\frac{1}{x^4}+8\left(x^2-\frac{1}{x^2}\right)+14$.

5. $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$.

6. $\frac{a^2x^2}{9}+\frac{b^2}{25}+\frac{9x^4}{4}-ax^3-\frac{3bx^2}{5}+\frac{2abx}{15}$.

7. $x^4+\frac{1}{x^2}+4+2x+4x^2+\frac{4}{x}$.

8. $x^6-4x^3+6-\frac{4}{x^3}+\frac{1}{x^6}$.

9. $x^2+2x+\frac{1}{3}-\frac{2}{3x}+\frac{1}{9x^2}$.

10. $4x^2-6x+\frac{43}{12}-\frac{1}{x}+\frac{1}{9x^2}$.

11. $x^4+\frac{4}{x^2}-x^2+4x+\frac{1}{4}-\frac{2}{x}$.

12. $4x^4+2x^3+\frac{17}{4}x^2-3x-\frac{2}{x}+\frac{1}{x^2}$.

13. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{6x}{y}-\frac{6y}{x}+7$.

14. $\frac{x^4}{4y^4}+\frac{4y^4}{x^4}+\frac{x^2}{y^2}+\frac{4y^2}{x^2}+3$.

15. $\frac{y^2}{x^2}+\frac{x^2}{4y^2}-\frac{y}{x}+\frac{x}{2y}-\frac{5}{4}$.

16. For what value of x will $x^4-12x^3+217x+320$ be a perfect square?

17. For what value of x will the expression $16x^4-24x^3-7x^2+18x-8$ become a perfect square?

18. For what value of m is the expression $4x^4-12x^3+25x^2-24x+m$ a perfect square?

CHAPTER XVIII

SURDS

1. When a root of a number *cannot be exactly obtained*, it is said to be *irrational.* If the root of an algebraical expression cannot be denoted without the use of a fractional index, it is also called an **irrational quantity.** All irrational quantities are otherwise called **surds.** Thus, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{x^2+xy^2+y^2}$, $\sqrt[3]{x^3+y^3}$, $\sqrt{\frac{x}{y}}$ are all surds.

Some numbers and expressions may be in the form of surds, but may not be really surds. $\sqrt{25}$, $\sqrt{x^2+2xy+y^2}$ are in *surd form*, but they are *not really surds* for they are equal to 5 and $x+y$ respectively.

Some quantities may be partly rationa and partly surd. They can be expressed as *complete surds* as shown below.

$3\sqrt{5}$ may be expressed as $(3^2)^{\frac{1}{2}} \times 5^{\frac{1}{2}} = (3^2 \times 5)^{\frac{1}{2}} = (45)^{\frac{1}{2}}$

$4\sqrt[3]{6} = (4^3)^{\frac{1}{3}}.6^{\frac{1}{3}} = (64 \times 6)^{\frac{1}{3}} = \sqrt[3]{384}$

By reversing this process, a complete surd can also be put in the form of a *mixed surd.* Thus

$\sqrt{50} = \sqrt{25 \times 2} = (5^2 \times 2)^{\frac{1}{2}} = (5^2)^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 5\sqrt{2}$

$\sqrt[3]{56} = \sqrt[3]{8 \times 7} = (2^3)^{\frac{1}{3}} \times 7^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} \times 7^{\frac{1}{3}}$

$= 2\sqrt[3]{7}.$

EXERCISE 149.

Express as a complete surd :

1. $4\sqrt{6}$. **2.** $5\sqrt[3]{4}$. **3.** $2\sqrt[4]{6}$.

4. $3\sqrt[4]{5}$. **5.** $2\sqrt[5]{3}$. **6.** $3\sqrt[5]{4}$.

7. $x\sqrt[n]{y}$. **8.** $x^2\sqrt[p]{y}$. **9.** $x^3\sqrt[5]{y^2}$.

Simplify :

10. $\sqrt{48}$. **11.** $\sqrt{63}$. **12.** $\sqrt{44}$.

13. $\sqrt[3]{250}$. **14.** $\sqrt[4]{567}$. **15.** $\sqrt[5]{80}$.

16. $\sqrt[3]{x^6y}$. **17.** $\sqrt[n]{x^{3n}.a}$. **18.** $\sqrt[a]{x^{4a}y^2}$.
19. $\sqrt[3]{-5120}$. **20.** $\sqrt[3]{250x^7y^4}$.

2. Similar Surds.

Surds are said to be **similar** when they have, or can be reduced so as to have the same irrational factors.

Thus, $3\sqrt{5}$, $7\sqrt{5}$ are similar surds, and $3\sqrt{125}$ and $7\sqrt{5}$ are also similar surds, for $3\sqrt{125}=15\sqrt{5}$.

For addition and subtraction, the surds have to be similar.

Example 1: Simplify $\sqrt[3]{128}+\sqrt[3]{54}$.

Sol.
$$\begin{aligned}\sqrt[3]{128}+\sqrt[3]{54} &= \sqrt[3]{64\times2}+\sqrt[3]{27\times2}\\ &= \sqrt[3]{4^3\times2}+\sqrt[3]{3^3\times2}\\ &= 4\sqrt[3]{2}+3\sqrt[3]{2}\\ &= (4+3)\sqrt[3]{2}=7\sqrt[3]{2}.\end{aligned}$$

Example 2: Simplify $\sqrt[4]{768}-\sqrt[4]{243}$.

Sol.
$$\begin{aligned}\sqrt[4]{768}-\sqrt[4]{243} &= \sqrt[4]{256\times3}-\sqrt[4]{81\times3}\\ &= \sqrt[4]{4^4\times3}-\sqrt[4]{3^4\times3}\\ &= 4\sqrt[4]{3}-3\sqrt[4]{3}\\ &= \sqrt[4]{3}(4-3)\\ &= \sqrt[4]{3}.\end{aligned}$$

EXERCISE 150.

Reduce to the simplest form:

1. $4\sqrt{5}+3\sqrt{20}-\sqrt{45}$. **2.** $\sqrt{72}-2\sqrt{32}+\sqrt{8}$.
3. $\sqrt{12}-\sqrt{27}+\sqrt{75}$. **4.** $2\sqrt[3]{4}+5\sqrt[3]{32}-\sqrt[3]{108}$.
5. $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$.
6. $4\sqrt[3]{-27}+\sqrt[3]{-8}+5\sqrt[3]{-64}$.
7. $\sqrt{20x^3}+\sqrt{80x^3}-45\sqrt{x^2y}$.
8. $\sqrt{a^2x^3}-\sqrt{a^4x}-\sqrt{9a^4x}$.
9. $3\sqrt[3]{8a^4}-\sqrt[3]{125a^4}+\sqrt[3]{27a^4}$.

3. Surds of the same order.

Surds are said to be of the **same order** when they have all got the same root-symbol. Thus, $\sqrt{7}$, $\sqrt{x-y}$, $\sqrt{5^3}$, $\sqrt{a^5}$ are all surds of the same (the *second*) order.

$\sqrt[3]{2}$, $\sqrt[3]{5}$, $\sqrt[3]{a^2}$, $\sqrt[3]{x+y}$ are all surds of the *third* order.

$\sqrt[n]{a}$, $\sqrt[n]{x^2}$, $\sqrt[n]{x^2-y^2}$ are all surds of the nth order.

Surds of the second order are called **quadratic surds** and those of the third order are called **cubic surds.**

Example 3: Express $\sqrt{a^3}$, $\sqrt[3]{a^4}$ and $\sqrt[4]{a^5}$ as surds of the same order.

Sol. The given surds are of the 2nd, 3rd and 4th orders. The required order will be indicated by the L.C.M. of 2, 3, 4, or it will be of the 12th order. Thus,

$$\sqrt{a^3} = a^{\frac{3}{2}} = a^{\frac{18}{12}} = \sqrt[12]{a^{18}},$$

$$\sqrt[3]{a^4} = a^{\frac{4}{3}} = a^{\frac{16}{12}} = \sqrt[12]{a^{16}},$$

$$\text{and } \sqrt[4]{a^5} = a^{\frac{5}{4}} = a^{\frac{15}{12}} = \sqrt[12]{a^{15}}.$$

Hence the required surds are

$$\sqrt[12]{a^{18}},\ \sqrt[12]{a^{16}} \text{ and } \sqrt[12]{a^{15}}.$$

Example 4: Which is greater $\sqrt[4]{5}$ or $\sqrt[3]{4}$?

Sol. The common order being 12*th*

$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}$$

$$\text{and} \quad \sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256}.$$

$$\therefore \quad \sqrt[3]{4} \text{ is greater than } \sqrt[4]{5}.$$

Example 5: Arrange $\sqrt{2}$, $\sqrt[3]{4}$ and $\sqrt[4]{6}$ in descending order of magnitude.

Sol. The common order being 12*th*

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64},$$

$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256},$$

$$\sqrt[4]{6} = 6^{\frac{1}{4}} = 6^{\frac{3}{12}} = \sqrt[12]{6^3} = \sqrt[12]{216}.$$

Hence the required arrangement is $\sqrt[3]{4}$, $\sqrt[4]{6}$ and $\sqrt{2}$.

EXERCISE 151.

Reduce to surds of the same order:

1. $\sqrt{6}, \sqrt[4]{3}, \sqrt[3]{2}$.

2. $\sqrt[3]{2}, \sqrt[3]{4}, \sqrt[6]{8}$.

3. $\sqrt{x^3}, \sqrt[3]{x^2}, \sqrt[4]{x}$.

Which is greater:

4. $\sqrt[3]{5}$ or $\sqrt[4]{6}$?

5. $\sqrt{2}$ or $\sqrt[3]{3}$?

6. $\sqrt[3]{4}$ or $\sqrt[4]{5}$?

7. $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Arrange according to descending order of magnitude:

8. $\sqrt{2}, \sqrt[3]{4}, \sqrt[4]{5}$.

9. $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{6}$.

10. $2\sqrt{3}, 3\sqrt{2}, \sqrt[5]{7}$.

11. $\sqrt[3]{6}, \sqrt[4]{8}, \sqrt[6]{12}$.

4. Multiplication and division of surds.

Case 1. When the surds are of the *same* order:

Example 6: $\sqrt[5]{8} \times \sqrt[5]{15} = 8^{\frac{1}{5}} \times 15^{\frac{1}{5}} = (8 \times 15)^{\frac{1}{5}}$

$= (120)^{\frac{1}{5}} = \sqrt[5]{120}$.

Example 7: $\sqrt[3]{10} \div \sqrt[3]{4} = 10^{\frac{1}{3}} \div 4^{\frac{1}{3}}$

$= \left(\frac{10}{4}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{5}{2}}$.

Case II. When the surds are of *different* orders:

Example 8: $\sqrt[6]{8} \times \sqrt[4]{3} = 8^{\frac{1}{6}} \times 3^{\frac{1}{4}} = 8^{\frac{2}{12}} \times 3^{\frac{3}{12}}$

$= (8^2)^{\frac{1}{12}} \times (3^3)^{\frac{1}{12}} = (8^2 \times 3^3)^{\frac{1}{12}}$

$= (64 \times 27)^{\frac{1}{12}} = \sqrt[12]{1728}$.

Example 9: $\sqrt[9]{8} \div \sqrt[6]{6} = 8^{\frac{1}{9}} \div 6^{\frac{1}{6}} = 8^{\frac{2}{18}} \div 6^{\frac{3}{18}}$

$= (8^2)^{\frac{1}{18}} \div (6^3)^{\frac{1}{18}}$

$= (64)^{\frac{1}{18}} \div (216)^{\frac{1}{18}}$

$= \left(\frac{64}{216}\right)^{\frac{1}{18}} = \sqrt[18]{\frac{8}{27}}$.

Case III. When the given surds have the *same* quantity under the radical sign.

Example 10: $\sqrt[4]{3} \times \sqrt[6]{3} = 3^{\frac{1}{4}} \times 3^{\frac{1}{6}} = 3^{\frac{1}{4}+\frac{1}{6}} = 3^{\frac{5}{12}}$

$= \sqrt[12]{3^5} = \sqrt[12]{243}.$

Example 11: $\sqrt[3]{5} \div \sqrt[4]{5} = 5^{\frac{1}{3}} \div 5^{\frac{1}{4}}$

$= 5^{\frac{1}{3}-\frac{1}{4}} = 5^{\frac{1}{12}} = \sqrt[12]{5}.$

Case IV. When simplification of surds is a necessary step before multiplication or division.

Example 12: $4.\sqrt{72} \times 2.\sqrt{45}$

$= 4.\sqrt{6^2.2} \times 2.\sqrt{3^2.5}$

$= 4.6.\sqrt{2} \times 2.3\sqrt{5}$

$= 24\sqrt{2} \times 6\sqrt{5}$

$= 144\sqrt{2} \times \sqrt{5} = 144\sqrt{10}.$

Example 13: $6\sqrt{75} \div 2\sqrt{63}$

$= 6\sqrt{5^2 \times 3} \div 2.\sqrt{3^2 \times 7}$

$= 6.5\sqrt{3} \div 2.3\sqrt{7}$

$= 30\sqrt{3} \div 6\sqrt{7}$

$= \frac{30\sqrt{3}}{6\sqrt{7}} = 5\sqrt{\frac{3}{7}}.$

EXERCISE 152.

Simplify :—

1. $\sqrt[3]{7} \times \sqrt[3]{9}.$ **2.** $\sqrt[4]{5} \times \sqrt[4]{12}.$ **3.** $\sqrt[5]{8} \times \sqrt[5]{18}.$

4. $\sqrt[5]{12} \div \sqrt[5]{8}.$ **5.** $\sqrt[6]{16} \div \sqrt[6]{12}.$ **6.** $\sqrt[4]{20} \div \sqrt[4]{15}.$

7. $\sqrt[3]{6} \times \sqrt[4]{9}.$ **8.** $\sqrt{15} \times \sqrt[3]{12}.$ **9.** $\sqrt[3]{21} \times \sqrt[5]{6}.$

10. $\sqrt[3]{20} \div \sqrt[4]{15}.$ **11.** $\sqrt[5]{18} \div \sqrt[6]{24}.$ **12.** $\sqrt[4]{48} \div \sqrt[8]{72}.$

13. $\sqrt{36} \times \sqrt{54}.$ **14.** $\sqrt{96} \times \sqrt{88}.$ **15.** $\sqrt[3]{128} \times \sqrt[3]{54}.$

16. $5\sqrt[3]{49} \times \sqrt[3]{14}.$ **17.** $\sqrt{18} \div \sqrt{200}.$ **18.** $5\sqrt[3]{6} \div 3\sqrt{10}.$

19. $\sqrt{3}.\sqrt[3]{2}.\sqrt[6]{4}.$ **20.** $\sqrt{8}.\sqrt[4]{3}.\sqrt[3]{4}.$ **21.** $\sqrt{2}.\sqrt[3]{3}.\sqrt[6]{5}.$

22. $5\sqrt{12} \div 10\sqrt{27}.$ **23.** $\sqrt[3]{56} \div \sqrt[3]{48}.$ **24.** $x.\sqrt[3]{x^4y^5}.$

5. Compound Surds.

An expression consisting of two or more simple surds connected by the signs + or − is called a **compound surd.** Thus, $6\sqrt[3]{7}$ and $2\sqrt[5]{8}$ being simple surds, $6\sqrt[3]{7}+2\sqrt[5]{8}$ and $6\sqrt[3]{7}-2\sqrt[5]{8}$ are compound surds.

The process of multiplication of compound surds is similar to that of compound algebraic expressions.

Example 14: Multiply $4\sqrt{3}+3\sqrt{2}$ by $2\sqrt{5}-5\sqrt{3}$.

Sol. $(4\sqrt{3}+3\sqrt{2})(2\sqrt{5}-5\sqrt{3})$.

$=4\sqrt{3}\times2\sqrt{5}-4\sqrt{3}\times5\sqrt{3}+3\sqrt{2}\times2\sqrt{5}-3\sqrt{2}\times5\sqrt{3}$.

$=8\sqrt{15}-60+6\sqrt{10}-15\sqrt{6}$.

Example 15: Find the square of $(5\sqrt{3}-6\sqrt{2})$.

Sol. $(5\sqrt{3}-6\sqrt{2})^2=(5\sqrt{3})^2-2.5\sqrt{3}.\ 6\sqrt{2}+(6\sqrt{2})^2$

$=75-60\sqrt{6}+72$

$=147-60\sqrt{6}$.

EXERCISE 153.

Multiply :—

1. $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}+\sqrt{7}$. **2.** $\sqrt{5}+\sqrt{6}$ by $\sqrt{3}-\sqrt{2}$.

3. $4\sqrt{2}-5\sqrt{3}$ by $2\sqrt{2}+3\sqrt{3}$.

4. $3\sqrt{5}+6\sqrt{3}$ by $4\sqrt{7}-2\sqrt{5}$.

5. $6-2\sqrt{6}$ by $5-2\sqrt{6}$. **6.** $5\sqrt{2}-2\sqrt{3}$ by $5\sqrt{2}-2\sqrt{3}$.

Find the continued product of:

7. $(2-\sqrt{3})$, $(\sqrt{3}+2)$, $(3-\sqrt{2})$.

8. $(4+\sqrt{6})$, $(\sqrt{3}+1)$, $(\sqrt{6}-\sqrt{3})$.

Find the square of:

9. $2\sqrt{6}+3\sqrt{5}$. **10.** $3\sqrt{7}-4\sqrt{6}$.

11. $3\sqrt{3}-4$. **12.** $x-\sqrt{x^2-1}$.

13. $a^2+\sqrt{a^2-b^2}$. **14.** $\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2}$.

15. $2\sqrt{8x-1}+3\sqrt{1-4x}$. **16.** $3\sqrt{a^2+b^2}+4\sqrt{a^2-b^2}$.

6. Rationalising Factor.

When the product of two irrational expressions or surds is rational, each is said to be the **rationalising factor** of the other.

Thus $\sqrt[3]{5}$ is the rationalising factor of $\sqrt[3]{25}$, and $\sqrt[3]{25}$ is the rationalising factor of $\sqrt[3]{5}$, for $\sqrt[3]{25} \times \sqrt[3]{5} = \sqrt[3]{125} = 5$, and $(\sqrt{5}+\sqrt{3})$ is the rationalising factor of $(\sqrt{5}-\sqrt{3})$, and $(\sqrt{5}-\sqrt{3})$ is the rationalising factor of $(\sqrt{5}+\sqrt{3})$, for $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3}) = 5-3 = 2$.

The rationalising factor of a monomial surd can be found easily, as illustrated below:

The rationalising factor of $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is obviously $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$, for $\sqrt[3]{a^2} \times \sqrt{a} = a^{\frac{2}{3}} \times a^{\frac{1}{3}} = a^{\frac{2}{3}+\frac{1}{3}} = a$.

Similarly, the rationalising factor of $\sqrt[3]{a^2}.\ \sqrt{b}.\ \sqrt[3]{c}$ or $a^{\frac{2}{3}} b^{\frac{1}{2}} c^{\frac{1}{3}}$ is $a^{\frac{1}{3}} b^{\frac{1}{2}} c^{\frac{2}{3}}$ or $\sqrt[3]{a}.\ \sqrt{b}.\ \sqrt[3]{c^2}$, for

$$\sqrt[3]{a^2}.\ \sqrt{b}.\ \sqrt[3]{c} \times \sqrt[3]{a}.\ \sqrt{b}.\ \sqrt[3]{c^2} = a^{\frac{2}{3}} b^{\frac{1}{2}} c^{\frac{1}{3}} \times a^{\frac{1}{3}} b^{\frac{1}{2}} c^{\frac{2}{3}}$$
$$= a^{\frac{2}{3}+\frac{1}{3}}\ b^{\frac{1}{2}+\frac{1}{2}}\ c^{\frac{1}{3}+\frac{2}{3}}$$
$$= a.b.c.$$

Example 16: Find the rationalising factor of $7\sqrt[5]{2^2}.\ \sqrt[4]{3^5}.\ \sqrt[3]{4^2}.$

Sol. Putting the given expression in the index form, we have

$$7.\ 2^{\frac{3}{5}}.\ 3^{\frac{5}{4}}.\ 4^{\frac{2}{3}}$$

The fractional indices $\frac{3}{5}$, $\frac{5}{4}$, $\frac{2}{3}$ will become integers by adding $\frac{2}{5}$, $\frac{3}{4}$, $\frac{1}{3}$ respectively.

$\therefore$ The required rationalising factor is

$$2^{\frac{2}{5}}.3^{\frac{3}{4}}.4^{\frac{1}{3}} \quad \text{or} \quad \sqrt[5]{2^2}.\ \sqrt[4]{3^3}.\ \sqrt[3]{4}.$$

The rationalising factor of a binomial quadratic surd can be found by merely *changing the sign of the irrational term* from $+$ to $-$ and from $-$ to $+$.

The rationalising factor of a binomial quadratic surd is called its *conjugate*.

Example 17: Find the rationalising factor of $a\sqrt{x}+b\sqrt{y}$.

Sol. Since $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y}) = (a\sqrt{x})^2-(b\sqrt{y})^2$
$= a^2x-b^2y,$

$\therefore$ $(a\sqrt{x}-b\sqrt{y})$ is the rationalising factor of $(a\sqrt{x}+b\sqrt{y})$.

EXERCISE 154.

Write down the rationalising factor of:

1. $2\sqrt{2}$, $\sqrt{45}$, $\sqrt{54}$, $\sqrt{a^3b}$, $\sqrt[3]{a}$, $\sqrt{x-1}$, $\sqrt[4]{a^2b^3}$.

2. $\sqrt{3}.\sqrt{2}$. **3.** $\sqrt[3]{4}.\sqrt{3}$. **4.** $\sqrt{2}.\sqrt{3}.\sqrt[3]{6}$.

5. $\sqrt{3}+\sqrt{2}$. **6.** $\sqrt{5}-\sqrt{3}$ **7.** $\sqrt{c}-\sqrt{b}$.

8. $a+\sqrt{b}$. **9.** $\sqrt{5}.\sqrt[3]{6}.\sqrt[4]{7^3}$. **10.** $\sqrt[3]{5}.\sqrt[3]{6^2}.\sqrt[4]{7^3}$.

11. $a\sqrt{x}-b\sqrt{y}$. **12.** $4\sqrt{3}+2\sqrt{5}$. **13.** $3\sqrt{2}-2\sqrt{3}$.

14. $\sqrt{a+x}-\sqrt{a-x}$. **15.** $x+\sqrt{x^2-1}$.

Example 18: Simplify $5\sqrt{3}\div 4\sqrt{2}$.

Sol. $5\sqrt{3}\div 4\sqrt{2} = \dfrac{5\sqrt{3}}{4\sqrt{2}}$.

Rationalising the denominator, we get

$$\frac{5\sqrt{3}}{4\sqrt{2}} = \frac{5\sqrt{3}\times\sqrt{2}}{4\sqrt{2}\times\sqrt{2}} = \frac{5\sqrt{6}}{4.2} = \frac{5\sqrt{6}}{8}.$$

Example 19: Given $\sqrt{3}=1{\cdot}732$, find the value of $\dfrac{8}{2\sqrt{3}}$ up to 3 places of decimals.

Sol. Rationalising the denominator of this fraction, we have

$$\frac{8}{2\sqrt{3}} = \frac{8\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}} = \frac{8\sqrt{3}}{2.3} = \frac{8\sqrt{3}}{6}$$
$$= \frac{8\times 1{\cdot}732}{6} = \frac{13{\cdot}856}{6} = 2{\cdot}309.$$

Note: $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$, $\dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$, $\dfrac{1}{\sqrt{x}} = \dfrac{\sqrt{x}}{x}$.

EXERCISE 155.

Simplify :

1. $5\sqrt{3} \div 7\sqrt{5}$.

2. $7\sqrt[3]{4} \div 5\sqrt[3]{2}$.

3. $4\sqrt[3]{5} \div 4\sqrt[3]{2}$.

4. $2\sqrt[4]{5} \div 6\sqrt[4]{3}$.

Given $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$ and $\sqrt{5} = 2{\cdot}236$ find the value of the following fractions up to 3 places of decimals :

5. $\frac{4}{5\sqrt{3}}$.

6. $\frac{3}{2\sqrt{2}}$.

7. $\frac{7}{4\sqrt{3}}$.

8. $\frac{1}{3\sqrt{5}}$.

9. $\frac{3}{2\sqrt{5}}$.

10. $\frac{6}{5\sqrt{3}}$.

7. Fractions with Irrational Denominators.

Example 20 : Given $\sqrt{3} = 1{\cdot}732$, find the value of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$.

Sol. Rationalising the denominator of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$, we have

$$\text{the fraction} = \frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$

$$= \frac{4+3+4\sqrt{3}}{4-3} = 7+4\sqrt{3}$$

$$= 7+4\times 1{\cdot}732 = 7+6{\cdot}928$$

$$= 13{\cdot}928.$$

Example 21 : Rationalise the denominator of $\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}$.

Sol. $$\text{The fraction} = \frac{(a+\sqrt{a^2-1})(a+\sqrt{a^2-1})}{(a-\sqrt{a^2-1})(a+\sqrt{a^2-1})}$$

$$= \frac{a^2+a^2-1+2a\sqrt{a^2-1}}{a^2-(a^2-1)}$$

$$= \frac{2a^2-1+2a\sqrt{a^2-1}}{a^2-a^2+1}$$

$$= 2a^2+2a\sqrt{a^2-1}-1.$$

EXERCISE 156.

Given $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$ and $\sqrt{6} = 2{\cdot}449$, find the value of:

1. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$. **2.** $\dfrac{1}{\sqrt{3}-2}$. **3.** $\dfrac{\sqrt{5}+2}{\sqrt{5}-2}$.

4. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. **5.** $\dfrac{7}{3\sqrt{3}-2\sqrt{2}}$. **6.** $\dfrac{15}{4\sqrt{3}-3\sqrt{2}}$.

7. Evaluate $\left(x+\dfrac{1}{x}\right)^2$ if $x = \sqrt{3}-\sqrt{2}$.

8. Evaluate $x^4+\dfrac{1}{x^4}$ when $x = 3+\sqrt{8}$.

Reduce to an equivalent fraction with a rational denominator :

9. $\dfrac{2+3\sqrt{2}}{3-2\sqrt{2}}$. **10.** $\dfrac{4\sqrt{2}+\sqrt{5}}{3\sqrt{3}-2\sqrt{3}}$. **11.** $\dfrac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}$.

12. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$. **13.** $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

8. Rationalising Factors of Expressions of the Form $a^{\frac{1}{n}} \pm b^{\frac{1}{n}}$.

(i) $(a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b^3$.

(ii) $(a-b)(a^{n-1}+a^{n-2}b+\ldots\ldots\ldots+b^{n-1}) = a^n - b^n$ for all values of n.

(iii) $(a+b)(a^{n-1}-a^{n-2}b+\ldots\ldots) = a^n - b^n$, when n is even.

(iv) $(a+b)(a^{n-1}-a^{n-2}b+\ldots\ldots) = a^n + b^n$, when n is odd.

Making use of the above identities the rationalising factors of certain binomial surds can be found.

Example 22: Find the rationalising factor of $\sqrt[3]{a}+\sqrt[3]{b}$.

$\sqrt[3]{a}+\sqrt[3]{b} = a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

Sol. Since $\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)\left\{\left(a^{\frac{1}{3}}\right)^2-a^{\frac{1}{3}}b^{\frac{1}{3}}+\left(b^{\frac{1}{3}}\right)^2\right\}$

$$=\left(a^{\frac{1}{3}}\right)^3+\left(b^{\frac{1}{3}}\right)^3 = a+b,$$

$\therefore$ $\left(a^{\frac{1}{3}}\right)^2-a^{\frac{1}{3}}b^{\frac{1}{3}}+\left(b^{\frac{1}{3}}\right)^2$ or $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$ is the required rationalising factor.

Example 23: Find the rationalising factor of $\sqrt[5]{a}-\sqrt[5]{b}$.

Sol. $\sqrt[5]{a}-\sqrt[5]{b}=a^{\frac{1}{5}}-b^{\frac{1}{5}}$.

In $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)\equiv x^5-y^5$,

if we put $a^{\frac{1}{5}}$ for x and $b^{\frac{1}{5}}$ for y, we get

$$(a^{\frac{1}{5}}-b^{\frac{1}{5}})(a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}})$$
$$=(a^{\frac{1}{5}})^5-(b^{\frac{1}{5}})^5=a-b.$$

$\therefore$ $(a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}})$, or

$\sqrt[6]{a^4}+\sqrt[5]{a^3b}+\sqrt[5]{a^2b^2}+\sqrt[5]{ab^3}+\sqrt[5]{b^4}$ is the required rationalising factor.

EXERCISE 157.

Write down the rationalising factor of:

1. $\sqrt[3]{5}-\sqrt[3]{2}$. **2.** $\sqrt[3]{4}+\sqrt[3]{3}$. **3.** $\sqrt[3]{6}-\sqrt[3]{5}$.
4. $\sqrt[3]{6}+\sqrt[3]{3}$. **5.** $\sqrt[3]{x}-\sqrt[3]{y}$. **6.** $\sqrt[3]{x}+\sqrt[3]{y}$.

Write down the rationalising factor of:

7. $\sqrt[4]{3}-\sqrt[4]{2}$. **8.** $\sqrt[5]{3}-\sqrt[5]{2}$.
9. $\sqrt[6]{5}-\sqrt[6]{3}$. **10.** $\sqrt[4]{3}+\sqrt[4]{2}$.
11. $\sqrt[5]{3}+\sqrt[5]{2}$. **12.** $\sqrt[6]{5}+\sqrt[6]{3}$.

Example 24: Find the rationalising factor of $1+\sqrt{2}+\sqrt{3}$.

Sol. $1+\sqrt{2}+\sqrt{3}=\{(1+\sqrt{2})+\sqrt{3}\}$.

Multiplying the expression by $(1+\sqrt{2})-\sqrt{3}$, we get

$$\{(1+\sqrt{2})+\sqrt{3}\}\ \{(1+\sqrt{2})-\sqrt{3}\}=(1+\sqrt{2})^2-(\sqrt{3})^2$$
$$=1+2+2\sqrt{2}-3$$
$$=2\sqrt{2}.$$

Again, multiplying $2\sqrt{2}$ by $\sqrt{2}$, we get

$$2\sqrt{2}\times\sqrt{2}=4$$

$\therefore$ the rationalising factor is $(1+\sqrt{2}-\sqrt{3})\sqrt{2}$.

EXERCISE 158.

Find the rationalising factor of:

1. $1-\sqrt{2}+\sqrt{3}$. **2.** $\sqrt{3}+\sqrt{2}-1$.
3. $\sqrt{3}-\sqrt{2}+\sqrt{5}$. **4.** $\sqrt{2}+\sqrt{3}-\sqrt{5}$.
5. $3\sqrt{5}-2\sqrt{3}+1$. **6.** $3\sqrt{2}+2\sqrt{3}-3\sqrt{6}$.

Example 25: Simplify $\frac{1}{2-\sqrt{3}}-\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{4}{3-\sqrt{5}}$.

Sol. The expression

$$=\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}-\frac{3-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$+\frac{4(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{2+\sqrt{3}}{4-3}-\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{4(3+\sqrt{5})}{9-5}$$

$$=2+\sqrt{3}-\sqrt{3}+\sqrt{2}+3+\sqrt{5}$$

$$=5+\sqrt{2}+\sqrt{5}.$$

EXERCISE 159.

Simplify :

1. $\frac{1}{a+\sqrt{a^2-1}}+\frac{1}{a-\sqrt{a^2-1}}$.

2. $\frac{1}{a-\sqrt{a^2-b^2}}-\frac{1}{a+\sqrt{a^2-b^2}}$.

3. $\frac{5}{3\sqrt{8}+\sqrt{12}-\sqrt{27}-\sqrt{32}}$.

4. $\frac{\sqrt{3}}{\sqrt{5}+1}-\frac{\sqrt{5}}{2(\sqrt{3}-1)}+\frac{1}{\sqrt{5}-\sqrt{3}}$.

5. $\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}}+\frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}$.

6. $\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}-\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

7. $\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

8. Find the value of $(2+\sqrt{3})^{-3}-(2-\sqrt{3})^{-3}$.

9. Properties of Quadratic Surds.

Theorem 1. *The square root of a rational quantity cannot be partly rational and partly irrational.*

For, if possible, let

$\sqrt{x}=a+\sqrt{b}$; where $\sqrt{b}$ is a *true surd.*

Squaring both sides, we get

$$x = a^2+b+2a\sqrt{b}.$$

Hence $\sqrt{b} = \dfrac{x-a^2-b}{2a}$, *i.e.*, a surd becomes equal to a rational quantity, which is impossible.

Theorem II. If $x+\sqrt{y} = a+\sqrt{b}$, *where* x *and* a *are both rational and* $\sqrt{y}$ *and* $\sqrt{b}$ *are true surds, then* $x = a$ *and* $y = b$.

For, if x is not equal to a than let x be greater than a and be equal to $a+h$.

Then $(a+h)+\sqrt{y} = a+\sqrt{b}$

$\therefore$ $h+\sqrt{y} = \sqrt{b}$

or $\sqrt{b} = h+\sqrt{y}$.

Thus, the square root of a rational quantity is partly rational and partly irrational, which is impossible by theorem 1.

Hence $x = a$ and consequently $\sqrt{y} = \sqrt{b}$.

Ex. Point out the flaw in the following process:

Since $3+\sqrt{25} = 6+\sqrt{4}$

$\therefore$ $3 = 6$ and $25 = 4$.

NOTE: It is important to remember that theorem II holds good only when $\sqrt{y}$ and $\sqrt{b}$ are *true surds* and not merely in *surd form*, as in the above example.

The method for finding the square root of expressions of the form $a+\sqrt{b}$, where $\sqrt{b}$ is a true surd, is based upon theorem II and is illustrated below by means of a few examples.

Example 26: Find the square root of $31+4\sqrt{21}$.

Sol. Let $\sqrt{(31+4\sqrt{21})} = \sqrt{a}+\sqrt{b}$.

Squaring both sides, we get

$$31+4\sqrt{21} = a+b+2\sqrt{ab}$$

$\therefore$ by theorem II, $a+b = 31$ ... ... (i)

and $2\sqrt{ab} = 4\sqrt{21}$ ... ... (ii)

Using the identity $a-b \equiv \sqrt{(a+b)^2-4ab}$, we have

$$a-b = \sqrt{(31)^2-(4.\sqrt{21})^2}$$

$$= \sqrt{961-336} = \sqrt{625}$$

$\therefore$ $a-b = 25$ ... ... (iii)

By adding (i) and (iii), we get $2a = 56$ or $a = 28$ and by substituting the value of a in (i), we get $b = 3$.

$\therefore$ $\sqrt{31+4\sqrt{21}} = \sqrt{28}+\sqrt{3} = 2\sqrt{7}+\sqrt{3}$.

Example 27: Find the square root of $21-8\sqrt{5}$.

Sol. Let $\sqrt{21-8\sqrt{5}} = \sqrt{a}-\sqrt{b}$.

Squaring both sides, we get

$$21-8\sqrt{5} = a+b-2\sqrt{ab}$$

$\therefore$ $a+b=21$ ... ... (i)

and $2\sqrt{ab} = 8\sqrt{5}$ ... ... (ii)

$$a-b = \sqrt{(a+b)^2-4ab}$$
$$= \sqrt{(21)^2-(8\sqrt{5})^2}$$
$$= \sqrt{441-320} = \sqrt{121}$$

$\therefore$ $a-b = 11.$ ... ... (iii)

From equations (i) and (iii), we get

$$a = 16 \text{ and } b = 5$$

$\therefore$ $\sqrt{21-8\sqrt{5}} = \sqrt{16}-\sqrt{5}$

$= 4-\sqrt{5}.$

NOTE: In most cases the square root of expressions of the form $a+\sqrt{b}$ can be found by inspection, as illustrated below.

Example 28: Find the square root of $7+2\sqrt{10}$.

Sol. Here, we have to find two numbers which when added give 7 and when multiplied give 10.

They are obviously 2 and 5.

Thus, $\sqrt{7+2\sqrt{10}} = \sqrt{5}+\sqrt{2}.$

EXERCISE 160.

Find the square root of:

1. $52+14\sqrt{3}$. **2.** $17-12\sqrt{2}$. **3.** $41+12\sqrt{5}$.

4. $19+\sqrt{83}$. **5.** $14-3\sqrt{20}$. **6.** $94+6\sqrt{245}$.

7. $43-30\sqrt{2}$. **8.** $27-10\sqrt{2}$. **9.** $87-12\sqrt{42}$.

Find by inspection the square root of:—

10. $12+2\sqrt{35}$. **11.** $8+2\sqrt{15}$. **12.** $5+2\sqrt{6}$.

13. $7+4\sqrt{3}$. **14.** $21-8\sqrt{5}$. **15.** $8-2\sqrt{7}$.

16. If $\sqrt{(21-6\sqrt{10})} = \sqrt{x}-\sqrt{y}$ find the value of x and y.

Example 29: Find the square root of $\sqrt{32}-\sqrt{24}$.

Sol. $\sqrt{32}-\sqrt{24} = \sqrt{2}(\sqrt{16}-\sqrt{12})$

$= \sqrt{2}(4-2\sqrt{3})$

$\therefore \quad \sqrt{(32-\sqrt{24})} = \sqrt[4]{2}\ \sqrt{(4-2\sqrt{3})}.$

Now proceeding as in example 27 we find that

$\sqrt{4-2\sqrt{3}} = \sqrt{3}-1$

$\sqrt{(\sqrt{32}-\sqrt{24})} = \sqrt[4]{2}(\sqrt{3}-1).$

EXERCISE 161.

Find the square root of:

1. $\sqrt{18}+\sqrt{10}$. **2.** $\sqrt{27}-\sqrt{24}$.

3. $\sqrt{75}+\sqrt{72}$. **4.** $\sqrt{128}-\sqrt{120}$.

5. $5\sqrt{3}+2\sqrt{18}$. **6.** $6\sqrt{5}-\sqrt{175}$.

Find the fourth root of:

7. $68+48\sqrt{2}$. **8.** $7-4\sqrt{3}$.

9. $28-16\sqrt{3}$. **10.** $124-32\sqrt{15}$.

Example 30. Simplify $\sqrt{2}(1+\sqrt{3})\sqrt{(2-\sqrt{3})}$.

Sol. Extracting the square root of $2-\sqrt{3}$, we find

$$\sqrt{2-\sqrt{3}} = \frac{1}{\sqrt{2}}(\sqrt{3}-1)$$

$$\therefore \text{ the expression} = \sqrt{2}(1+\sqrt{3})\frac{1}{\sqrt{2}}(\sqrt{3}-1)$$

$$= (\sqrt{3}+1)(\sqrt{3}-1) = 3-1 \qquad = 2$$

Example 31: Find the square root of $12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15}$.

Sol. Let $\sqrt{(12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15})} = \sqrt{x}+\sqrt{y}-\sqrt{z}$.

Squaring both sides, we have

$12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15} = x+y+z+2\sqrt{xy}-2\sqrt{xz}-2\sqrt{yz}$

$\therefore \quad x+y+z = 12 \quad \ldots \quad \ldots \quad \ldots$ (i)

$2\sqrt{xy} = 4\sqrt{3} \quad \ldots \quad \ldots \quad \ldots$ (ii)

$2\sqrt{xz} = 4\sqrt{5} \quad \ldots \quad \ldots \quad \ldots$ (iii)

$2\sqrt{yz} = 2\sqrt{15} \quad \ldots \quad \ldots \quad \ldots$ (iv)

From (ii) and (iii), we get

$$4x\sqrt{yz} = 16\sqrt{15} \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

But $2\sqrt{yz} = 2\sqrt{15}$ $\quad 2x = 8$, or $\quad x = 4$.

From (ii) $y = 3$, and from (iii), $z = 5$.

Also $x+y+z = 4+3+5 = 12$.

$$\therefore \quad \sqrt{(12+4\sqrt{3}-4+5-2\sqrt{15})} = 2+\sqrt{3}-\sqrt{5}.$$

EXERCISE 162.

Simplify :

1. $\sqrt{6-\sqrt{(17+12\sqrt{2})}}$.

2. $\dfrac{2-\sqrt{3}}{\sqrt{(7-4\sqrt{3})}}$.

3. $(1+\sqrt{3})(\sqrt{2}-\sqrt{3})(\sqrt{(5+2\sqrt{6})}$.

4. $\dfrac{\sqrt{(9+2\sqrt{20})}+\sqrt{(9-2\sqrt{20})}}{\sqrt{(9+2\sqrt{20})}-\sqrt{(9-2\sqrt{20})}}$.

5. $\dfrac{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}$

$$+\frac{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}$$

6. $\dfrac{3}{\sqrt{(7+2\sqrt{10})}}+\dfrac{2}{\sqrt{(8-2\sqrt{15})}}-\dfrac{5}{\sqrt{(7+2\sqrt{6})}}$.

7. Evaluate $\sqrt{x}-\dfrac{1}{\sqrt{x}}$ when $x = 3+2\sqrt{2}$.

8. Find the value of $\dfrac{1+x}{1+\sqrt{(1+x)}}+\dfrac{1-x}{1-\sqrt{(1-x)}}$

when $x = \dfrac{\sqrt{3}}{2}$.

[*Hint*: $1+x = 1+\dfrac{\sqrt{3}}{2} = \dfrac{2+\sqrt{3}}{2}$ or $\dfrac{4+2\sqrt{3}}{4}$

and $\sqrt{1+x} = \sqrt{\left(\dfrac{4+2\sqrt{3}}{4}\right)} = \dfrac{\sqrt{3}+1}{2}$.

Similarly, $1-x = \dfrac{2-\sqrt{3}}{2}$ or $\dfrac{4-2\sqrt{3}}{4}$, *and* $\sqrt{1-x} = \dfrac{\sqrt{3}-1}{2}$.]

Find the square root of:

9. $6+\sqrt{8+\sqrt{12+\sqrt{24}}}$.

10. $10-\sqrt{24}+2\sqrt{10}-2\sqrt{15}$.

11. $9+2\sqrt{3}-2\sqrt{5}-2\sqrt{15}$.

12. $15-4\sqrt{5}-4\sqrt{6}-2\sqrt{30}$.

SELECTED QUESTIONS—J

1. Find the value of

$$\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}.$$

2. If $a=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ and $b=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, find the value of a^2+b^2.

3. If $a=\dfrac{2-\sqrt{5}}{2+\sqrt{5}}$ and $b=\dfrac{2+\sqrt{5}}{2-\sqrt{5}}$, evaluate a^2-b^2.

4. If $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$, find the value of x^2+xy+y^2.

5. If $x=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ and $y=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, show that $x^2+y^2=\dfrac{1}{x^2}+\dfrac{1}{y^2}$.

6. Given $m=\sqrt{7}+\sqrt{5}$ and $n=\sqrt{7}-\sqrt{5}$, find the value of m^4+n^4.

7. Evaluate $x^3-\dfrac{1}{x^3}$, when $x=2+\sqrt{5}$.

8. If $x=3+\sqrt{8}$, find the value of $x^4+\dfrac{1}{x^4}$.

9. If $x=3-\sqrt{8}$, find the value of $x^6+\dfrac{1}{x^6}$.

CHAPTER XIX

ELIMINATION

1. Let us solve the equations $\left.\begin{matrix}\text{(i)}\ x+5=9\\ \text{(ii)}\ x+3=8.\end{matrix}\right\}$

Obviously, the first equation is true when $x=4$ and the second is true when $x=5$, but both of them are never true *simultaneously* for any value of x.

Thus, these two equations are *not consistent* for one and the same value of x.

Now, let us solve the equations $\left.\begin{matrix}\text{(i)}\ x+a=9\\ \text{(ii)}\ x+b=8.\end{matrix}\right\}$

The first equation is true when $x=9-a$ and the second is true when $x=8-b$ and both are *simultaneously* true for the same value of x if $9-a=8-b$ or if $a-b=1$.

Thus, these two equations are *consistent* for one and the same value of x if $a-b=1$. The condition discovered here is *general* and helps us in constructing pairs of consistent equations, for example:

$\left.\begin{matrix}x+3=9\\ x+2=8\end{matrix}\right\}$ here $3-2=1$ and $x=6$,

$\left.\begin{matrix}x+4=9\\ x+3=8\end{matrix}\right\}$ here $4-3=1$ and $x=5$,

$\left.\begin{matrix}x+5=9\\ x+4=8\end{matrix}\right\}$ here $5-4=1$ and $x=4$,

and so on.

It is by removing x from the equations $x+a=9$ and $x+b=8$ that we have arrived at the condition $a-b=1$, which is *necessary* for the consistency of these two equations.

The process involved in *throwing off* the unknown from the given equations and arriving at a condition or relation necessary for their consistency is called **elimination** and the relation thus obtained is called an **eliminant.**

NOTE: Since, to find the value of n unknown quantities, we require n equations, for their elimination we *generally require one more* equation or in all $n+1$ equations.

Now we shall discuss a few methods of elimination.

2. **Method of Substitution.**

Find the value of the variable to be eliminated from one of the equations—preferably from the linear equation—and substitute this value in the second equation.

Example 1: Eliminate x from the equations

$$mx+n=0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$ax^2+bx+c=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. From (i), we get $x=-\frac{n}{m}$.

Substituting $-\frac{n}{m}$ for x in (ii), we get

$$a.\left(-\frac{n}{m}\right)^2+b.\left(-\frac{n}{m}\right)+c=0$$

or $an^2-bmn+cm^2=0$, which is the required eliminant.

Example 2: Eliminate t from the equations

$$x=at \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$y=b\sqrt{1-t^2} \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. From (i), we get $t=\frac{x}{a}$

Substituting this value of t in (ii), we have

$$y=b\sqrt{1-\frac{x^2}{a^2}} \quad \text{or} \quad \frac{y}{b}=\sqrt{1-\frac{x^2}{a^2}}$$

Squaring, $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

which is the required eliminant.

EXERCISE 163.

Eliminate x from the equations:

1. $x=2$, $mx=n$.

2. $ax=b$, $x^2=m^2$.

3. $ax+b=0$, $a'x+b'=0$.

4. $x+m=0$, $ax^2+bx+c=0$.

5. $ax+b=0$, $px^2+qx+r=0$.

6. $px+q=0$, $ax^3+bx^2+cx+d=0$.

Eliminate t from the equations:

7. $x = at^2$ }
$y = 2at.$ }

8. $v = u+ft$ }
$s = ut+\frac{1}{2}ft^2.$ }

9. Eliminate r from the equations:
$l(r-a) = m(r-b)$ }
$l(r-c) = m(r-d).$ }

10. Eliminate x and y from the equations:
$ax+by = 0$ }
$lx^2+mxy+ny^2 = 0.$ }

[*Hint*: Take $\frac{x}{y}$ as the variable.]

3. **Method of comparison.**

Find the values of the variable from each equation and then equate them.

Example 3: Eliminate x from the equations:

$$ax^m - b = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$cx^n - d = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. From (i), $x^m = \frac{b}{a}$ and from (ii), $x^n = \frac{d}{c}$

$$\therefore \quad \left(x^m\right)^n = \left(\frac{b}{a}\right)^n \text{ and } \left(x^n\right)^m = \left(\frac{d}{c}\right)^m$$

$$\therefore \quad x^{mn} = \left(\frac{b}{a}\right)^n \text{ and } \quad x^{mn} = \left(\frac{d}{c}\right)^m$$

$$\therefore \quad \left(\frac{b}{a}\right)^n = \left(\frac{d}{c}\right)^m, \text{ or } a^n d^m = b^n c^m \text{ is the required eliminant.}$$

EXERCISE 164.

Eliminate x from the equations:

1. $ax+b = 0$
$px+q = 0.$

2. $m = 2ax$
$n = ax^3.$

3. $ax^2+b = 0$
$cx^3+d = 0.$

4. $px^3-q = 0$
$mx^4-n = 0.$

5. $ax^4+b = 0,\ cx^5+d = 0.$

4. Application of Formulae.

Example 4: Eliminate x from the equations $\left.\begin{aligned} x^2+\frac{1}{x^2}&=a^2\\ x+\frac{1}{x}&=b.\end{aligned}\right\}$

Sol. Since $x+\frac{1}{x}=b.$ $\therefore$ $\left(x+\frac{1}{x}\right)^2=b^2$

Or $\left(x^2+\frac{1}{x^2}\right)+2=b^2.$

Substituting the value of $x^2+\frac{1}{x^2}$, we get

$$a^2+2=b^2$$

or $a^2-b^2+2=0$, which is the required eliminant.

EXERCISE 165.

Eliminate x from the equations:

1. $\left.\begin{aligned} x^2+\frac{1}{x^2}&=p^2\\ x-\frac{1}{x}&=q.\end{aligned}\right\}$

2. $\left.\begin{aligned} x^3+\frac{1}{x^3}&=p^3\\ x+\frac{1}{x}&=q.\end{aligned}\right\}$

3. $\left.\begin{aligned} x^3-\frac{1}{x^3}&=p^3\\ x-\frac{1}{x}&=q.\end{aligned}\right\}$

4. $\left.\begin{aligned} x^4+\frac{1}{x^4}&=p^4\\ x^2+\frac{1}{x^2}&=q^2\end{aligned}\right\}$

5. $\left.\begin{aligned} x^4+\frac{1}{x^4}&=p^4\\ x+\frac{1}{x}&=q.\end{aligned}\right\}$

6. $\left.\begin{aligned} x^4+\frac{1}{x^4}&=p^4\\ x-\frac{1}{x}&=q.\end{aligned}\right\}$

Example 5: Eliminate x from the equations $\left.\begin{aligned} ax+\frac{b}{x}&=m\\ ax-\frac{b}{x}&=n.\end{aligned}\right\}$

Sol. $ax+\frac{b}{x}=m$ ... (i) $ax-\frac{b}{x}=n$ ... (ii)

By adding (i) and (ii), we get $2ax = m+n.$

By subtracting (ii) from (i), we get $\frac{2b}{x} = m-n.$

Multiplying these results, we get the eliminant

$$4ab = m^2-n^2.$$

Example 6: Eliminate t from the equations

$$x = \frac{a(1+t^2)}{1-t^2} \quad \ldots \quad \text{(i)} \qquad y = \frac{2bt}{1-t^2} \quad \ldots \quad \text{(ii)}$$

Sol. From (i), we have $\frac{x}{a} = \frac{1+t^2}{1-t^2}$ ... ... (iii)

From (ii), we have $\frac{y}{b} = \frac{2t}{1-t^2}$ ... ... (iv)

Adding (iii) and (iv), we get

$$\frac{x}{a}+\frac{y}{b} = \frac{1+t^2}{1-t^2}+\frac{2t}{1-t^2} = \frac{1+t^2+2t}{1-t^2}$$

$$= \frac{(1+t)^2}{1-t^2} \qquad \text{or} \quad \frac{x}{a}+\frac{y}{b} = \frac{1+t}{1-t}.$$

Subtracting (iv) from (iii), we get

$$\frac{x}{a}-\frac{y}{b} = \frac{1+t^2}{1-t^2}-\frac{2t}{1-t^2} = \frac{1+t^2-2t}{1-t^2}$$

$$= \frac{(1-t)^2}{1-t^2} \qquad \text{or} \quad \frac{x}{a}-\frac{y}{b} = \frac{1-t}{1+t}$$

Multiplying these results, we get

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right) = \left(\frac{1+t}{1-t}\right)\left(\frac{1-t}{1+t}\right)$$

or $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, which is the required eliminant.

EXERCISE 166.

Eliminate x from

1. $\left.\begin{aligned} x+\frac{1}{x} &= a \\ x-\frac{1}{x} &= b. \end{aligned}\right\}$ **2.** $\left.\begin{aligned} 2x+\frac{3}{x} &= m \\ 2x-\frac{3}{x} &= n. \end{aligned}\right\}$

Eliminate m from the equations:

3. $\left.\begin{aligned}\frac{a}{m}+\frac{m}{a}&=p\\ \frac{a}{m}-\frac{m}{a}&=q.\end{aligned}\right\}$ **4.** $\left.\begin{aligned}2m+\frac{3}{m}&=4a-5b\\ 2m-\frac{3}{m}&=4a+5b.\end{aligned}\right\}$

Eliminate t from the equations:

5. $x=\frac{a}{2}\left(t+\frac{1}{t}\right),\ y=\frac{b}{2}\left(t-\frac{1}{t}\right)$. **6.** $t^2+1=2at,\ t^2-1=2bt.$

7. $x=\frac{1+t^2}{2at},\ y=\frac{1-t^2}{2bt}$. **8.** $x=a\left(\frac{1-t^2}{1+t^2}\right),\ y=\frac{2bt}{1+t^2}$.

9. $x=\frac{2at}{1+t^2},\quad y=\frac{2bt}{1-t^2}$. [*Hint*: *Add and subtract the values of* $\frac{2a}{x}$ and $\frac{2b}{y}$. *Multiply the results.*]

10. Eliminate x from

$$x^3+\frac{3}{x}=14a^3,\qquad 3x+\frac{1}{x^3}=13a^3.$$

[*Hint*: *Add and subtract these equations.*]

Example 7: Eliminate x, y from the equations $\left.\begin{aligned}ax+by&=c\\ bx-ay&=d\\ x^2+y^2&=1.\end{aligned}\right\}$

Sol. Squaring the first and the second equations and adding the results, we have

$$a^2x^2+b^2y^2+2abxy+b^2x^2+a^2y^2-2abxy=c^2+d^2$$

$$\therefore\quad a^2(x^2+y^2)+b^2(x^2+y^2)=c^2+d^2.$$

Substituting the value of x^2+y^2, we have

$a^2+b^2=c^2+d^2$, which is the required eliminant.

Example 8: Eliminate x, y from the equations:

$x-y=a,\ x^2-y^2=b^2$ and $x^3-y^3=c^3$.

Sol. Dividing the second equation by the first, we get

$$x+y=\frac{b^2}{a}$$

$$\therefore\quad (x+y)^2-(x-y)^2 \quad \text{or} \quad 4xy=\left(\frac{b^2}{a}\right)^2-a^2$$

$$\therefore \qquad xy = \tfrac{1}{4}\left(\frac{b^4-a^4}{a^2}\right).$$

Also $$x^3-y^3 = (x-y)^3+3xy(x-y)$$

$$\therefore \qquad c^3 = a^3+\tfrac{3}{4}\left(\frac{b^4-a^4}{a^2}\right)a$$

$a^4-4ac^3+3b^4 = 0$ is the required eliminant.

EXERCISE 167.

Eliminate x, y from the equations:

1. $\left.\begin{aligned} ax+y &= m \\ x-ay &= n \\ x^2+y^2 &= 1. \end{aligned}\right\}$

2. $\left.\begin{aligned} ax-by &= 0 \\ bx+ay &= r \\ x^2+y^2 &= 1. \end{aligned}\right\}$

Eliminate x and y from the equations:

3. $x+y = a, \quad x^2+y^2 = b^2, \quad x^3+y^3 = c^3.$

4. $x+y = a, \quad x^2-y^2 = b^2, \quad x^3-y^3 = c^3.$

5. $x+y = a, \quad x^3+y^3 = b^3, \quad x^4+y^4 = c^4.$

6. $\left.\begin{aligned} x+y+m &= a \\ xy+xm+ym &= b^2 \\ xym &= c^3. \end{aligned}\right\}$

[*Hint.* (i) *From the first equation find the value of* $x+y$ *and from the third the value of* xy, *and substitute them in the second equation.*

(ii) $(m-x)(m-y)(m-m) \equiv 0$

or $m^3-(x+y+m)m^2+(xy+xm+ym)m-xym \equiv 0.$]

5. Method of Cross-multiplication.

Example 9: Eliminate x and y from the equations

$$a_1x+b_1y+c_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$a_2x+b_2y+c_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$a_3x+b_3y+c_3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Sol. From (i) and (ii), by the method of cross-multiplication,

$$\frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

$$\therefore \quad x = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \; y = \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Substituting these values of x and y in (iii), we have

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0,$$

or $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0,$

which is the required eliminant.

Example 10: Eliminate x from the equations

$$ax^2+bx+c=0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$a'x^2+b'x+c'=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. From (i) and (ii), by the method of cross-multiplication,

$$\frac{x^2}{bc'-b'c}=\frac{x}{ca'-c'a}=\frac{1}{ab'-a'b}$$

$$\therefore \quad \frac{x^2}{bc'-b'c}\times\frac{1}{ab'-a'b}=\left(\frac{x}{ca'-c'a}\right)^2$$

$\therefore$ $(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2$, which is the required eliminant.

Example 11: Eliminate x from the equations

$$ax^3+bx+c=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$x^2+px+q=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Sol. Multiplying (ii) by ax and subtracting the resulting equation from (i),

$$-apx^2+(b-aq)x+c=0$$

or
$$apx^2+(aq-b)x-c=0 \quad \text{(iii)}$$

Now, taking (ii) and (iii) and proceeding as in example 10, we find the required eliminant.

EXERCISE 168.

Eliminate x and y from:

1. $ax+by+c=0$
$px+qy+r=0$
$x+y+1=0$

2. $ax+hy+g=0$
$hx+by+f=0$
$gx+fy+c=0$

Eliminate t from the equations:

3. $t^2+ft+x=0$
$t^2+gt+y=0.$

Eliminate l from the equations:

4. $xl^2+yl+z=0$
$x'l^2+y'l+z'=0.$

Eliminate x from the equations:

5. $\left.\begin{aligned} x^2+d &= 0 \\ ax^2+bx+c &= 0. \end{aligned}\right\}$ (*Hint.* Put the first equation in the form of $x^2+0x+d=0$.)

6. $\left.\begin{aligned} ax^3+bx+c &= 0 \\ a'x^3+b'x+c' &= 0. \end{aligned}\right\}$ **7.** $\left.\begin{aligned} ax^3+bx^2+c &= 0 \\ a'x^3+b'x^2+c' &= 0. \end{aligned}\right\}$

8. $\left.\begin{aligned} x^2+px+q &= 0 \\ x^3+qx+r &= 0. \end{aligned}\right\}$ **9.** $\left.\begin{aligned} ax^2+bx+c &= 0 \\ x^3+3x^2+4 &= 0. \end{aligned}\right\}$

Example 12: Eliminate x, y, z from the equations:

$$x+y+z=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$lx+my+nz=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$(l+m)x+(m+n)y+(n+l)z=0 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Sol. From (i) and (ii), by the method of cross-multiplication,

$$\frac{x}{n-m}=\frac{y}{l-n}=\frac{z}{m-l}=k \text{ (say)}$$

$\therefore \quad x=k(n-m), \quad y=k(l-n), \quad z=k(m-l)$

Substituting the values of x, y, z in (iii), we get

$(l+m)\,k(n-m)+(m+n)\,k\,(l-n)+(n+l)\,k\,(m-l)=0$

or $(l+m)(n-m)+(m+n)(l-n)+(n+l)(m-l)=0$

or $nl-lm+mn-m^2+lm-mn+nl-n^2+mn-nl+lm-l^2=0$

or $-l^2-m^2-n^2+lm+mn+nl=0$

or $l^2+m^2+n^2-lm-mn-ln=0$

which is the required eliminant.

EXERCISE 169.

Eliminate x, y, z from the equations:

1. $\left.\begin{aligned} ax+by+cz &= 0 \\ a'x+b'y+c'z &= 0 \\ a''x+b''y+c''z &= 0. \end{aligned}\right\}$ **2.** $\left.\begin{aligned} ax+by &= z \\ by+cz &= x \\ cz+ax &= y. \end{aligned}\right\}$

Eliminate l, m, n from the equations:

3. $\left.\begin{aligned} mz+ny &= l \\ nx+lz &= m \\ ly+mx &= n. \end{aligned}\right\}$

Eliminate x, y, z from the equations:

4. $\dfrac{x}{y-z}=a, \quad \dfrac{y}{z-x}=b, \quad \dfrac{z}{x-y}=c.$

5. $\frac{x}{y+z}=a, \quad \frac{y}{z+x}=b, \quad \frac{z}{x+y}=c.$

6. **Miscellaneous Methods.**

Example 13: Eliminate x, y, z from

$$\frac{y}{z}+\frac{z}{y}=a, \quad \frac{z}{x}+\frac{x}{z}=b, \quad \frac{x}{y}+\frac{y}{x}=c.$$

Sol. Multiplying all the three equations, we get

$$\begin{aligned} abc &= \left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right) \\ &= \left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{y}+\frac{x^2}{yz}+\frac{yz}{x^2}+\frac{y}{z}\right) \\ &= \left(\frac{y}{z}+\frac{z}{y}\right)\left\{\left(\frac{y}{z}+\frac{z}{y}\right)+\frac{yz}{x^2}+\frac{x^2}{yz}\right\} \\ &= \left(\frac{y}{z}+\frac{z}{y}\right)^2+\frac{y^2}{x^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2} \\ &= \left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z^2}{x^2}+\frac{x^2}{z^2}+2\right)+\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\right)-4 \\ &= \left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)^2-4 \end{aligned}$$

or $abc = a^2+b^2+c^2-4$

$\therefore \quad a^2+b^2+c^2 = abc+4$ which is the required eliminant.

EXERCISE 170.

Eliminate x, y, z from the equations:

1. $x-y=a, \quad y-z=b, \quad z-x=c.$

2. $\frac{x}{p+q}=y, \quad \frac{y}{q+r}=z, \quad \frac{z}{r+p}=x.$

3. $\frac{y+z}{y-z}=a, \quad \frac{z+x}{z-x}=b, \quad \frac{x+y}{x-y}=c.$

[*Hint*: Apply. componendo-dividendo and multiply the results.]

4. $x+y+z=a, \; x^2+y^2+z^2=b^2, \; xy+yz+zx=c^2.$

5. Eliminate x from the equations:

$$\frac{a-x}{1+c} = \frac{b}{1+x} = \frac{a+x}{x(1+c)}.$$

$$\left[\textit{Hint}: \quad \frac{(a-x)+(a+x)}{(1+c)+x(1+c)} = \frac{b}{1+x}\right]$$

6. Eliminate x and y from the equations:

$$\frac{3x+y}{p} = \frac{x+y}{q} = \frac{3y+x}{r}$$

[*Hint*: Proceed as in Q. 5.]

7. Eliminate x from the equations:

$$x+x^2 = \frac{p}{m}, \quad x^3 = \frac{q}{m}.$$

[*Hint*: Add these two equations, take out x common and substitute the value of $x+x^2$.]

8. Eliminate m and n from the equations:

$$y = mx + \frac{a}{m}, \quad y = nx + \frac{a}{n}, \quad mn = -1.$$

[*Hint*: Subtracting the 2nd equation from the first, factorising the results and substituting the value of mn, we get the eliminant.]

9. Eliminate x and y from the equations:

$$x.y^{a-1} = p, \; xy^{b-1} = q, \; xy^{c-1} = r.$$

[*Hint*: Divide the first equation by the 2nd, the 2nd by the 3rd.]

10. Eliminate x, y, z from the equations:

$x^2(y+z) = a^3$, $y^2(z+x) = b^3$, $z^2(x+y) = c^3$ and $xyz = abc$.

[*Hint*: Multiply the first three equations.]

SELECTED QUESTIONS — K

Eliminate x from the equations:

1. $ax^3+bx^2+cx+d = 0, \; lx+m = 0.$

2. $x + \frac{1}{2x} = a, \; x^2 + \frac{1}{4x^2} = b^2.$

3. $x+\frac{1}{x}=a,\ x^3+\frac{1}{x^3}=b^3+3b.$

Eliminate t from the equations:

4. $mt=1+t^2$ and $t^2-nt=1.$

5. $x=\frac{t}{1+t^2},\ y=\frac{t}{1-t^2}.$

Eliminate x, y from:

6. $ax+by+c=0,\ a'x+b'y+c'=0$ and $x^2+y^2=1.$

7. $ax+by+c=0,\ a'x+b'y+c'=0$ and $x^2+xy+y^2=0.$

8. Eliminate y from $x+\frac{1}{y}=a,\ y+\frac{1}{z}=b.$

9. Eliminate x from the equations:

$$a+c=\frac{b}{x}-dx,\ a-c=\frac{d}{x}-bx.$$

CHAPTER XX

INTEGRAL FUNCTION, DETACHED CO-EFFICIENTS, INDETERMINATE CO-EFFICIENTS

1. Variables and constants.

Consider the expression $2p^2+3p+5$. In it, if p is given the values 1, 2, 3, etc., then the expression has the values 10, 19, 32, etc. Thus the value of the expression $2p^2+3p+5$ varies as p varies. p is the *variable* in this expression. Similarly, in the expression $3x^2-4x+6$, x is the *variable*.

In each of these expressions, *only one letter occurs*. Now, let us consider the expression $a^2x^2-abxy+b^2y^2$. There are four letters in it. If the value of the expression is supposed to depend upon x only, then x is the *variable*; a, b and y are *constants*. If its value is supposed to depend upon x and y, then x and y are the *variables*; a and b are *constants*.

In a combination of a, b, c, x, y, z, it is usual to treat x, y, z as *variables* and a, b, c as *constants* or *literal coefficients*.

2. Function.

An algebraical expression which contains x as the variable is called a **function of x**, and is denoted as $f(x)$, $F(x)$, $\phi(x)$, &c. If it contains x and y both as the variables, it is called a *function of x and y* and is denoted as $f(x,y)$, $F(x,y)$, etc.

For instance, x^3+4x^2-5x+7 is a function of x and may be denoted by $f(x)$ and $3x^2+4xy+2y^2$ is a function of x and y and may be denoted by $f(x, y)$.

A function is said to be **integral** when none of the variables appears in any denominator. It is said to be **fractional** if any variable occurs in any denominator.

Thus, when x, y, z are variables and a, b, c are constants, $ax^2+bz^2-cz^2+dxy$ and $\frac{x^2}{ab}-\frac{y^2}{bc}+\frac{xz}{abc}$ are integral functions; and $\frac{a}{x^2}+bx+c$ and $ax^3-\frac{b}{y^3}+cz^3+dxyz$ are fractional functions.

A function is said to be **rational** if it contains no roots of the variables, and **irrational** if it contains their roots; for instance, $ax^2+2bxy+cy^2$ is rational and $ax+2b\sqrt{xy}+cy$ is irrational.

Here we shall consider only rational, integral functions.

In a particular discussion, if $f(x)$ stands for, say, $3x^2-4x+5$, then $f(y)$ would stand for $3y^2-4y+5$,

$f(p)$,, ,, ,, $3p^2-4p+5$,
$f(-m)$,, ,, ,, $3(-m)^2-4(-m)+5$,
$f(1)$,, ,, ,, $3.1^2-4.1+5$,
$f(0)$,, ,, ,, $3.0^2-4.0+5$,
$f(p+1)$,, ,, ,, $3(p+1)^2-4(p+1)+5$.

Example 1: If $f(x)=x^3-3x^2+3x-1$, find the value of (i) $f(-x)$, (ii) $f(p-1)$.

Sol. $f(x)=x^3-3x^2+3x-1$.

(i) Putting $-x$ for x in the given expression, we have

$$f(-x)=(-x)^3-3(-x)^2+3(-x)-1.$$
$$=-x^3-3x^2-3x-1.$$

(ii) Putting $(p-1)$ for x in the given expression, we have

$$f(p-1)=(p-1)^3-3(p-1)^2+3(p-1)-1$$
$$=p^3-3p^2+3p-1-3(p^2-2p+1)+3p-3-1$$
$$=p^3-3p^2+3p-1-3p^2+6p-3+3p-3-1$$
$$=p^3-6p^2+12p-8.$$

EXERCISES 171.

Taking x, y, z as the variables, state which of the following expressions 1 to 6 are

(i) integral and rational, (ii) integral and irrational,
(iii) fractional and rational, (iv) fractional and irrational.

1. $a\sqrt{x}+\dfrac{b}{\sqrt{y}}+cz$.

2. $ax^2+\dfrac{b}{x}+\dfrac{1}{c}$.

3. $p\sqrt{x^3}+qx$.

4. $\left(x+\dfrac{1}{x}\right)\left(y+\dfrac{1}{y}\right)\left(z+\dfrac{1}{z}\right)$

5. $\left(x+\dfrac{1}{a}\right)^2+\left(y+\dfrac{1}{b}\right)^2+\left(z+\dfrac{1}{c}\right)^2$.

6. $\dfrac{ax-b\sqrt{x^3}}{c\sqrt{x}}$.

7. If $f(x)=3x^2-4x+5$, find the value of
(i) $f(1)$, (ii) $f(2)$, (iii) $f(-1)$, (iv) $f(0)$, (v) $2f(x)$, (vi) $3f(-k)$, (vii) $f(m+1)$.

8. if $f(x)=2x^3-x^2+3x+1$, find the value of
(i) $f(2)-f(1)$, (ii) $f(1)+f(0)$, (iii) $f(p)+f(-p)$.

9. if $f(x)=ax^2+bx+c$, find the value of $f(x+1)-f(x-1)$

10. If $f(x)=\dfrac{2x+1}{2x-1}$, find the value of $3f(2)-2f(0)$.

11. If $y=f(x)=\dfrac{2x-3}{3x-2}$, find $f(y)$ in terms of x.

12. If $y=\phi(x)=\dfrac{ax+b}{cx-a}$, find $\phi(y)$ in terms of x.

3. Degree of Function.

The degree or dimension of an integral function is the same as the degree of the highest term in it. Thus, when x is the variable, the degree of $2x^3-5x^6+4x-7x^2+8$ is 6, and the degree of $ax^2+abx^5+c^6x+9$ is 5.

Functions of the second and third degree are also called **quadratic** and **cubic** functions respectively.

Examples of general expressions in x, of various degrees, are given below. In them, a_1, a_2, a_3, etc. are coefficients.

a_1x+a_2	... (1st degree)	2 terms
$a_1x^2+a_2x+a_3$	... (2nd degree)	3 terms
$a_1x^4+a_2x^3+a_3x^2+a_4x+a_5$	... (4th degree)	5 terms.

Similarly, the general expression in x, y of various degrees are as follows :

1st degree	$a_1x+a_2y+a_3$	... (3 terms)
2nd degree	$a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6$	... (6 terms)
3rd degree	$a_1x^3+a_2x^2y+a_3xy^2+a_4y^3+a_5x^2$ $+a_6xy+a_7y^2+a_8x+a_9y+a_{10}$	... (10 terms)

If an expression of the 3rd degree is multiplied/divided by an expression of the 2nd degree, the result is an expression of $(3+2)$ or $(3-2)$ th degree.

If an expression of the m th degree is multiplied/divided by another expression of the n th degree, the result is an expression of the $(m+n)$ th or $(m-n)$ th degree.

4. Method of Detached Co-efficients.

When two integral functions of the *same variable* (s) are to be multiplied together or one is to be divided by another, the process may be abbreviated, by taking the following steps.

(1) Arrange the integral functions in descending or ascending powers of the variable.

(2) Supply the missing terms, if any, by introducing a zero to each such term as its co-efficient.

(3) Drop out the variable and copy down only the co-efficients without changing the order.

(4) Perform the multiplication or division.

(5) To the terms of the product or the quotient, affix the variable with proper degree.

Example 2: Multiply x^3+3x-1 by x^2-4x+2.

Sol. The first expression when completed $=x^3+0x^2+3x-1$. As both the expressions are already arranged in descending powers of x, we take their co-efficients in the given order.

$$
\begin{array}{r}
1+0+3-\ 1 \\
1-4+2 \\
\hline
1+0+3-\ 1 \\
\quad -4-0-12+\ 4 \\
\qquad +2+\ 0+\ 6-2 \\
\hline
1-4+5-13+10-2
\end{array}
$$

As the given expressions are of the 3rd and 2nd degrees respectively, their product must be of the 5th degree.

$\therefore$ the product $=x^5-4x^4+5x^3-13x^2+10x-2$.

Example 3: Divide x^4-2x^3+6x-9 by x^2-3.

Sol. The dividend when completed $=x^4-2x^3+0x^2+6x-9$ and the divisor is x^2+0x-3.

Taking the co-efficients in order, we have

$$
\begin{array}{r}
1+0-3\ \big)\ 1-2+0+6-9\ \big(\ 1-2+3 \\
\underline{1+0-3} \\
\quad -2+3+6 \\
\quad \underline{-2-0+6} \\
\qquad +3+0-9 \\
\qquad \underline{+3+0-9} \\
\qquad \times\ \ \times\ \ \times
\end{array}
$$

It is obvious that the quotient must be of the 2nd degree.
Hence the quotient $=x^2-2x+3$.
The method employed in these examples is called the **method of detached co-efficients.**

EXERCISE 172.

1. Write down a general expression in x of the 8th degree.
2. Write down a general expression in x, y of the 4th degree.
Multiply, using the method of detached co-efficients:
3. x^2-3x+2 by $2x^2+x-3$.
4. x^2-5x+3 by $2x^3-3x+1$.
5. $5x^3-2x-1$ by $3x^3+x^2+1$.
6. $6x^4+x^3-1$ by $2x^4-3x^2+4$.
7. Expand using detached co-efficients:
$(x^2+2x+1)(x^3-3x+1)(2x^3+x^2+3)$.
8. Expand $(2x-3)(3x+2)(x-a)$ and find the value of a for which the co-efficient of x^2 vanishes.
Divide, using the method of detached co-efficients:
9. $x^5-2x-x^4+6x^2-4x^3$ by $2+x^3-4x$.
10. $8x^3-1$ by $4x^2+2x+1$.
11. x^4-1 by $x+1$.
12. Find the expression of the 4th degree which multiplied by x^3+x^2+x+1 gives x^7+x^6+x+1 as the product.

5. Principle of Indeterminate Co-efficients.

Let us consider the following statements:

$$x+3=8 \quad \ldots \quad \text{(i)}$$
$$2x-1=7 \quad \ldots \quad \text{(ii)}$$
$$2(x-2)+(3x-2)=5x-6 \quad \ldots \quad \text{(iii)}$$
$$(x+1)(x-1)=x^2-1 \quad \ldots \quad \text{(iv)}$$
$$\frac{2x^2+3x+7}{x+1}=(2x+1)+6 \quad \ldots \quad \text{(v)}$$

(i) is true for $x=5$ and for no other value of x,
(ii) is true for $x=4$ and for no other value of x,
(iii) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
(iv) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
(v) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
Equalities (i) and (ii) are true for particular values of x. They are ordinary equations.

Equalities (iii), (iv) and (v) are true for any value of the variable. These are called **Identities.**

Symbol $\equiv$ is used for '*identically equal to*'.

If two expressions are identically equal, then it follows that the co-efficients of the like powers of the variable in both are equal.

This is known as the **principle of indeterminate co-efficients.**

A formal proof is given below.

Let $a_0+a_1x+a_2x^2+a_3x^3+\ldots \equiv A_0+A_1x+A_2x^2+A_3x^3+\ldots$ be an identify and hence true for any value of x.

Put $x=0$, then $a_0=A_0$.

Since $a_0=A_0$,

$$a_1x+a_2x^2+a_3x^3+\ldots \equiv A_1x+A_2x^2+A_3x^3+\ldots\ldots\ldots$$

Dividing both sides by x, we get

$$a_1+a_2x+a_3x^2+\ldots \equiv A_1+A_2x+A_3x^2+\ldots$$

Put $x=0$, then $a_1=A_1$.

Similarly, $a_2=A_2$, $a_3=A_3$, &c.

Hence the *co-efficients of like powers of x are equal.*

The principle established here is of great importance.

Example 4: If $a_0x^3+a_1x^2+a_2x+a_3 \equiv (x^2-3x+2)(x+3)$, find the values of a_0, a_1, a_2, a_3.

Sol. $a_0x^3+a_1x^2+a_2x+a_3 \equiv (x^2-3x+2)(x+3)$

i.e. $a_0x^3+a_1x^2+a_2x+a_3 \equiv x^3+0x^2-7x+6$.

Equating the co-efficients of terms of the same degree, we get $a_0=1$, $a_1=0$, $a_2=-7$, $a_3=6$.

The co-efficient of the missing term is taken to be 0.

Example 5: If $2x^2+9x+6 \equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$. find the values of A, B and C.

Sol.

$$2x^2+9x+6 \equiv A(x^2+x)+B(x^2+2x)+C(x^2+3x+2)$$

or $$2x^2+9x+6 \equiv (A+B+C)x^2+(A+2B+3C)x+2C.$$

Equating the co-efficients of terms of equal powers, we get

$A+B+C=2$ ... ... (i)

$A+2B+3C=9$ ... ... (ii)

$2C=6$ ... ... (iii)

Solving these simultaneous equations, we get

$$A=-2,\ B=1 \text{ and } C=3.$$

Alternative Method

Instead of equating the co-efficients of terms of equal powers, it is sometimes more convenient to find the values of the constants, by assigning suitable values to x, as illustrated below :

In $2x^2+9x+6 \equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$,

put $x=0$; then $2\times0+9\times0+6 = A\times0+B\times0+C\times1\times2$

or $6 = 2C$

$\therefore$ $C = 3$.

Again, by putting -1 for x in the above identity, we get

$2(-1)^2+9(-1)+6 = A(-1)(0)+B(-1)(-1+2)+C(0)(-1+2)$

or $2-9+6 = -B$

$\therefore$ $B = 1$.

Again, by putting -2 for x in the above identity, we get

$2(-2)^2+9(-2)+6 = A(-2)(-2+1)+B(-2)(0)+C(-2+1)(0)$

or $8-18+6 = 2A$

$\therefore$ $A = -2$.

$\therefore$ $A = -2$, $B = 1$ and $C = 3$.

Sometimes the first method is convenient and sometimes the second. It is only by practice that we can find out which method is to be employed in a particular case.

Example 6: Express $\dfrac{x+4}{(x+1)(x+2)}$ in the form $\dfrac{A}{x+1}+\dfrac{B}{x+2}$.

Sol.

Putting $\dfrac{x+4}{(x+1)(x+2)} \equiv \dfrac{A}{x+1}+\dfrac{B}{x+2}$, we get

$x+4 \equiv A(x+2)+B(x+1)$

or $x+4 \equiv Ax+2A+Bx+B$

or $x+4 \equiv x(A+B)+(2A+B)$.

Comparing the co-efficients, we have

$A+B = 1$ ... ... (i)

and $2A+B = 4$ ... ... (ii)

From (i) and (ii), $A = 3$ and $B = -2$.

Hence the required form is $\dfrac{3}{x+1}-\dfrac{2}{x+2}$.

EXERCISE 173

1. If $ax^2+bx+c \equiv 2(x+3)(x+1)-9$, find the values of a, b and c.

2. If $ax^3+bx^2+cx+d \equiv 3(x+1)(x-2)^2+3$, find the values of a, b, c and d.

3. If $lx^3+mx^2+nx+k \equiv (x+2)(x+3)(x-4)+3x+1$, find the values of l, m, n and k.

4. If $ax^4+bx^3+cx^2+dx+e \equiv (x+1)^4+(x-1)^4$, find the values of a, b, c, d and e.

5. For what value of m will $x+3$ be a factor of x^2+9x+m?

6. If $x^3-6x^2+12x+k \equiv (x-a)^3$, find the values of k and a.

Find the numerical values of A, B, C and D in the following identities:

7. $9x-7 \equiv A(x-1)+B(4x-3)$.

8. $12x+5 \equiv A(3x+4)-B(2x-1)$.

9. $2(3x^2-13x+13) \equiv A(x-1)(x-2)+B(x-1)(x-3)+C(x-2)(x-3)$.

10. $2x^3-x^2+x-2 \equiv A(x^2-1)(x+2)+B(x-1)(x^2-4)+C(x+1)(x^2-4)+D(x^2-1)(x-2)$.

11. Express $3x^2-4x+8$ in the form, $A(x-1)^2+B(x+2)+C$.

12. Express $3x^3+9x^2+7x+2$ in the form, $A(x+1)^3+B(x+1)+C$.

13. Express x^3-x^2-4x-6 in the form, $A(x+1)^3+B(x+1)^2+C(x+1)+D$.

14. Express $\dfrac{2(1-x)}{(1+x)(x^2-x-6)}$ in the form of $\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{x+2}$.

15. Express $\dfrac{2(x+5)}{8x^3+1}$ in the form of $\dfrac{A}{2x+1}+\dfrac{Bx+C}{4x^2-2x+1}$.

16. Express $\dfrac{9}{(x-2)(x+1)^2}$ in the form of $\dfrac{A}{x-2}+\dfrac{Bx+C}{(x+1)^2}$.

17. If $\dfrac{2x^2-x-3}{(x-1)^2(x-2)} = \dfrac{A}{x-2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, find the values of A, B and C.

18. If $\dfrac{x^2+x+2}{(x^2+x+1)(x+1)^2} \equiv \dfrac{A}{(x+1)}+\dfrac{B}{(x+1)^2}+\dfrac{Cx+D}{x^2+x+1}$, find the values of A, B, C and D.

CHAPTER XXI

HOMOGENEITY, SYMMETRY, CYCLIC ORDER

1. Homogeneous expression

An expression is said to be **homogeneous** when all its terms are of the same degree. Thus,

x^2-y^2+3xy is a homogeneous expression of the 2nd degree in x and y.

$x^2-y^2+z^2-2xy+3xz-yz$ is a homogeneous expression of the 2nd degree in x, y and z.

$ax^3+by^3+cz^3-dxyz$ is a homogeneous expression of the 3rd degree in x, y and z.

2. Complete Homogeneous Expression.

$5x^3+4x^2y+3xy^2+y^3$ is a homogeneous expression of the 3rd degree in x and y and is **complete** in itself, for no term is missing; whereas $5x^3+3xy^2+y^3$ is also a homogeneous expression of the 3rd degree in x and y but is **incomplete,** for a term of the type x^2y is missing. It can be made complete by inserting $+0x^2y$ in it. The insertion of $+0x^2y$ does not change its value, for $0x^2y=0$.

Similarly, any incomplete expression can be put in the form of a complete expression by inserting the missing terms with zero as the co-efficient of each such term.

Let us examine the forms of terms in a few complete homogeneous expressions in x and y:

a_0x+a_1y ... ... (i)
$a_0x^2+a_1xy+a_2y^2$ ... ... (ii)
$a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$... ... ... (iii)
$a_0x^4+a_1x^3y+a_2x^2y^2+a_3xy^3+a_4y^4$ ... ... (iv)
$a_0x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3+a^4xy^4+a_5y^5$ ... (v)

and classify the terms in each, according to the forms or types.

In the first expression the terms are of *one* type only, *i.e.*, a_0x.

In the 2nd expression the terms are of *two* types, *i.e.*, a_0x^2 and a_1xy.

In the 3rd expression the terms are of *two* types, *i.e.*, a_0x^3 and a_1x^2y.

In the 4th expression the terms are of *three* types, *i.e.*, a_0x^4, a_1x^3y and $a_2x^2y^2$.

In the 5th expression the terms are of *three* types, *i.e.*, a_0x^5, a_1x^4y and $a_2x^3y^2$.

If in these expressions the terms of the same type be grouped together, we have

(a_0x+a_1y) ... ... (i)

$(a_0x^2+a_2y^2)+a_1xy$ ... ... (ii)

$(a_0x^3+a_3y^3)+(a_1x^2y+a_2xy^2)$ ... ... (iii)

$(a_0x^4+a_4y^4)+(a_1x^3y+a_3xy^3)+a_2x^2y^2$ ... (iv)

$(a_0x^5+a_5y^5)+(a_1x^4y+a_4xy^4)+(a_2x^3y^2+a_3x^2y^3)$ ... (v)

Thus, the best way of writing down a homogeneous expression is first to think out all the possible types and then to write down all the terms in each type.

Example 1: Write down a homogeneous expression of the 3rd degree in x, y and z.

The types of terms in the required expression are:

(i) ax^3, (ii) bx^2y, (iii) $cxyz$.

$\therefore$ the required expression is

$(a_0x^3+a_1y^3+a_2z^3)+(a_3x^2y+a_4x^2z+a_5y^2x+a_6y^2z+a_7z^2x+a_8z^2y)+a_9xyz$.

3. Absolute Symmetry

In Algebra, an expression is said to be *symmetrical* with respect to a pair of letters in it, when it remains unaltered if these two letters are interchanged.

$2x^2+3xy+2y^2$ is symmetrical with respect to x and y, for, when x and y are interchanged, we get $2y^2+3yx+2x^2$ which is the same as the original expression. This sort of symmetry is known as **absolute symmetry.**

The expression $2x^2+2y^2+2z^2+3xy+3xz+3yz$ is symmetrical with respect to (x, y), (y, z) and (z, x), for it remains unaltered when (i) x, y are interchanged, (ii) y, z are interchanged and (iii) when z, x are interchanged.

The expression $2x^2+2y^2+5z^2+3xy+7xz+7yz$ is symmetrical with respect to (x, y) but *not* with respect to (x, z) and (y, z), for if we interchange x and y, its value remains unaltered, but its value is altered when x and z or y and z are interchanged.

Think out the condition which will make the above expression symmetrical
(i) with respect to x, z ; (ii) with respect to y, z.

Let us consider the expressions :

$a_0x^5+a_1y^5$ ... ... (i)
$a_2x^4y+a_3xy^4$ ... ... (ii)
$a_4x^3y^2+a_5x^2y^3$ ... ... (iii)

Expression (i) will be symmetrical with respect of x, y if $a_0=a_1$.

Expression (ii) will be symmetrical with respect to x, y if $a_2=a_3$.

Expression (iii) will be symmetrical with respect to x, y if $a_4=a_5$.

Consequently, $(a_0x^5+a_1y^5)+(a_2x^4y+a_3xy^4)+(a_4x^3y^2+a_5x^2y^3)$ will be symmetrical, with respect to x, y if the terms of the type x^5 have equal co-efficients, the terms of the type x^4y have equal co-efficients and the terms of the type x^3y^2 have equal co-efficients.

In a symmetrical expression all the terms of the same type must have equal co-efficients.

Let us consider the *sum*, the *difference*, the *product* and the *quotient* of symmetrical expressions.

Take the following 4 expressions, which are symmetrical with respect to x, y :

$x+y$ ... ... (i)
x^2+xy+y^2 ... ... (ii)
x^2-xy+y^2 ... ... (iii)
$x^4+x^2y^2+y^4$ ... ... (iv)

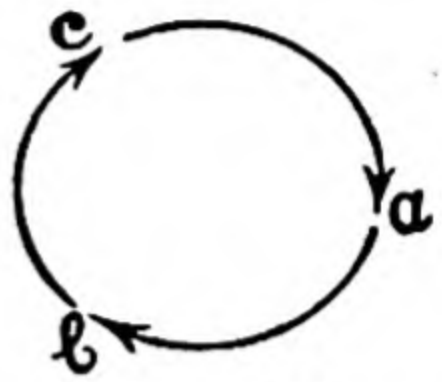
and apply the test of symmetry on the results we get by adding (i) and (ii), (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv), (iii) and (iv) ;
by subtracting (i) from (ii), (i) from (iii), (i) from (iv), (ii) from (iii), (ii) from (iv) and (iii) from (iv) ;
by multiplying (i) and (ii), (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv), (iii) and (iv) ;
and by dividing (iv) by (ii) and (iv) by (iii).

From the examination of the above results, we establish the following law :

The sum, the difference, the product and the quotient of two symmetrical expressions are symmetrical.

4. Cyclic Order

In this figure, the letters a, b, c, are so arranged round the circumference of a circle that if we start from a and go round the circumference in the direction of the arrows, we meet with b and after b with c and after c with a and so on. Such an arrangement of letters is said to be **cyclic.**



Let us consider the arrangement of terms in the following expressions :

$$a(b+c)+b(c+a)+c(a+b)$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

In these expressions, if in the first terms a be replaced by b, b by c and c by a, we get the second terms ; if in the second terms b be replaced by c, c by a and a by b, we get the third terms ; if in the third terms c be replaced by a, a by b and b by c we get the first terms. Such expressions are said to be written in **cyclic order.**

An expression is said to be written in **cyclic order** when each term in it can be derived from the preceding term by changing the first letter into the second, the second into the third and the third into the first.

Example 2: Write $a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)$ in cyclic order.

Sol. The first factors of the terms *i.e.* a b, c are already in cyclic order, but the second factors are not.

Arranging the second factors of the terms in cyclic order, we get

$$-a(b^2-c^2)-b(c^2-a^2)-c(a^2-b^2)$$
$$=-[a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)].$$

Example 3: Write down the cyclic expression in a, b, c, whose first term is $\frac{a}{a(b-c)}$.

Sol. Putting b for a, c for b and a for c, we get $\frac{b}{b(c-a)}$ as the second term.

Putting in the second term c for b, a for c and b for a we get $\frac{c}{c(a-b)}$ as the third term.

$\therefore$ the expression $= \frac{a}{a(b-c)} + \frac{b}{b(c-a)} + \frac{c}{c(a-b)}$.

5. Sigma Notation

Whenever an expression consists of the sum of terms of the same type it is quite easy to derive all the terms from the first term, as illustrated in example 3. Such an expression can be put in an abbreviated form by prefixing the Greek letter Σ (Sigma) to the first term. The symbol Σ stands for *the sum of the terms of the same type.*

Thus with variables a, b, c,

$\Sigma\, a$ means $a+b+c$,

$\Sigma\, a^2$ means $a^2+b^2+c^2$,

$\Sigma\, ab$ means $ab+bc+ca$,

$\Sigma\,(a^2+ab)$ means $(a^2+ab)+(b^2+bc)+(c^2+ca)$,

$3\,\Sigma\, ab$ means $3(ab+bc+ca)$,

and $2\,\Sigma\, a^2-3\,\Sigma\, ab$ means $2(a^2+b^2+c^2)-3(ab+bc+ca)$.

6. Cyclic Symmetry.

Let us examine the nature of the expression $a^2(b-c)+b^2(c-a)+c^2(a-b)$. It appears to be absolutely symmetrical with respect to the letters $(a, b,)$, (b, c) and (c, a) but on interchanging a, b, we get the expression

$b^2(a-c)+a^2(c-b)+c^2(b-a)$ which is equal to $-a^2(b-c)-b^2(c-a)-c^2(a-b)$.

This expression differs from the original expression. So it *does not possess absolute symmetry*. But the expression remains unaltered by putting b for a, c for b and a for c, all at the same time. So, it possesses a certain kind of symmetry which is known as **cyclic symmetry.**

Example 4: Simplify by the cyclic symmetry

$$(a+b-2c)(a+c-b)+(b+c-2a)(b+a-c)+(c+a-2b)(c+b-a).$$

Sol. By actual multiplication,

$$(a+b-2c)(a+c-b) = a^2-b^2-2c^2-ac+3bc.$$

By cyclic symmetry,

$$(b+c-2a)(b+a-c) = b^2-c^2-2a^2-ba+3ca$$

and $(c+a-2b)(c+b-a) = c^2-a^2-2b^2-cb+3ab)$

$\therefore$ the expression $= -2(a^2+b^2+c^2)+2(ab+ac+bc)$

$$= -2\,\Sigma\, a^2+2\,\Sigma\, ab.$$

EXERCISE 174.

Taking x, y, z as variables state which of the functions given below are

(i) homogeneous but not symmetrical,
(ii) symmetrical but not homogeneous,
(iii) homogeneous and symmetrical,
(iv) neither homogeneous nor symmetrical.

In cases (ii) and (iii), tell also whether the symmetry is absolute or cyclic :

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(x-y)(x+y-6)$.
4. $x^3-2x^2y+2xy^2+y^3$.
5. $x^3+ax^2y+bxy^2+y^3$.
6. $(x^2+y^2)(x^3-y^3)$.
7. $x^3+y^3+z^3-3xyz$.
8. $2(x^2+y^2)+5xy+a(x+y)$.
9. $x^3+3x^2y+3xy^2-y^3$.
10. $x^2+y^2+z^2-a(xy+yz+zx)$.
11. $x^2+y^2+z^2-2(x+y+z)$.
12. $xy(x-y)+yz(y-z)+zx(z-x)$.

13. Write down all the possible *types* of terms in a complete homogeneous expression of the 5th degree in x and y.

14. Write down all the possible types of terms in a complete homogeneous expression of the 4th degree in x, y and z.

15. Write down a homogeneous and symmetrical expression of the 4th degree in x, y, and z, using a, b, c, &c. as co-efficients.

16. Construct a homogeneous and symmetrical expression of the 2nd degree in x and y, which shall be equal to 16 when $x=1$ and $y=2$, and which shall be equal to 4 when $x=2$ and $y=-1$.

17. Construct a homogeneous and symmetrical expression of the 3rd degree in x and y, which shall be equal to 10 when $x=1$ and $y=1$, and equal to 36 when $x=2$ and $y=1$.

Write down all the types of terms in the following expressions, a, b, c being the variables and x being constant:

18. $\Sigma\,(x+a)(x+b)$.

19. $\Sigma\,(x-a)(b-a)(c-a)$.

20. $\Sigma\,(x+a)(x+b)(x-c)$.

21. $\Sigma\,\dfrac{x^2(a+b)}{ab}$.

22. $\Sigma\,\dfrac{bc}{(a-b)(a-c)}$.

23. $\Sigma\,\dfrac{1}{(a-b)(a-c)(x-a)}$.

Simplify the following expressions by the method of cyclic symmetry :

24. $(x+b)(x+c)+(x+c)(x+a)+(x+a)(x+b)$.

25. $(a+b-c)(b-c)+(b+c-a)(c-a)+(c+a-b)(a-b)$.

26. $(x+b+c)(x+b-c)+(x+c+a)(x+c-a)$
$+(x+a+b)(x+a-b)$.

27. $(a-b)(x+a)(x+b)+(b-c)(x+b)(x+c)$
$+(c-a)(x+c)(x+a)$.

28. $\Sigma\,(x+y)(x+z)$.

29. $\Sigma\,(x+y)(x-z)$.

30. $\Sigma\,xy(x-y)(x-z)$.

31. $\Sigma\,(a+b-c)^2$.

Under what conditions will the following expressions be symmetrical :

32. (i) $7x^2+ay^2+bz^2-3xy+cyz+dzx$,
(ii) $4x^3+py^3-2x^2y+qxy^2$,
(iii) $2x^4+my^4-nz^4+7x^2yz-py^2zx+qz^2xy$.

Without actual multiplication, state why the following relations are wrong :

33. $(a+b)(a^2-ab+b^2)=a^3-b^3$.

34. $a^3+b^3=(a-b)(a^2+ab+b^2)$.

35. $(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$
$=(a^3+b^2+c^3-2ab-2bc-2ca)$.

36. $(x^2+y^2)(x+y)=x^3+2xy^2+3x^2y+y^3$.

37. $(x^3+y^3)(x+y)=x^4+x^3y+3xy^3+y^4$.

38. $(x+y+z)(x^2+y^2+z^2)=(x^3+y^3+z^3+xy+xz+zy)$.

39. $(x+y)(x^2+xy+y^2)=x^3+3xy+3xy^2+y^4$.

40. $(x+y+1)(x+y+z)=x^2+xy+xz+y^2$.

Example 5 : Simplify $(a+b)^3+(b+c)^3+(c+a)^3$ $-3(a+b)(b+c)(c+a)$ by the principles of homogeneity, symmetry and indeterminate co-efficients.

Sol. Since the given expression is symmetrical, homogeneous and of the 3rd degree in a, b, c, the result will also satisfy these conditions.

The types of terms in the result will be
(i) a^3, (ii) a^2b and (iii) abc.
$\therefore$ $(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$
$\equiv p(a^3+b^3+c^3)+q(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)$
$+ r.abc$, where p, q, r are the co-efficients.
Putting $a=0$, $b=0$ and $c=1$, we get
$(0)^3+(1)^3+(1)^3-3(0)(1)(1) \equiv p(1)+q(0)+r(0)$
$\therefore$ $p=2$.
Putting $a=0$, $b=1$ and $c=1$, we get
$(1)^3+(2)^3+(1)^3-3(1)(2)(1) \equiv p(2)+q(1+1)+r(0)$
$10-6=2p+2q$
or $4=2\times2+2q$
$\therefore$ $q=0$.
Again, putting $a=2$, $b=1$ and $c=1$, we get
$(3)^3+(2)^3+(3)^3-3(3)(2)(3) \equiv p(8+1+1)$
$+q(4+4+1+2+2+1)+r.\ 2.\ 1.\ 1.$
or $27+8+27-54 \equiv p\times10+q\times14+r\times2$.
Substituting the values of p and q, we get
$8 \equiv 2\times10+0\times14+2r$
or $r=-6$.
$\therefore$ $p=2$, $q=0$ and $r=-6$.
Hence the given expression $=2(a^3+b^3+c^3)-6abc$.

Example 6: Factorise x^3-5x^2-5x-6.

Sol. As the expression is of the 3rd degree, it can have either three linear factors or one linear and one quadratic, *i.e., it must have at least one linear factor.*

As the first term of the expression is x^3,
$\therefore$ the first term of the linear factor is x.
As the last term of the expression is -6,
$\therefore$ the last term of the linear factor may be one of
$\pm 1, \pm 2, \pm 3, \pm 6$.
Substituting $\mp 1, \mp 2, \mp 3$, for x in the given expression, we find that it does not vanish, but on substituting $+6$ for x, we see that it vanishes.
$\therefore$ $x-6$ is one of the factors.
The given expression can be put as
$x^2(x-6)+x(x-6)+(x-6)$
$=(x-6)\ (x^2+x+1)$.
As x^2+x+1 cannot be further factorised,
$\therefore$ $x^3-5x^2-5x-6=(x-6)(x^2+x+1)$

Example 7: Factorise x^3+x^2+4x-6.

Sol. As the expression is of the 3rd degree, it must have at least one linear factor.

As the first term of the expression is x^3, therefore the first term of the linear factor is x.

As the last term of the expression is -6, therefore the last term of the linear factor may be one of ±1, ±2, ±3, ±6.

On trial we find that the given expression vanishes when we substitute $+1$ for x,

$\therefore$ $(x-1)$ is one of the factors.

Now, the expression $= x^2(x-1)+2x(x-1)+6(x-1)$

$= (x-1)(x^2+2x+6)$.

As x^2+2x+6 cannot be further factorised,

$\therefore$ $(x-1)$, (x^2+2x+6) are the required factors.

Example 8: Factorise x^3-5x^2+2x+8.

Sol. As the expression is of the third degree, with x^3 as its first term and $+8$ as its last term, therefore the first term of the linear factor is x and the last term of the linear factor may be one of ±1, ±2, ±4, ±8.

On trial we find that the expression vanishes when $x=-1$, $+2$ and $+4$.

$\therefore$ $(x+1)$, $(x-2)$, $(x-4)$ are the three factors of the expression.

As these three linear factors form a cubic expression and the given expression is also cubic, therefore there will be no other factor.

$\therefore$ $x^3-5x^2+2x+8=(x+1)(x-2)(x-4)$.

EXERCISE 175

1. If $(x^2+y^2+z^2)(x+y+z) \equiv A\,\Sigma x^3+B\,\Sigma x^2y+C\,\Sigma xyz$, find the values of A, B and C.

Simplify :

2. $(x+y+z)^3-x^3-y^3-z^3$.

3. $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3$.

4. $(y+z-2x)(y-z)^2+(z+x-2y)(z-x)^2$
$+(x+y-2z)(x-y)^2+(y+z-2x)(z+x-2y)(x+y-2z)$.

Factorise :

5. x^3+x^2+x-3. **6.** $x^3-6x^2+11x-6$.

7. $x^3+x^2-17x+15$. **8.** $x^3+3x^2-10x-24$.

9. $x^3+10x^2+31x+30$. **10.** x^3+5x^2-x-5.

11. $x^3-19x-30$. **12.** x^3-x^2+12.

13. $x^4-2x^3-3x^2+2x+2$. **14.** $x^4-2x^3-20x^2+39x+36$.

15. If n denote any odd number, prove that $(b+c)(c+a)(a+b)$ is a factor of $(a+b+c)^n-a^n-b^n-c^n$ and that $(b-c)(c-a)(a-b)$ is a factor of $(b-c)^n+(c-a)^n+(a-b)^n$.

16. If n denote any odd number which is not a multiple of 3, prove that $ab(a+b)(a^2+ab+b^2)$ is a factor of $(a+b)^n-a^n-b^n$.

Example 9: Factorise $(x+y+z)^3-x^3-y^3-z^3$.

Sol. As the given expression is symmetrical and of the 3rd degree in x, y, z, the possible forms of the linear factor are

(i) $(x+y+z)$, (ii) $(x-y)$, (iii) $(x+y)$, (iv) x.

Putitng $-(y+z)$ for x in the given expression, we find that it does not vanish.

$\therefore$ $x+y+z$ is not a factor of the expression.

Putting y for x in the given expression, we find that it does not vanish.

$\therefore$ $x-y$ is not a factor of the expression.

Putting $-y$ for x in the given expression, we find that it vanishes.

$\therefore$ $x+y$ is a factor of the expression.

As the given expression is symmetrical and homogeneous, therefore $(y+z)$ and $(z+x)$ must *necessarily* be two other factors.

As the given expression is of the 3rd degree, it cannot have any other factor excepting a numerical one.

$\therefore$ $(x+y+z)^3-x^3-y^3-z^3 \equiv A(x+y)(y+z)(z+x)$ ... (i)

where A s ands for the numerical factor.

If in (i) $x=0$, $y=1$ and $z=1$, then

$$(0+1+1)^3-0^3-1^3-1^3 = A(0+1)(1+1)(1+0)$$

$\therefore$ $$A = 3.$$

$\therefore$ $$(x+y+z)^3-x^3-y^3-z^3 \equiv 3(x+y)(y+z)(z+x).$$

Example 10: Factorise $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Sol. Arranging the terms, according to descending powers of a,

$$\begin{aligned}\text{the expression} &= a^3(b-c)-a(b^3-c^3)+(b^3c-c^3b)\\ &= a^3(b-c)-a(b^2+bc+c^2)(b-c)+bc(b+c)(b-c)\\ &= (b-c)\{a^3-ab^2-abc-ac^2+b^2c+bc^2\}.\end{aligned}$$

Arranging the 2nd factor, according to descending powers of b,

$$\begin{aligned}\text{the expression} &= (b-c)\{(bc^2-ab^2)+(bc^2-abc)-(ac^2-a^3)\}\\ &= (b-c)\{b^2(c-a)+bc(c-a)-a(c+a)(c-a)\}\\ &= (b-c)(c-a)\{b^2+bc-ac-a^2\}.\end{aligned}$$

Arranging the 3rd factor, according to descending powers of c,

$$\begin{aligned}\text{the expression} &= (b-c)(c-a)\{(bc-ac)+(b^2-a^2)\}\\ &= (b-c)(c-a)\{c(b-a)+(b+a)(b-a)\}\\ &= (b-c)(c-a)(b-a)(a+b+c)\\ &= -(a-b)(b-c)(c-a)(a+b+c).\end{aligned}$$

Example 11: Factorise $(xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2$.

Sol. As the given expression is symmetrical and of the fourth degree in x, y, z, the possible forms of the linear factors, if any, are (i) x, (ii) $x-y$, (iii) $x+y$, (iv) $x+y+z$.

Putting 0 for x, we find that the expression vanishes.

$\therefore$ x is one of the factors.

As the expression is symmetrical and homogeneous,

y and z must be two other factors.

As the given expression is of the 4th degree, symmetrical and homogeneous, and the expression xyz formed by the factors x, y, z is of the 3rd degree, symmetrical and homogeneous, the remaining factor must be of the 1st degree, symmetrical and homogeneous, which cannot be other than $A(x+y+z)$ where A is the numerical co-efficient.

$\therefore$ $(xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2 \equiv Axyz(x+y+z)$ ... (i)

Putting in (i) $x=1$, $y=1$, $z=1$, we get $3A=6$ or $A=2$.

$\therefore$ $(xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2=2xyz(x+y+z)$.

Example 12: Factorise $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.

Sol. As the given expression is symmetrical and of the 5th degree in a, b, c, the possible forms of the linear factors, if any, are

(i) a, (ii) $a-b$, (iii) $a+b$, (iv) $a+b+c$.

Putting 0 for a, we find that the expression does not vanish.
$\therefore$ a is not a factor of the expression.
Putting b for a, we find that the expression vanishes.
$\therefore$ $(a-b)$ is one of the factors.
As the expression is symmetrical and homogeneous,
$\therefore$ $(b-c)$ and $(c-a)$ must also be factors.

As the expression is of the 5th degree, symmetrical and homogeneous, and the expression $(a-b)(b-c)(c-a)$ formed by the factors is of the 3rd degree, there will be one more factor of the 2nd degree, in a, b, c which will be symmetrical, homogeneous and of the form $[A(a^2+b^2+c^2)+B(ab+bc+ca)]$.

$$\therefore \quad a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2) \equiv (a-b)(b-c)(c-a)\{A(a^2+b^2+c^2)+B(ab+bc+ca)\}.$$

Comparing like terms, say a^3b^2, on both sides, we get
$a^3b^2 = -a^2b \times Bab$ or $a^3b^2 = -Ba^3b^2$ $B=-1$.

Again, comparing terms containing a^4b, we get
$0 = -a^2b \times Aa^2 = -Aa^4b$ $\therefore$ $A=0$.

$$\therefore \quad a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$
$$=(a-b)(b-c)(c-a)\{(0)(a^2+b^2+c^2)+(-1)(ab+bc+ca)\}$$
$$=-(a-b)(b-c)(c-a)(ab+bc+ca).$$

NOTE: (*i*) In the process of this example, the evaluation of the constants, by giving different values to the variables is tedious, therefore we have adopted the method of *comparison of like terms*.

(*ii*) In finding out the value of the constants, care should be taken not to give such values to the variables as to reduce the two sides to the form $0=0$.

EXERCISE 176

1. Prove that $a^2(b-c)+b^2(c-a)+c^2(a-b) \equiv -(a-b)(b-c)(c-a)$.

2. Prove that $ab(a-b)+bc(b-c)+ca(c-a) \equiv -(a-b)(b-c)(c-a)$.

Resolve into factors:

3. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

4. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

5. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

6. $(a-b)^3+(b-c)^3+(c-a)^3$.

7. $(a+b+c)(ab+ac+bc)-abc.$

8. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$

9. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3.$

10. $a^3(b-c)+b^3(c-a)+c^3(a-b).$

11. $a(b-c)^3+b(c-b)^3+c(a-b)^3.$

12. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc.$

13. $(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3.$

14. $a^4(b-c)+b^4(c-a)+c^4(a-b).$

15. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$

16. $(x-y)^5+(y-z)^5+(z-x)^5.$

17. $a^4(b-c)+b^4(c-a)+c^4(a-b).$

18. $(x+y)^5-x^5-y^5.$

19. $(x+y+z)^5-x^5-y^5-z^5.$

20. $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2).$

21. $a^5(b-c)+b^5(c-a)+c^5(a-b).$

CHAPTER XXII

CONDITIONAL IDENTITIES

1. Now it is proposed to discuss a few important conditional identities.

2. **Conditional identities based on $a+b+c=0$.**

Example 1: If $a+b+c=0$, show that $(a+b)(b+c)(c+a) = -abc$.

Sol. Since $a+b+c=0$,

$$\therefore \quad a+b=-c,\ b+c=-a,\ c+a=-b.$$

Multiplying these three results, we get

$$(a+b)(b+c)(c+a) = (-c)(-a)(-b) = -abc.$$

Example 2: If $a+b+c=0$, show that $a^2-bc = b^2-ca = c^2-ab$.

Sol. Since $a+b+c=0$, $\quad\therefore\quad a=-b-c$

$$a^2-bc = a\times a-bc = a(-b-c)-bc = -ab-ac-bc = -(ab+bc+ca).$$

Similarly, $\quad b^2-ca = -(ab+bc+ca)$,

and $\quad c^2-ab = -(ab+bc+ca)$.

$$\therefore \quad a^2-bc = b^2-ca = c^2-ab.$$

Example 3: If $a+b+c=0$, show that $a^3+b^3+c^3=3abc$.

Sol. Since $a+b+c=0$, $\quad\therefore\quad a+b=-c$

$$\therefore \quad (a+b)^3 = (-c)^3$$

$$\therefore \quad a^3+b^3+3ab(a+b) = -c^3$$

or

$$a^3+b^3+c^3 = -3ab(a+b) = -3ab(-c) = 3abc.$$

Example 4: If $a+b+c=0$, show that $a^5+b^5+c^5 = -5abc(ab+ac+bc)$.

Sol. Since $a+b+c=0$, $\quad\therefore\quad a+b=-c$.

$$\therefore \quad (a+b)^5 = (-c)^5$$

$$\therefore \quad a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 = -c^5$$

$$\begin{aligned} a^5+b^5+c^5 &= -5ab(a^3+2a^2b+2ab^2+b^3) \\ &= -5ab\{(a+b)^3-ab(a+b)\} \\ &= -5ab\{(-c)^3-ab(-c)\} \\ &= -5ab(-c^3+abc) \\ &= -5abc(ab-c^2) \\ &= -5abc(ab-c\times c) \\ &= -5abc\{ab+c(a+b)\} \\ &= -5abc(ab+ac+bc). \end{aligned}$$

Example 5: If $a+b+c=0$, show that

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)=9$$

Sol. Since $a+b+c=0$

$$\therefore \quad a+b=-c, \qquad b+c=-a, \qquad c+a=-b.$$

$$\text{L.H.S.} = \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)$$

$$= \left\{\frac{a}{-a}+\frac{b}{-b}+\frac{c}{-c}\right\}\left\{\frac{-c}{c}+\frac{-a}{a}+\frac{-b}{b}\right\} \left[\begin{array}{l}\text{Substituting}\\ \text{the values}\end{array}\right.$$

$$= (-1-1-1)(-1-1-1) = (-3)(-3) = 9.$$

Example 6: If $a+b+c=0$, show that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

Sol. Since $a+b+c=0$,

$$\therefore \quad a+b=-c \qquad \therefore \quad a^2+2ab+b^2=c^2$$

$$\therefore \quad a^2+b^2-c^2 = -2ab.$$

Similarly, $\quad b^2+c^2-a^2 = -2bc,$

and $\quad c^2+a^2-b^2 = -2ac.$

$$\text{Hence the expression} = \frac{1}{-2bc}+\frac{1}{-2ac}+\frac{1}{-2ab}$$

$$= \frac{-a-b-c}{2abc} = \frac{-(a+b+c)}{2abc}$$

$$= \frac{0}{2abc} = 0.$$

EXERCISE 177

If $a+b+c=0$, show that

1. $ab(a+b) = bc(b+c) = ca(c+a)$.

2. $a^2+b^2+c^2 = -2(ab+ac+bc)$.

3. $a(b^2+c^2-a^2) = b(c^2+a^2-b^2) = c(a^2+b^2-c^2) = -2abc$.

4. $a^2+ab+b^2 = b^2+bc+c^2 = c^2+ca+a^2$.

5. $(a^4+b^4+c^4) = \frac{1}{2}(a^2+b^2+c^2)^2$.

6. $a^4+b^4+c^4 = 2(ab+bc+ca)^2$.

7. $\dfrac{a^5+b^5+c^5}{5} = \dfrac{a^3+b^3+c^3}{3} \cdot \dfrac{a^2+b^2+c^2}{2}$.

8. $a^5+b^5+c^5 = 5abc(c^2-ab)$.

9. $\dfrac{a^2+b^2-c^2}{2ab} + \dfrac{b^2+c^2-a^2}{2bc} + \dfrac{c^2+a^2-b^2}{2ca} = -3$.

10. $\dfrac{a^2}{2a^2+bc} + \dfrac{b^2}{2b^2+ca} + \dfrac{c^2}{2c^2+ab} = 1$.

[*Hint.* $2a^2+bc = a^2+a(-b-c)+bc$. Factorise it.]

Example 7: If $2s = a+b+c$, then

$$s^2+(s-a)^2+(s-b)^2+(s-c)^2 = a^2+b^2+c^2.$$

Sol. The left-hand side

$$\begin{aligned}
&= s^2+s^2-2sa+a^2+s^2-2sb+b^2+s^2-2sc+c^2 \\
&= 4s^2-2s(a+b+c)+a^2+b^2+c^2 \\
&= 4s^2-2s.2s+a^2+b^2+c^2 \\
&= 4s^2-4s^2+a^2+b^2+c^2 \\
&= a^2+b^2+c^2.
\end{aligned}$$

Example 8: If $2s = a+b+c$, show that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s} = \frac{abc}{s(s-a)(s-b)(s-c)}.$$

Sol. The left-hand side

$$\begin{aligned}
&= \left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right) \\
&= \frac{(s-b)+(s-a)}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)} \\
&= \frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)} \\
&= \frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)} \qquad [\because\ 2s-a-b=c.]
\end{aligned}$$

$$= c\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$

$$= c\left\{\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}$$

$$= c\left\{\frac{s^2-sc+s^2-sa-sb+ab}{s(s-a)(s-b)(s-c)}\right\}$$

$$= c\left\{\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}\right\}.$$

$$= c\left\{\frac{2s^2-2s^2+ab}{s(s-a)(s-b)(s-c)}\right\} \qquad [\because\ a+b+c=2s.]$$

$$= \frac{abc}{s(s-a)(s-b)(s-c)}.$$

Example 9: If $x+\frac{1}{y}=1$, $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$.

Sol. Since $y+\frac{1}{z}=1$, $\quad\therefore\quad y=1-\frac{1}{z}=\frac{z-1}{z}$.

Substituting the value of y in $x+\frac{1}{y}=1$, we get

$$x+\frac{z}{z-1}=1 \qquad \text{or} \qquad \frac{zx-x+z}{z-1}=1$$

or $\qquad xz-x+z=z-1 \quad$ or $\quad xz+1=x.$

Dividing both sides by x, we get

$$z+\frac{1}{x}=1. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

From (i), $\qquad z=1-\frac{1}{x}$

$$\therefore \qquad xyz=xy\left(1-\frac{1}{x}\right). \quad =xy-y=y(x-1)$$

$$=y\times\left(-\frac{1}{y}\right) \qquad \left[\because\ x-1=-\frac{1}{y}\right]$$

$$=-1.$$

$$\therefore\ xyz+1=0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

Example 10: If $xy+yz+zx=1$, show that

$$\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{1}{xyz}.$$

Sol. Since $xy+yz+zx=1$, $\therefore$ $1-xy=z(x+y)$.

$$\therefore \quad \frac{x+y}{1-xy}=\frac{x+y}{z(x+y)}=\frac{1}{z} \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

Similarly, $$\frac{y+z}{1-yz}=\frac{1}{z} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

and $$\frac{z+x}{1-zx}=\frac{1}{y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$\therefore$ Adding (i), (ii) and (iii), we get

$$\frac{x+y}{1-xv}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{yz+xz+xy}{xyz}$$

$$=\frac{1}{xyz} \quad [\because \quad yz+xz+xy=1.]$$

EXERCISE 178.

If $2s=a+b+c$, show that :

1. $\dfrac{a}{s-a}+\dfrac{b}{s-b}+\dfrac{c}{s-c}+2=\dfrac{abc}{(s-a)(s-b)(s-c)}$.

2. $1-\left[\dfrac{b^2+c^2-a^2}{2bc}\right]^2=\dfrac{4s(s-a)(s-b)(s-c)}{b^2c^2}$.

3. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

4. $s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)$
$=(s-a)(s-b)(s-c)+abc$.

5. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)$
$+c(s-a)(s-b)=abc$.

6. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$
$=\frac{1}{2}(a^3+b^3+c^3-3abc)$.

7. If $a=1-\dfrac{1}{b}$, $b=1-\dfrac{1}{c}$, prove that $c=1-\dfrac{1}{a}$.

8. If $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$.
prove that $\dfrac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=4$.

9. If $x = a(b-c)$, $y = b(c-a)$, $z = c(a-b)$,
show that $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + \left(\frac{z}{c}\right)^3 = \frac{3xyz}{abc}$.

10. If $\frac{y+x}{y-x} + \frac{y+z}{y-z} = 2$, prove that $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$.

11. If $x+y = 2z$, prove that $\frac{x}{x-z} + \frac{y}{y-z} = 2$.

12. If $b^2 = ac$, $x = \frac{1}{2}(a+b)$, $y = \frac{1}{2}(b+c)$, prove that
$\frac{a}{x} + \frac{c}{y} = 2$.

13. If $3(a^2+b^2+c^2) = (a+b+c)^2$, prove that $a = b = c$.

14. If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$, prove that $\frac{1}{a^5} + \frac{1}{b^5} + \frac{1}{c^5}$
$= \frac{1}{(a+b+c)^5}$.

15. If $x = by+cz$, $y = cz+ax$ and $z = ax+by$.
show that $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$.

CHAPTER XXIII

FRACTIONS WITH DENOMINATORS IN CYCLIC ORDER.

It is useful to learn by heart the following formulae, after proper verification :

If a,b,c are the variables

(i) $\Sigma a = a+b+c.$

(ii) $\Sigma (a+b) = 2(a+b+c).$

(iii) $\Sigma (a-b) = 0.$

(iv) $\Sigma (a^2-b^2) = 0.$

(v) $\Sigma a(b-c) = 0.$

(vi) $\Sigma a^2(b-c) = -(a-b)(b-c)(c-a).$

(vii) $\Sigma ab(a-b) = -(a-b)(b-c)(c-a).$

(viii) $\Sigma a(b^2-c^2) = (a-b)(b-c)(c-a).$

(ix) $\Sigma (a-b)^3 = 3(a-b)(b-c)(c-a).$

(x) $\Sigma a^3(b-c) = -(a-b)(b-c)(c-a)(a+b+c).$

(xi) $\Sigma a^4(b-c) = -(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca).$

(xii) $\Sigma a^3(b^2-c^2) = -(a-b)(b-c)(c-a)(ab+bc+ca).$

Example 1 : Simplify :—

$$\frac{1}{(a-b)\ (a-c)}+\frac{1}{(b-c)\ (b-a)}+\frac{1}{(c-a)\ (c-b)}.$$

Sol. Putting the denominators in cyclic order, the expression

$$= \frac{1}{-(a-b)(c-a)}+\frac{1}{-(b-c)(a-b)}+\frac{1}{-(c-a)(b-c)}$$

$$= \frac{(b-c)+(c-a)+(a-b)}{-(a-b)(b-c)(c-a)}$$

$$= \frac{0}{-(a-b)(b-c)(c-a)} = 0.$$

NOTE : $\frac{a-b}{b-a} = -1$; $\frac{b-c}{c-b} = -1$; $\frac{c-a}{a-c} = -1.$

Example 2. Simplify :

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}.$$

Sol. Putting the denominators in cyclic order, the expression

$$=\frac{a^3}{-(a-b)(c-a)}+\frac{b^3}{-(a-b)(b-c)}+\frac{c^3}{-(c-a)(b-c)}$$

$$=\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{-(a-b)(b-c)(c-a)(a+b+c)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{+(a-b)(b-c)(c-a)(a+b+c)}{(a-b)(b-c)(c-a)}=a+b+c.$$

EXERCISE 179.

Simplify :

1. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$

2. $\frac{a^2}{(a-b)(a-c)}$ + two similar terms.

3. $\frac{bc}{(a-b)(a-c)}$ + two similar terms.

4. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}.$

5. $\frac{1}{bc(a-b)(a-c)}+\frac{1}{ca(b-c)(b-a)}+\frac{1}{ab(c-a)(c-b)}.$

6. $\Sigma\,\frac{a}{bc(a-b)(a-c)}.$

7. $\frac{x-a}{(a-b)(a-c)}+\frac{x-b}{(b-c)(b-a)}+\frac{x-c}{(c-a)(c-b)}.$

8. $\Sigma\,\frac{x^2-yz}{(x-y)(x-z)}.$

9. $\Sigma\,\frac{x^2+yz}{(x-y)(x-z)}.$

10. $\Sigma\,\frac{b^2c^2}{(a-b)(a-c)}$

11. $\Sigma\,\frac{a^4}{(a-b)(a-c)}.$

12. $\Sigma \dfrac{bc(b+c)}{(a-b)(a-c)}$.

13. $\Sigma \dfrac{b^2+c^2}{(a^2-b^2)(a^2-c^2)}$.

14. $\Sigma \dfrac{bc}{a(a^2-b^2)(a^2-c^2)}$.

15. $\Sigma \dfrac{a(a+b)(a+c)}{(a-b)(a-c)}$.

16. $\Sigma \dfrac{(a+1)^2}{(a-b)(a-c)}$.

17. $\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{a-b}{a+b}$.

18. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}$ + two similar terms.

19. $\dfrac{(a+b)(b+c)(c+a)}{abc}-\dfrac{a+b}{c}-\dfrac{b+c}{a}-\dfrac{c+a}{b}$.

20. $\dfrac{b^2+c^2-2a^2}{(a-b)(a-c)}+\dfrac{c^2+a^2-2b^2}{(b-c)(b-a)}+\dfrac{a^2+b^2-2c^2}{(c-a)(c-b)}$.

[*Hint.* $b^2+c^2-2a^2 = b^2+c^2+a^2-3a^2$].

21. $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$.

Example 3: Simplify $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-c)(b-a)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}$

Sol. Putting the denominators in cyclic order, we have

$$\text{the expression} = \frac{a}{-(a-b)(c-a)(x-a)}+\frac{b}{-(a-b)(b-c)(x-b)}+\frac{c}{-(c-a)(b-c)(x-c)}$$

$$= \frac{a(b-c)(x-b)(x-c)+ \text{ two similar terms}}{-(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}.$$

$$\text{The numerator} = a(b-c)\{x^2-x(b+c)+bc\} + \text{two similar terms}$$

$$= x^2\Sigma a(b-c) - x\Sigma a(b^2-c^2) + abc\Sigma(b-c)$$

$$= x^2\times 0 - x\times(a-b)(b-c)(c-a) + abc\times 0$$

$$= -x(a-b)(b-c)(c-a).$$

$$\therefore \quad \text{the expression} = \frac{-x(a-b)(b-c)(c-a)}{-(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}$$

$$= \frac{x}{(x-a)(x-b)(x-c)}.$$

EXERCISE 180

Simplify :

1. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$.

2. $\dfrac{a^2}{(a-b)(a-c)(x-a)}+\dfrac{b^2}{(b-c)(b-a)(x-b)}+\dfrac{c^2}{(c-a)(c-b)(x-c)}$.

3. $\dfrac{1}{(a-b)(a-c)(x+a)}+\dfrac{1}{(b-c)(b-a)(x+b)}+\dfrac{1}{(c-a)(c-b)(x+c)}$.

4. $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}$.

5. $\dfrac{1+a}{(a-b)(a-c)(x-a)}+\dfrac{1+b}{(b-c)(b-a)(x-b)}+\dfrac{1+c}{(c-a)(c-b)(x-b)}$.

[*Hint.* Decompose the expression into $\dfrac{1}{(a-b)(a-c)(x-a)}+$ two similar terms $+\dfrac{a}{(a-b)(a-c)(x-a)}+$ two similar terms.]

6. $\dfrac{1+pa+qa^2}{(a-b)(a-c)(x-a)}+\dfrac{1+pb+qb^2}{(b-c)(b-a)(x-b)}+\dfrac{1+pc+qc^2}{(c-a)(c-b)(x-c)}$.

Example 4: Simplify $\dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}$.

Sol. The exp. $= \dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{(ab-ac)^3+(bc-ba)^3+(ca-cb)^3}$.

Since $(a-b)+(b-c)+(c-a) = 0$

and $(ab-ac)+(bc-ba)+(ca-cb) = 0$,

$\therefore$ the expression $= \dfrac{3(a-b)(b-c)(c-a)}{3(ab-ac)(bc-ba)(ca-cb)}$

$= \dfrac{(a-b)(b-c)(c-a)}{abc(a-b)(b-c)(c-a)} = \dfrac{1}{abc}$.

EXERCISE 181

Simplify :

1. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

2. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b).}$.

3. $\dfrac{27(a+b+c)^3-(a+2b)^3-(b+2c)^3-(c+2a)^3}{(a+3b+2c)(b+3c+2a)(c+3a+2b)}$.

[*Hint.* Apply $(a+b+c)^3-a^3-b^3-c^3 = 3(a+b)(b+c)(c+a)$.]

4. $\dfrac{(a+b)^3+(c+d)^3-(a+c)^3-(b+d)^3}{(a+b)^2+(c+d)^2-(a+c)^2-(b+d)^2}$.

5. $\dfrac{(a-b)(a+b)^3+(b-c)(b+c)^3+(c-a)(c+a)^3}{(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3}$.

6. $\dfrac{x^2\left(\frac{1}{y}-\frac{1}{z}\right)+y^2\left(\frac{1}{z}-\frac{1}{x}\right)+z^2\left(\frac{1}{x}-\frac{1}{y}\right)}{x\left(\frac{1}{y}-\frac{1}{z}\right)+y\left(\frac{1}{z}-\frac{1}{x}\right)+z\left(\frac{1}{x}-\frac{1}{y}\right)}$

7. Show that $\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}+\dfrac{c-a}{1+ca}$

$= \dfrac{a-b}{1+ab} \times \dfrac{b-c}{1+bc} \times \dfrac{c-a}{1+ca}$.

8. Show that $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}$

$= m.\dfrac{a-b}{m+ab}.\ \dfrac{b-c}{m+bc}.\ \dfrac{c-a}{m+ca}$.

9. Prove that $\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$

$$=\left\{\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right\}^2.$$

SELECTED QUESTIONS—L

Simplify :

1. $\frac{y+z}{(x-y)(x-z)}+\frac{z+x}{(y-z)(y-x)}+\frac{x+y}{(z-x)(z-y)}$.

2. $\frac{1}{a^2-(b-c)^2}+\frac{1}{b^2-(c-a)^2}+\frac{1}{c^2-(a-b)^2}$.

3, $\frac{y+z-x}{(x-y)(x-z)}+\frac{z+x-y}{(y-z)(y-x)}+\frac{x+y-z}{(z-x)(z-y)}$.

4. $\frac{a(b+c)}{(a-b)(a-c)}+\frac{b(c+a)}{(b-c)(b-a)}+\frac{c(a+b)}{(c-a)(c-b)}$.

5. $\frac{bc(x-a)}{(a-b)(a-c)}+\frac{ca(x-b)}{(b-c)(b-a)}+\frac{ab(x-c)}{(c-a)(c-b)}$.

6. $\frac{bc(x-a)^2}{(a-b)(a-c)}+\frac{ca(x-b)^2}{(b-c)(b-a)}+\frac{ab(x-c)^2}{(c-a)(c-b)}$.

CHAPTER XXIV

MISCELLANEOUS FACTORS

1. **Factors of expressions of the form ax^2+bx+c which involve the application of the equations:**

$$p+q=m\ ;\qquad pq=n.$$

Example 1: Factorise $20x^2+xy-30y^2$.

Sol. Here we have to find out two numbers, p and q, such that

$p+q=+1$ ... ... (i)

and $pq=20\times(-30)=-600$ ... (ii)

Now, $(p-q)^2=(p+q)^2-4pq$ ...

$=1^2-4(-600)=2401$

$\therefore\quad p-q=\sqrt{2401}=49$ ... (iii)

From (i) and (iii), we get

$p=25$ and $q=-24$.

$\therefore\quad 20x^2+xy-30y^2=20x^2+25xy-24xy-30y^2$

$=5x(4x+5y)-6y(4x+5y)$

$=(4x+5y)(5x-6y)$.

EXERCISE 182

Factorise:

1. $18x^2-51xy+35y^2$.
2. $27a^2-33ab-20b^2$.
3. $55x^2-13xy-12y^2$.
4. $120x^4y^2+73x^3y^3-28x^2y^4$.
5. $44x^2-71xy+20y^2$.
6. $21x^2-58xy+21y^2$.
7. $28x^2-73xy-20y^2$.

2. **Factors of expressions of the form ax^2+bx+c, with literal co-efficients.**

Example 2: Factorise $a^2-(x+y)a+(x+1)(y-1)$.

Sol. Here we have to find out two quantities whose product $=(x+1)(y-1)$, and whose sum $=-(x+y)$.

Obviously, $-(x+1)$ and $-(y-1)$ are the required quantities.

$\therefore\quad a^2-(x+y)a+(x+1)(y-1)$

$=a^2-(x+1)a-(y-1)a+(x+1)(y-1)$

$=a\{a-(x+1)\}-(y-1)\{a-(x+1)\}$

$=(a-x-1)(a-y+1)$.

EXERCISE 183

Resolve into factors :

1. $a^2+a(x+y)+(x-2)(2+y)$.
2. $(a-1)x^2-x-a$.
3. $x^2+x-(a+1)(a+2)$.
4. $x^2-(1+a+b)x+a(1+b)$.
5. $(b^2-1)x^2+(b^2+1)x+b$.
6. $a^2-(2b+1)a+b^2+b-6$.
7. $x^2+2px+p^2-1$.
8. $a^2+(a+b+c)x+ab+ac$.

3. Factors of cubic expressions with one term missing.

Example 3: Factorise $x^3-19x-30$.

Sol. $$\begin{aligned} x^3-19x-30 &= x^3-19x-38+8 \\ &= (x)^3+(2)^3-19(x+2) \\ &= (x+2)(x^2-2x+4)-19(x+2) \\ &= (x+2)(x^2-2x+4-19) \\ &= (x+2)(x^2-2x-15) \\ &= (x+2)(x-5)(x+3). \end{aligned}$$

Example 4: Factorise $2x^3-3x^2-4$.

Sol. $$\begin{aligned} 2x^3-3x^2-4 &= 2x^3-3x^2-16+12 \\ &= (2x^3-16)-3(x^2-4) \\ &= 2(x^3-8)-3(x+2)(x-2) \\ &= 2(x-2)(x^2+2x+4)-3(x+2)(x-2) \\ &= (x-2)\{2x^2+4x+8-3x-6\} \\ &= (x-2)(2x^2+x+2). \end{aligned}$$

EXERCISE 184

Resolve into factors :

1. $x^3-13x+12$.
2. x^3-6x^2+32.
3. $x^3-3x-18$.
4. $x^3-5x+12$.
5. $x^3-3x^2y+20y^3$.
6. $8x^3+4x-3$.
7. $2a^3-a^2b-b^3$.
8. $27x^3+12x-5$.

4. Factors of expressions of the form $ax^4+bx^3+cx^2+bx+a$ or $ax^4+bx^3y+cx^2y^2+bxy^3+ay^4$.

Example 5: Factorise $8x^4-6x^3+7x^2-6x+8$.

Sol.

$$\text{The expression} = x^2\left(8x^2-6x+7-\frac{6}{x}+\frac{8}{x^2}\right)$$

$$= x^2\left[8\left(x^2+\frac{1}{x^2}\right)-6\left(x+\frac{1}{x}\right)+7\right]$$

$$= x^2\left[8\left(x^2+2+\frac{1}{x^2}\right)-6\left(x+\frac{1}{x}\right)+7-16\right]$$

$$= x^2\left[8\left(x+\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)-9\right]$$

Putting A for $\left(x+\frac{1}{x}\right)$, the expression

$$= x^2[8A^2-6A-9]$$
$$= x^2[8A^2-12A+6A-9]$$
$$= x^2[4A(2A-3)+3(2A-3)]$$
$$= x^2(2A-3)(4A+3).$$

Substituting the value of A, we have

$$\text{the expression} = x^2\left[2\left(x+\frac{1}{x}\right)-3\right]\left[4\left(x+\frac{1}{x}\right)+3\right]$$

$$= x^2\left[\frac{2x^2-3x+2}{x}\right]\left[\frac{4x^2+3x+4}{x}\right]$$

$$= (2x^2-3x+2)(4x^2+3x+4).$$

NOTE: It may be noted that in such expressions, the co-efficients of terms equi-distant from the middle term are equal, irrespective of signs, and their special characteristic is that when they are equated to zero, the equations thus formed have reciprocal roots: hence such expressions are known as **Reciprocal Expressions.**

EXERCISE 185

Resolve into factors:

1. $x^4-x^3-8x^2+x+1$.
2. $x^4+4x^3-10x^2+4x-1$.
3. $x^4-5x^3+6x^2-5x+1$.
4. $2x^4+3x^3+5x^2+3x+2$.
5. $3x^4-x^3y-8x^2y^2+xy^3+3y^4$.
6. $4x^4+8x^3y+3x^2y^2+8xy^3+4y^4$.
7. $6x^4-25x^3y+7x^2y^2+25xy^3+6y^4$.

5. **Factors of homogeneous expressions of the second degree involving three letters.**

Example 6: Factorise $2a^2+3ab-9ac-2b^2+7bc-5c^2$.

Sol.

If $c = 0$, the expression is reduced to

$$2a^2+3ab-2b^2 = (2a-b)(a+2b) \quad \ldots \quad \text{(i)}$$

If $b = 0$, the expression is reduced to

$$2a^2-9ac-5c^2 = (2a+c)(a-5c) \quad \ldots \quad \text{(ii)}$$

If $a = 0$, the expression is reduced to

$$-2b^2+7bc-5c^2 = (-2b+5c)(b-c)$$
$$= (2b-5c)(-b+c) \quad \ldots \quad \text{(iii)}$$

Factors in (i), (ii) and (iii) are incomplete and complementary. Their proper association can be determined by comparing the co-efficients and signs of like terms.

Here combining (i) and (ii) and comparing the results with (iii), we have

$$2a^2+3ab-9ac-2b^2+7bc-5c^2$$
$$= (2a-b+c)(a+2b-5c).$$

EXERCISE 186

Resolve into factors:

1. $2a^2-ab-6b^2+ac+19bc-15c^2$.
2. $2x^2-2y^2-3z^2+3xy-5xz+5yz$.
3. $x^2-xy+4zx+yz-2y^2+3z^2$.
4. $a^2+(x+y)a-2x^2+5xy-2y^2$.
5. $x(x+a)-2a^2+3b(a+x)+2b^2$.
6. $x^2-xy+4xz-2y^2-5yz+3z^2$.
7. $2x^2+5xy-3xz+5yz-3y^2-2z^2$.

6. **Factors of expressions which satisfy the condition $a+b+c = 0$.**

Example 7: Factorise $(a-y)^3+(y-z)^3+(z-x)^3$.

Sol.

Since $(x-y)+(y-z)+(z-x) = 0$.

$\therefore$ $(x-y)^3+(y-z)^3+(z-x)^3 = 3(x-y)(y-z)(z-x)$.

Example 8: Factorise $(x-2y)^5+(2y-3z)^5+(3z-x)^5$.

Sol.

Since $(x-2y)+(2y-3z)+(3z-x) = 0$,

$\therefore$ $(x-2y)^5+(2y-3z)^5+(3z-x)^5$

$$= -5(x-2y)(2y-3z)(3z-x) \times$$
$$\{(x-2y)(2y-3z)+(x-2y)(3z-x)+(2y-3z)(3z-x)\}.$$

EXERCISE 187

Resolve into factors:

1. $(2a-b)^3+(b-3c)^3+(3c-2a)^3$.
2. $(a-2b+1)^3+(a+b-1)^3+(b-2a)^3$.
3. $(a-1)^3+(a-2)^3-(2a-3)^3$.
4. $(a+b+1)^3+(a-b-1)^3-8a^3$.
5. $(2x+y)^3-(x-y)^3-(x+2y)^3$.
6. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.
7. $(x-y)^5+(y-z)^5+(z-x)^5$.
8. $(x-1)^5+(x-2)^5-(2x-3)^5$.

SECTIONAL REVISION V

TEST PAPERS

PAPER 1

1. (*i*) Multiply, using the method of detached co-efficients, $5x^4-2x^2+x-4$ and $3x^3+x+2$.

(ii) Without actual multiplication find the value of a for which the co-efficient of x^2 in $(4x-1)(2x+3)(x-a)$ vanishes.

2. Find by expansion the approximate value of
(i) $(1\cdot0001)^4$, (ii) $(1\cdot004)^5$.

3. Expand (i) $(3p+2q)^4$ (ii) $(3p-2q)^4$.

4. Find the values of a and b for which $x^4-4x^3-2x^2+ax+b$ may be a perfect square.

5. Find the square root of $57-28\sqrt{2}$.

6. If $x \equiv A(x-2)+B(x-1)$, find the values of A and B.

PAPER 2

1. (i) If $f(x)=4x^3-5x^2+3x-1$, find the value of $f(0)$, $f(1)$ and $f(-1)$.
(ii) Divide by the method of detached co-efficients $3x^3-2x+2-x^2$ by $3x^2+2-4x$.

2. Write down all the types of terms in the expression $\Sigma\frac{ab}{x^2(a-b)}$, where a, b, c are variables and x is constant.

3. If $a^x = m$, $a^y = n$ and $a^2 = (m^y n^x)^z$, show that $xyz = 1$.

4. Simplify $(1+x)^6 - (1-x)^6$.

5. Find the square root of $31 - 12\sqrt{3}$.

6. Simplify $\Sigma \frac{a^2}{(a-b)(a-c)(x-a)}$ where a, b, c are variables and x is constant.

PAPER 3

1. Exemplify the distinction between absolute symmetry and cyclic symmetry.

2. Write down a cyclic expression in x, y, z whose first term is $\frac{x^2-y^2}{x(y-z)}$.

3. Without actual multiplication, state why $(a^2+b^2)(a+b) = a^3+2a^2b+3ab^2+b^3$ is wrong.

4. Express $\frac{x^2-2x-14}{(x-1)(x-2)(x+3)}$ in proper partial fractions in the form of $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3}$.

5. Find the co-efficient of x^3 in the expansion of $(x+1)^5 (x-2)^4$.

6. Find the square root of $5\sqrt{5} + \sqrt{120}$.

PAPER 4

1. Write down $a^3(c-b) + b^3(a-c) + c^3(b-a)$ in cyclic order.

2. Express $\frac{5x^4-12x^3}{(x-1)(x-2)}$ in the form of

$$Ax^2+Bx+C+\frac{D}{x-1}+\frac{E}{x-2}.$$

3. A homogeneous and symmetrical expression of the 2nd degree in x, y and z has the value 14 when $x = -3$, $y = 4$ and $z = 3$ and the value 16 when $x = -2$, $y = 1$, $z = -5$; find it.

4. Rupee one with compound interest @ 5% per annum after 12 years amounts to Rs. $(1+\cdot 05)^{12}$. Calculate the value of the amount to 3 places of decimals.

5. Simplify $\frac{2+\sqrt{3}}{\sqrt{2+\sqrt{2+\sqrt{3}}}} + \frac{2-\sqrt{3}}{\sqrt{2-\sqrt{2-\sqrt{3}}}}$.

6. Eliminate x and y from the equations:

$$x+\frac{1}{x}=a, \qquad y+\frac{1}{y}=b, \qquad xy+\frac{1}{xy}=c.$$

PAPER 5

1. Simplify (i) $\sqrt{(2+\sqrt{3})}-\sqrt{(2-\sqrt{3})}$.

(ii) $\dfrac{\sqrt{5}+2}{\sqrt{(9-4\sqrt{5})}}$.

2. Find the square root of $x^2+\frac{1}{x^2}+6\left(x-\frac{1}{x}\right)+7$.

3. Find the values of A, B and p, in order that
$$A(x-1)^2+B(x-p)^2 \equiv x^2-8x+10.$$

4. Show that $p+q+r+s$ is a factor of
$$(p+r)(p+s)(q+r)(q+s)-(pq-rs)^2.$$

5. Eliminate x, y and z from the equations:
$(y+z)^2=ayz$, $(z+x)^2=bzx$, $(x+y)^2=cxy$.

6. If $x+y+z=0$, show that $\Sigma(x-y)^2=3\Sigma x^2$ where x, y, z are variables.

PAPER 6

1. Show that $l+m+n$ is a factor of
$l^2(m+n)+m^2(n+l)+n^2(l+m)$ if $l^3+m^3+n^3=0$.

2. Simplify $\Sigma\dfrac{1}{a(a+b)(a+c)}$.

3. Find the square root of:
$9+3x+2y+\frac{1}{3}xy+\frac{1}{4}x^2+\frac{1}{9}y^2$.

4. If $2s=a+b+c$, show that

(i) $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$.

(ii) $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$.

(iii) $s^2+(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)=bc+ca+ab$.

5. (i) Rationalise the denominator of $\dfrac{\sqrt{5}-\sqrt{3}-2}{\sqrt{5}+\sqrt{3}+2}$.

(ii) Simplify $\dfrac{1}{2(\sqrt{5}-\sqrt{3})}+\dfrac{\sqrt{3}}{(\sqrt{5}+1)}-\dfrac{\sqrt{5}}{2(\sqrt{3}-1)}$.

6. Eliminate y from $m = y^x$ and $n = x^y$

PAPER 7

1. Expand (i) $(1+x)^6$, (ii) $(1-x)^6$.

2. Find the square root of $x^4+1+2x+x^3+\frac{1}{x^2}+\frac{x^2}{4}$.

3. Evaluate $\sqrt[4]{(17-12\sqrt{2})}$ correct up to 3 decimal places.

4. Eliminate x from $x^3+\frac{1}{x^3}=a^3$ and $x+\frac{1}{x}=b$.

5. Simplify $\Sigma\frac{a}{ab(a-b)(b-c)}$, where a, b, c are variables.

6. If $a+b+c=0$, show that $\Sigma a^3 = 3abc$,
where a, b, c are variables.

PAPER 8

1. Expand (i) $(x+a)^5$, (ii) $(x-a)^5$.

2. Find the square root of $4x^4-12x^3+29x^2-30x+25$.

3. Eliminate x and y from the equations:
$a=\frac{1}{x}-x$, $b=\frac{1}{y}-y$ and $x^2+y^2=1$.

4. Simplify $\frac{(x-b)(x-c)}{(a-b)(a-c)}$ +two similar terms.

5. Find the value of p so that $x^2+(p-3)x-3p$ and $x^3+8x^2+17x+10$ may have a common linear factor.

6. If $\frac{7x^2+x+1}{x^3-1}\equiv\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$, find A, B, and C.

OBJECTIVE TYPE ITEMS — V

1. Fill up the blanks in the following:

(i) The process of finding the roots of quantities is called ————————.

(ii) The number of equations required to eliminate three unknown quantities is————————.

(iii) An expression is said to be————————when all its terms are of the same degree.

(iv) An expression involving———or more———surds is called a compound surd.

(v) The rationalising factor of a binomial quadratic surd is its———.

(vi) In reciprocal expressions, the coefficients of terms ———from the middle term are equal, without regard to signs.

2. Write 'T' or 'F' according as the statement is true or false, in the space provided in the following cases:

——— (i) If the sum of any three numbers is zero, the sum of their cubes is equal to three times their product.

——— (ii) A surd may sometimes be expressed as the product of a rational quantity and a surd.

———(iii) Evolution is the process of finding the powers of quantities.

———(iv) A negative quantity cannot have a real square root.

——— (v) The square root of a rational quantity cannot be partly rational and partly irrational.

———(vi) An identity is a relation of equality which is true for all values of the letters involved.

3. Put an *X* in the space provided against the correct answer in each of the following.

(i) If $p+q+r=0$, then $\frac{p^2}{6qr}+\frac{q^2}{6rp}+\frac{r^2}{6pq}$ is equal to

———*A.* Zero ———*B.* $\frac{1}{2}$
———*C.* 1 ———*D.* $\frac{1}{6}$

(ii) If $\sqrt{5}=2{\cdot}2$, the value of $\frac{1}{\sqrt{5}}$ is

———*A.* ·11 ———*B.* ·22
———*C.* ·44 ———*D.* 1·1

(iii) If $\frac{1}{x}+x=p$ and $\frac{1}{x}-x=q$, then

———*A.* $p^2+q^2=2$ ———*B.* $p^2-q^2=2$
———*C.* $p^2+q^2=4$ ———*D.* $p^2-q^2=4$

(iv) The square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$ is

——A. $\pm\left(x+\frac{1}{x}+2\right)$ ——B. $\pm\left(x+\frac{1}{x}-2\right)$

——C. $\pm\left(x-\frac{1}{x}+2\right)$ ——D. $\pm\left(x-\frac{1}{x}-2\right)$

4. Put a tick ($\surd$) against the correct results in the following cases.

——— (i) $\Sigma c(a-b) = 0$

——— (ii) $\Sigma a^2(b-c) = (a-b)\ (b-c)(c-a)$

——— (iii) $\Sigma a(b^2-c^2) = -(a-b)(b-c)(c-a)$

——— (iv) $\Sigma(a-b)^3 = 3(a-b)(b-c)(c-a)$

——— (v) $\Sigma a^3(b-c) = -(a-b)(b-c)(c-a)(a+b+c)$.

MISCELLANEOUS QUESTIONS

(A)

1. Find the H.C.F. of x^5-x^3+8 and x^5-x^2+4.

2. (i) $2b = a+c$ and $\frac{2}{c} = \frac{1}{b}+\frac{1}{d}$; show that $a : b = c : d$.

(ii) If a, b, c are in continued proportion, prove that $a^2+ab : b^2 :: b^2+bc : c^2$.

3. Resolve into factors:

(i) $a^2-b^2+ab(b-a)$.

(ii) $c(a^2+b^2-c^2)+b(c^2+a^2-b^2)+a(b^2+c^2-a^2)+6abc$.

(iii) $\left(a+\frac{1}{a}\right)^2-\left(b+\frac{1}{b}\right)^2$.

4. Simplify $\frac{(x+1)^2}{(x-y)(x-z)}$ + two similar terms.

5. Solve the equations (i) $x+y=8$, $y+z=10$, $z+x=12$.

(ii) $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3$.

6. If $a = \frac{b}{1-b}$ and $b = \frac{c}{1-c}$, prove that $c = \frac{a}{1+2a}$.

7. Express $\frac{3x+11}{(x+2)(x+3)}$ in the form of $\frac{A}{x+2}+\frac{B}{x+3}$.

8. The perimeter of a rectangular field is 72 m. ; find its dimensions if it contains 315 sq. m.

(B)

1. Find the H.C.F. of $x^4+9x-20$ and $5x^4+9x^3-64$.

2. If $a:b::b:c$, prove that

(i) $a-2b+c=\frac{(a-b)^2}{a}=\frac{(b-c)^2}{c}$.

(ii) $(a+b+c):(a-b+c)=(a+b+c)^2:(a^2+b^2+c^2)$.

3. Resolve into factors:

(i) $a^2b^2-a^2-b^2+1$.

(ii) $x(y^2-a)-y(x^2-a)$.

(iii) $324x^4+1$, and *hence* find out the factors of 3240001.

4. Simplify $\frac{bc(x-a)^2}{(a-b)(a-c)}$ + two similar terms.

5. Solve the equations (i) $\frac{x-1}{2}+\frac{2x-7}{3}=x-2$.

(ii) $\sqrt{4x+5}+\sqrt{4x-11}=8$.

6. Eliminate h, k and l from the equations:

$h=k^a,\ l=k^b,\ h^b l^a=\sqrt[c]{k^2}$.

7. Express $\frac{11x+5}{(x-2)(x+1)^2}$ in the form of $\frac{A}{x-2}+\frac{Bx+C}{(x-1)^2}$.

8. The difference between the length and breadth of a rectangle is 18 m. and its area is 448 sq. m. ; find

(i) its semi-perimeter,

(ii) its length and breadth.

(C)

1. Find the value of:

(i) $128^{\frac{4}{7}}$, $\sqrt[4]{81^3}$ and $\left(\frac{1}{27}\right)^{-\frac{4}{3}}$.

(ii) $\left\{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}\right\}^2$

2. Show that (i) 7^n-1 is divisible by 6, and

(ii) $45^{2n+1}+1$ is divisible by 46,

if n is a positive integer,

3. If $x:a=y:b=z:c$, then

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

4. Resolve into factors:

(i) $x^2-x-1332$.

(ii) $z(x+y)^2-y(z+x)^2$.

5. Simplify $\dfrac{3\sqrt{3}-2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

6. Eliminate t from the equations $x=t+\dfrac{1}{t}$, $y^2=t^2-\dfrac{1}{t^2}$.

7. Find the values of A, B, C in the following identity:

$$\frac{x^2}{(x-1)\ x-2)(x-3)}\equiv\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

8. A certain sum was divided equally among a certain number of persons; had there been 1 person more, each would have received 5 paise less, and had there been 4 persons fewer, each would have received 25 paise more; find the sum of money and the number of men.

(D)

1. (*a*) Find the value $\sqrt[5]{243^3}$ and $1296^{-\frac{3}{4}}$.

(*b*) If $x+\dfrac{1}{y}=1$, $y-\dfrac{1}{z}=1$, prove that $xyz=1$.

2. (*a*) Show that $5^{3x}-4^{3x}$ is always divisible by 61.

(*b*) Divide $2x^3+5x^2-mx+4$ by x^2+2x-1, and find the value of m for which the given divisor will be a factor of the given dividend.

3. Find the H. C. F. of $8x^4+3x+10$ and $10x^4+3x^3+8$.

4. If $\dfrac{a^2+c^2}{ab+cd}=\dfrac{ab+cd}{b^2+d^2}$, show that $\dfrac{a}{b}=\dfrac{c}{d}$.

5. Resolved into factors :

(i) $a(a+1)x^2+x-a(a-1)$.

(ii) $x^4+y^4+1+2x^2y^2-2x^2-2y^2$.

6. Simplify $\frac{x^3+6x^2+2x-15}{x^3+5x^2-2x-10}$.

7. If $\frac{x^2+3x+7}{(x^2+x+1)(x+1)^2} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+x+1}$, find the values of A, B, C and D.

8. A person bought 13 horses and 9 cows for Rs. 4,185 and at the same rates 8 horses and 6 cows for Rs. 2,640. Find the price of 9 horses and 5 cows.

(E)

1. (*a*) Simplify by the shortest possible method.

$(5x-8)^3-(3x-8)^3-6x(5x-8)(3x-8)$.

(*b*) Without actual multiplication, work out the co-efficient of x^3 in the product of $x^4-5x^3-2x^2+7x-4$ and x^2-3x+1.

2. If $\frac{1}{b+c}+\frac{1}{c+a}=\frac{2}{a+b}$, shew that $a^2+b^2=2c^2$.

3. Resolve into factors :

(i) x^3+64.

(ii) $(x+y+z)(xy+yz+zx)-xyz$.

4. Simplify $\frac{3^{2x+1}.\ 5^{2x-y}.\ 7^{2x+y+2},\ 15^y}{15^x.\ 21^{y+x+2}.\ 35^x}$.

5. Solve the equations (i) $\frac{x-1}{x-2}+\frac{x-5}{x-6}=\frac{x-2}{x-3}+\frac{x-4}{x-5}$

(ii) $y-\frac{1}{x}=x-\frac{1}{y}=\frac{3}{2}$.

6. Eliminate x, y, z from $y=az$, $z=bx$, $x=cy$.

7. Find the values of A, B, C in $(2x-3)^2 \equiv Ax^2+2Bx-3C$.

8. A railway journey of 240 km would take half an hour less if the speed of the train were increased by 2 km an hour. Find the speed in km per hour.

(F)

1. (i) If $2s = a+b+c$, show that
$$(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3.$$

(ii) If $a^x = b^y = c^z$ and $ac = b^2$, then $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$.

2. Extract the square root of:
$$x^4+4x^2+\frac{1}{x^2}-2x-\frac{4}{x}+4.$$

3. If $a=\sqrt{\frac{x-1}{x+1}}$, express $\frac{1-a}{1+a}$ in terms of x.

4. Resolves into factors:
(i) x^4+64.
(ii) $x^2+(a+b+c)x+ab+ac$.

5. Simplify $\Sigma \frac{1}{(a-b)(a-c)(x-a)}$.

6. Draw the graph of $\frac{3x-7}{6}$. From the graph find the value of the function when $x=3{\cdot}5$; also find for what value of x the function becomes equal to $1{\cdot}1$?

7. Determine the values of p, q, r in
$$(x-1)(x-2)(x-3) \equiv (x-4)^3+p(x-4)^2+q\,(x-4)+r.$$

8. A farmer bought an equal number of two kinds of sheep, one kind @ Rs. 24 each and the other @ Rs. 32 each ; if he had spent his money equally on the two kinds, he would have had 3 sheep more than he had. How many of each kind did he buy ?

(G)

1. (i) If $a+b+c=15$ and $ab+ac+bc=71$, find the value of $a^2+b^2+c^2$.

(ii) If $a^2+b^2+c^2=155$ and $ab+ac+bc=143$, find the value of $a+b+c$.

2. The H.C.F. and L.C.M. of two integral expressions A and B, each of the second degree, are $x+3$ and x^3-7x+6; find A and B.

3. If $a:b::b:c$ show that
$$(a+b+c)(a-b+c) = a^2+b^2+c^2.$$

4. Resolve into factors:
(i) $16x^8-1$.
(ii) $(5x+3y)^3-(3x+5y)^3-8(x-y)^3$.

5. Simplify $\dfrac{x^3-1}{x^2-1} \times \dfrac{x^2+3x+2}{x^4+x^2+1} \div \dfrac{x^2+x-2}{x^4+x}$.

6. If $x-\dfrac{1}{x}=k$, find the value of $x^3-\dfrac{1}{x^3}$.

7. Find the values of k, l, m in
$$2x^2-11x+5 \equiv (kx+l)(x-3)+m(x^2+2x-5).$$

8. A number consists of two digits. When the number is divided by the sum of its digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number.

(H)

1. (*a*) For what value of x will both the expressions x^3-x^2-5x-3 and $x^3-4x^2-11x-6$ vanish?

(*b*) To make $9x^4-12x^3+10x^2-3x-3$ a perfect square, (i) what should be added to it, (ii) what should be subtracted from it, (iii) what values should be given to x?

2. The H.C.F. of two expressions is x^2+3x+2 and their L.C.M. is $(x^2+3x+2)(x-3)(x+5)$, one expression is $x^3+8x^2+17x+10$; find the other.

3. (i) If $ax=y$ and $by=x$, show that $\dfrac{1}{1+a}+\dfrac{1}{1+b}=1$.

(ii) If $a:b=c:d=e:f$, prove that
$$(a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2.$$

4. If $x+y+z=6$ and $xy+yz+zx=9$, shew that
$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0.$$

5. Simplify $\dfrac{(x^2)^{a+b} \times (x^2)^{b+c} \times (x^2)^{c+a}}{x^{4a} \times x^{4b} \times z^{4c}}$.

6. Solve the equations (i) $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}=\dfrac{3}{7}$

(ii) $x^2-y^2=18,\ x-y=3.$

7. If $f(x)=x^3-3x^2+3x-1$ find the values of $f(0)$, $f(1)$ and $f(x+1)-f(x-1)$.

8. A man, who went out for a walk between 5 & 6 A.M. & returned between 6 & 7, found that the hands of his watch had exactly changed places. When did he go out ?

[*Hint.* Suppose he went out at x minutes past 5 and returned at y minutes past 6. From these suppositions, we get $x=30+\dfrac{y}{12}$ and $y=25+\dfrac{x}{12}$.]

(I)

1. (*a*) If $x^2+y^2=1$, complete $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{1}{\quad}$.

(*b*) Fill in the blank in $\dfrac{x^2+y^2}{a}=\dfrac{xy}{b}=\dfrac{x^2-y^2}{\quad}$.

2. (*a*) Find the value of $x^3+y^3-z^3+3xyz$ when $x=3{\cdot}461$, $y=2{\cdot}314$, $z=5{\cdot}775$.

(*b*) Find the continued product of x^2+x+1, x^2-x+1, x^4-x^2+1.

3. For what value of x will the expression $2x^4-7x^3+13x^2-9x-3$ be divisible by x^2-2x+3 ?

4. If $x+y=a$ and $xy=b$, express x^4+y^4 in terms of a and b.

5. Simplify $\dfrac{x^3+1}{x^2-5x+6}\times\dfrac{x^2-7x+10}{x^4+x^2+1}\div\dfrac{x^2-4x-5}{x^3+x^2+x}$.

6. Solve the equation $\dfrac{1}{x-1}+\dfrac{4}{x-4}=\dfrac{2}{x-2}+\dfrac{3}{x-3}$.

7. If $\dfrac{x}{a-b}=\dfrac{y}{b-c}=\dfrac{z}{c-a}$, prove that $x(a+b)+y(b+c)+z(c+a)=0$.

8. Simplify $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$ by the principles of homogeneity, symmetry and indeterminate co-efficients.

(J)

1. (*a*) Simplify $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a}$.

(*b*) Show that $7^{2n+1}+1$ is exactly divisible by 8 for all integral values of n, even or odd.

2. (*a*) If $x = b+c-a$, $y = c+a-b$, $z = a+b-c$, then
$$x^3+y^3+z^3-3xyz = 4(a^3+b^3+c^3-3abc).$$

(*b*) Find the continued product of
$(1+x)(1+x^2)(1+x^4)$, without actual multiplication.

3. Find the L. C. M. of $3a^2-ab-2b^2$, $8a^3-27b^3$ and $6a^2-5ab-6b^2$.

4. Resolve into factors (i) $a^3-19ab^2+30b^3$,
(ii) $a^3-17x+26$.

5. Simplify $\frac{2^6 \times 16^{-2} \times 256}{4^{-9} \times 2^{23}}$.

6. Find the square root of $22-12\sqrt{2}$.

7. If $a+b+c = 0$, show that
$$a^2+ab+b^2 = b^2+bc+c^2 = c^2+ac+a^2.$$

8. Prove that $(a+b+c)^3-a^3-b^3-c^3 \equiv 3(a+b)(b+c)(c+a)$.

PART III

CHAPTER XXV

QUADRATIC EQUATIONS AND PROBLEMS

1. Equations involving the square of the variable but no higher power are called quadratic equations. $3x^2 = 12$; $3x^2 - 5x = 0$; $x^2 - x + 7 = 0$ are all quadratic equations in x.

The general form of a quadratic equation is

$$ax^2 + bx + c = 0.$$

where a, b and c do not contain x.

When there is no term in x, as in $3x^2 - 12 = 0$, the equation is called a **pure quadratic.** When there is a term in x, as in $3x^2 - 5x = 0$ or $x^2 - x - 7 = 0$, the equation is called an **adfected quadratic.**

2. Solution of Pure Quadratic.

Example 1: Solve the equation $7x^2 = 63$.

Sol. Dividing both sides by 7, $x^2 = 9$.

Taking the square root of each side, we have $x = \pm 3$.

For when $x = +3$ or -3, $7x^2 = 7(\pm 3)^2 = 63$.

NOTE: The value of x which satisfies the equation is called a **root** of the equation. Quadratic equations have two roots.

If $x^2 = 9$, then it seems necessary to write the answer as $\pm x = \pm 3$; which apparently contains 4 different answers:

(i) $+x = +3$, (ii) $+x = -3$, (iii) $-x = +3$, (iv) $-x = -3$,

But, (i) and (iv) give the same solution *viz.*, $x = +3$

and (ii) ,, (iii) ,, ,, $x = -3$

Hence $x = \pm 3$ is the same as $\pm x = \pm 3$.

Example 2: Solve the equation $5(x-3)^2 = 180$.

Sol. Dividing both sides by 5, we have $(x-3)^2 = 36$.

Taking the square root of both sides, we have

$$x - 3 = \pm 6$$

i.e. $x - 3 = +6$, $\therefore\ x = +9$

and $x - 3 = -6$, $\therefore\ x = -3$.

$\therefore$ the two roots are $+9$ and -3.

Example 3: Solve the equation $\dfrac{1}{1+x} + \dfrac{1}{1-x} = \dfrac{4}{1+x^2}$.

Sol. Simplifying the left-hand side, we have

$$\frac{1-x+1+x}{1-x^2} = \frac{4}{1+x^2}$$

or $$\frac{2}{1-x^2} = \frac{4}{1+x^2}.$$

Dividing both sides by 2, we have

$$\frac{1}{1-x^2} = \frac{2}{1+x^2}.$$

By cross-multiplication, we get $1+x^2 = 2-2x^2.$

By transposition, $3x^2 = 1$ or $x^2 = \frac{1}{3}.$

Taking the square root of both sides, $x = \pm\frac{1}{\sqrt{3}}.$

EXERCISE 188

Solve the following equations:

1. $3x^2+7 = 55.$

2. $10-2x^2 = \frac{1}{2}x^2.$

3. $(x-5)^2 = 25.$

4. $(x+1)^2 = 2x+5.$

5. $(x-2)(x-5) = 15-7x.$

6. $(2x-3)(x-1) = 12-5x.$

7. $\frac{ax^2}{b} = \frac{a^3}{b}.$

8. $15x^2-7x^2 = 20+3x^2.$

9. $8x^2+11 = 5x^2+14.$

10. $29-10x^2 = 83-16x^2.$

11. $\frac{x^2+7}{x^2-5} = \frac{x^2-15}{x^2-3}.$

12. $\frac{4}{x-3} - \frac{4}{x+3} = \frac{1}{3}.$

13. $\frac{x^2-1}{5} + \frac{1-2x^2}{3} = x^2.$

14. $\frac{1}{1+x} + \frac{1}{1-x} = \frac{6}{1+x^2}.$

15. $\frac{1}{a+x} + \frac{1}{a-x} = 1.$

16. $12\left(\frac{x}{2}+1\right)^2 = 108.$

3. Solution by Factorisation.

The solution of adfected quadratic equations depends on the principle that if $a \times b = 0$, then either $a = 0$, or $b = 0$.

Example 4: Solve the equation $(x+2)(x-3) = 0.$

Sol. Since the product of two factors $(x+2)$ and $(x-3)$ is equal to zero, either $x+2 = 0$, or $x-3 = 0.$

If $x+2=0$, $x=-2$; if $x-3=0$, $x=+3$.
$\therefore$ the required roots are -2 and $+3$.

For when $x=-2$, we have $(-2+2)(-2-3)=0\times-5=0$
and when $x=3$, „ „ $(3+2)(3-3)=5\times0=0$.

Example 5: Solve the equation $2x+5=3x^2$.

Sol. Reducing it to the standard from $ax^2+bx+c=0$, we have by transposition $3x^2-2x-5=0$.
Factorising, $(x+1)(3x-5)=0$
whence, $x+1=0$ or $3x-5=0$
$\therefore$ the required roots are -1 and $+\frac{5}{3}$.

Verification.
(1) When $x=-1$,
$3x^2-2x-5=3(-1)^2-2(-1)-5$
$=3+2-5=0$.
(2) When $x=\frac{5}{3}$,
$3x^2-2x-5=3(\frac{5}{3})^2-2.\frac{5}{3}-5$
$=\frac{25}{3}-\frac{10}{3}-5=0$.

Example 6: Solve the equation $x-\frac{9}{x}=15\left(1-\frac{3}{x}\right)$.

Sol. Multiplying both sides by x, we have
$x^2-9=15x-45$.
By transposition, $x^2-15x+36=0$.
Factorising, $(x-12)(x-3)=0$,
whence $x-12=0$, or $x-3=0$.
$\therefore$ the required roots are $+12$ and $+3$.

This can be verified by substitution.

EXERCISE 189.

Write down the roots of the following equations:

1. $(x-4)(x-6)=0$.
2. $(x+4)(x+6)=0$.
3. $(x-5)(x+7)=0$.
4. $(x+8)(x-9)=0$.
5. $(1-x)(3-x)=0$.
6. $(2-x)(5+x)=0$.
7. $(x-a)(x+b)=0$.
8. $(2x-3)(3x-4)=0$.
9. $(6x-1)(5x+2)=0$.
10. $(ax-b)(bx+c)=0$.
11. $\left(\frac{x}{3}+2\right)\left(3x-\frac{2}{3}\right)=0$.
12. $x(x+5)=0$.
13. $(x+4)^2=0$.
14. $(3x-7)^2=0$.

15. $(3x+a)^2 = 0$. **16.** $(ax-b)^2 = 0$.

17. $\{x-(p+q)\}^2 = 0$. **18.** $\{x-(p+q)\}\{x+(p-q)\}=0$

Solve the following equations:

19. $x^2-9x+14 = 0$. **20.** $x^2+10x+24 = 0$.

21. $x^2-x-12 = 0$. **22.** $1+7x+12x^2 = 0$.

23. $27+12x+x^2 = 0$. **24.** $5x^2+4x-1 = 0$.

25. $6x^2+6 = 13x$. **26.** $3x^2+3 = 10x$.

27. $x^2-5x = 36$. **28.** $16x^2-159x = 10$.

29. $8x^2+3 = 14x$. **30.** $5(3x^2-4) = 44x$.

NOTE: During the process of solving an equation, we are likely to divide both sides by an expression containing the variable. By doing so, we lose some roots. Suppose it is required to solve

$(x-1)(x-3) = 2x^2-9x+9$.

Dividing by $x-3$, $x-1 = 2x-3$ or $x = 2$.

But the solutions are 3 and 2. We have lost the other root $x = 3$.

Hence, if you remove a factor which contains the variable, equate it to zero and get a root.

4. Method of Completing the Square.

Quadratic equations which cannot be easily done by the method of factorisation can be solved by the method of completing the square.

The method is illustrated in the following examples.

Example 7: Solve the equation $x^2-3x = 18$.

Sol. Completing the square by adding to each side the square of half the co-efficient of x, we have

$$x^2-3x+(\tfrac{3}{2})^2 = 18+(\tfrac{3}{2})^2$$

or $$(x-\tfrac{3}{2})^2 = 8\tfrac{1}{4}.$$

Taking the square root of both sides, we have

$$(x-\tfrac{3}{2}) = \pm\tfrac{9}{2}$$

$$\therefore \quad x = \tfrac{3}{2}\pm\tfrac{9}{2}.$$

$\therefore$ the required roots are $+6$, -3.

Example 8: Solve the equation $6x^2+7x-20 = 0$.

Sol. By transposition, $6x^2+7x = 20$.

Making the co-efficient of x^2 unity by dividing both sides by 6, we have

$$x^2+\tfrac{7}{6}x = \tfrac{10}{3}.$$

Adding the square of half the co-efficient of x to each side, *i.e.*, $(\frac{7}{12})^2$, we have

$$x^2+\tfrac{7}{6}x+(\tfrac{7}{12})^2 = \tfrac{10}{3}+(\tfrac{7}{12})^2$$

$$\therefore \quad (x+\tfrac{7}{12})^2 = \tfrac{289}{144}.$$

Taking the square root of both sides, we have

$$x+\tfrac{7}{12} = \pm\tfrac{17}{12}$$

$$\therefore \quad x = -\tfrac{7}{12}\pm\tfrac{17}{12}.$$

Hence the required roots are $-\frac{5}{2}$, $+\frac{4}{3}$.

EXERCISE 190

Solve the following equations by completing the square:

1. $x^2-4x = 45$.
2. $x^2-6x-65 = 0$.
3. $3x^2+13x = 30$.
4. $6x^2-13x+6 = 0$.
5. $8x^2-14x-9 = 0$.
6. $15x^2-11x-12 = 0$.
7. $5x^2-2x-3 = 0$.
8. $4x^2-48x-25 = 0$.
9. $3x^2-51x+216 = 0$.
10. $ax^2-bx-c = 0$.

5. Method of Solving by Formula.

The method of completing the square, when applied to the general form $ax^2+bx+c = 0$, gives us a well-known formula for finding the roots of the quadratic equation,

$$ax^2+bx+c = 0.$$

Dividing the equation by a, we get

$$x^2+\frac{b}{a}x+\frac{c}{a} = 0.$$

$$\therefore \quad x^2+\frac{b}{a}x = -\frac{c}{a}$$

$$\therefore \quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\therefore \quad \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\therefore \quad x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula should be committed to memory.

NOTE: Instead of dividing $ax^2+bx+c=0$ by a and completing the square, we can also multiply by $4a$, complete the square and obtain the result thus:

Multiply by $4a$; $ax^2+bx+c=0$

$4a^2x^2+4abx=-4ac$

or $4a^2x^2+4abx+b^2 = b^2-4ac$

or $(2ax+b)^2 = b^2-4ac$

$\therefore\ 2ax+b = \pm\sqrt{b^2-4ac}$

or $2ax = -b\pm\sqrt{b^2-4ac}$

$\therefore\ x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Example 9: Solve the equation $5x^2-31x+30=0$ by the formula.

Sol. If the given equation is put in the general form $ax^2+bx+c=0$, then $a=+5$, $b=-31$, $c=30$, and

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{31\pm\sqrt{(-31)^2-4.\ 5.\ 30}}{2.5}$$

$$= \frac{31\pm\sqrt{961-600}}{10} = \frac{31\pm\sqrt{361}}{10}$$

$$= \frac{31\pm19}{10} = 5,\ \frac{6}{5}.$$

EXERCISE 191

Solve the following equations by the formula:

1. $3x^2-5x-2=0$.
2. $3x^2-2x-1=0$.
3. $3x^2-7x+2=0$.
4. $12x^2-17x+6=0$.
5. $4x^2-65x+126=0$.
6. $10x+11=\dfrac{6}{x}$.
7. $\dfrac{5}{5-x}+\dfrac{8}{8-x}=3$.
8. $7x^2+17x+6=0$.
9. $5x^2-23x+12=0$.
10. $15x^2+2px-8p^2=0$.
11. $x^2+px=2(x+p)$.
12. $pq(x^2-1)=(p^2-q^2)x$.

6. PROBLEMS INVOLVING QUADRATIC EQUATIONS

Since a quadratic equation has always two roots, the problems involving quadratic equations give us two different solutions. The student has to verify to see which of the two roots satisfies the particular problem.

Generally one of the two solutions will satisfy the given problem and the other solution will satisfy a similar problem, if properly interpreted.

Example 10: Find two consecutive even numbers such that the sum of their squares is 100.

Sol. Let x and $x+2$ be the two consecutive even numbers.

$$\therefore \quad x^2+(x+2)^2 = 100$$
$$\therefore \quad x^2+x^2+4x+4 = 100$$
$$\therefore \quad 2x^2+4x-96 = 0$$
$$\therefore \quad x^2+2x-48 = 0$$
$$\therefore \quad (x+8)(x-6) = 0$$

Hence the roots are -8 and $+6$.

Verification.

(1) When $x = -8$, $x+2 = -6$ and $(-8^2)+(-6)^2 = 100$.
(2) When $x = +6$, $x+2 = +8$ and $(+6)^2+(+8)^2 = 100$.

Example 11: The perimeter of a rectangle is 54 metres and its area is 180 sq. metres. Find its length and breadth.

Sol. Since the perimeter = 54 metres

$\therefore$ its semi-perimeter = 27 metres

Let the length be x metres

$\therefore$ the breadth $= (27-x)$ metres.

and the area $= x(27-x)$ sq. metres

$$\therefore \quad x(27-x) = 180$$
$$\text{or} \quad x^2-27x+180 = 0$$
$$\therefore \quad (x-15)(x-12) = 0$$
$$\therefore \quad x = 15 \text{ or } 12.$$

$\therefore$ The length and the breadth are 15m and 12m.

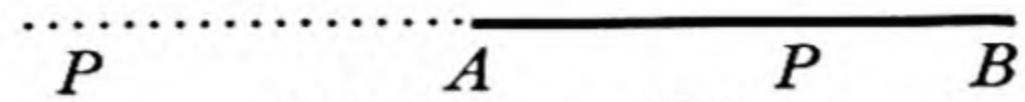
Verification :

(1) When the length is 15, breadth $= 27-15 = 12$ and the area $= 15\times 12 = 180$.

(2) When the length is 12, breadth $= 27-12 = 15$ and the area $= 12\times 15 = 180$.

Example 12: AB is a straight line whose length is a units. Find a point P in it such that $AB.PB = AP^2$.

Sol. Let $AP = x$ units, then $PB = (a-x)$ units.



$$\because \quad AB.PB = AP^2$$
$$\therefore \quad a(a-x) = x^2$$
$$\text{or} \quad x^2+ax-a^2 = 0$$
$$x = \frac{-a\pm\sqrt{a^2+4a^2}}{2} = \frac{-a\pm a\sqrt{5}}{2} = \frac{a(-1\pm\sqrt{5})}{2}.$$

Hence the two roots are $\frac{1}{2}(\sqrt{5}-1)a$ and $-\frac{1}{2}(\sqrt{5}+1)a$.

The positive value of x corresponds to the point P of internal division.

The negative value of x corresponds to the point P of external division.

EXERCISE 192

1. Find a number which is equal to three times its square.

2. The sum of a number and its reciprocal is $2\frac{5}{12}$. Find it.

3. The sum of the squares of two consecutive numbers is 365. Find them.

4. The product of two consecutive even numbers exceeds their sum by 142. Find them.

5. The sum of the squares of two consecutive odd numbers is 290. Find them.

6. Two numbers differ by 7 and their reciprocals differ by $\frac{7}{60}$. Find them.

7. The area of a rectangle is 255 sq. metres. If its length be diminished by 1 metre and breadth increased by 1 metre, it becomes a square. Find its dimensions.

8. A straight line AB is a units in length, find a point P in it such that

(i) $AP^2 = 2\ AB.\ PB.$ (ii) $AP^2 = 2\ PB^2.$

(iii) $AP.\ PB = 3\ PB^2.$ (iv) $2\ AP.\ PB = (\frac{1}{2}\ AB)^2.$

9. The area of a rectangle is 240 sq. metres and its diagonal is 26 metres. Find its sides.

10. The area of a right-angled triangle is 240 sq. cm. and its hypotenuse is 34 cm. Find its sides.

11. The area of a rectangle is equal to the area of a square whose side is 12 metres longer than the breadth of the rectangle.

If the length of the rectangle be increased by 25 metres and the breadth diminished by 6 metres, its area remains unaltered. Find its dimensions.

12. A lawn is of rectangular shape. Its length is 24 metres and breadth 16 metres. A path of uniform width whose area is equal to that of the lawn, goes round it. Find the width of the path.

13. The hypotenuse of a right-angled triangle is less than the sum of the other sides by 8 cm and its area is 120 sq. cm. Find its sides.

14. A cyclist rode 132 km. in a number of hours which was less by 1 than the number of km he rode per hour. Find the number of hours he rode.

15. A cyclist rode 75 km. at a uniform rate. If he had gone 3 km. per hour slower, he would have taken one hour and 15 minutes more. What was his rate ?

16. If a cyclist had gone 3 km. per hour faster, he would have taken 1 hour and 20 minutes less to ride 80 km. What time did he take ?

17. The circumference of one wheel is 1 metre more than that of another. If the larger wheel makes 250 revolutions less in 1.5 km. than the smaller, find the circumference of each wheel.

18. The front wheel of a carriage makes 60 revolutions more than the hind wheel in going 1,080 m. If the circumference of the front wheel be increased by 2 m., it will make 15 revolutions more than the hind wheel. Find the circumference of each wheel.

19. The length, breadth and height of a rectangular room are in the ratio of 8 :6 :5 : ; if each of the dimensions be increased by a metre, the area of its four walls would be 1,408 sq. metres. Find the dimensions of the room.

20. A battalion of soldiers is formed into a solid square. If the number of soldiers be reduced by 16, it can be formed into a hollow square 6 deep, having in the front 16 soldiers more than before. What is the number of soldiers ?

21. By selling a fan for Rs. 144, a person gained as much per cent as it cost him. What was the cost ?

22. By auctioning a car for Rs. 2,100, a person lost as much per cent as it had cost him. Find the cost price.

CHAPTER XXVI

IMAGINARY QUANTITIES

The need for an imaginary quantity: In Algebra we started with the concept of positive integer and the four fundamental operations denoted by $+$, $-$, $\times$ and $\div$. Little by little, fraction, involution, evolution and irrational numbers came to be introduced.

In like manner, subtraction of one positive integer from a smaller positive integer led to the adoption of negative integers. Having accepted the negative integers, we were led to apply to them all the fundamental rules. When we could raise a negative quantity to any power, we should also be able to apply the rules of evolution to negative numbers. This necessitates the introduction of *imaginary quantities*.

What is an imaginary quantity? The root of a positive quantity is either positive or negative. The cube root, fifth root etc. of a negative quantity are negative. But it is impossible to find the square root, the fourth root of a negative number; for, any number when raised to an even power becomes positive. The square root of a negative number is *impossible*.

We come across such numbers in the solution of quadratic equations. For example, the roots of $x^2+2x+7=0$ are $-1+\sqrt{-6}$ and $-1-\sqrt{-6}$. We cannot think of any quantity which when multiplied by itself gives -6. Such a quantity like $\sqrt{-6}$ is called an *Imaginary Quantity*. It is opposed to *real*.

However, we have to interpret it. So, we shall assign to it a meaning consistent with all the fundamental laws proved for other numbers; and assume that imaginary quantities obey the rules of algebra quite like any real quantity.

Accordingly, $\sqrt{-6}$ is a quantity which when multiplied by itself gives -6.

How to express: Since $\sqrt{xy}=\sqrt{x}\times\sqrt{y}$, $\sqrt{-6}$ can be expressed as $\sqrt{6}\times\sqrt{-1}$. In fact, the square root of any negative quantity can be expressed as the product of $\sqrt{-1}$

and the root of the corresponding positive quantity. $\sqrt{-a} = \sqrt{1} \times \sqrt{a}$. The single expression $\sqrt{-1}$ appears to be sufficient to denote all imaginary quantities.

By $\sqrt{-1}$ is meant an expression which, when multiplied by itself, produces -1.

$\sqrt{-1}$ is indicated by the symbol i.

Powers of i:

$(i)^2 = -1$; $(i)^3 = (i)^2 \times i = -1 \times i = -i$

$(i)^4 = (i^2)^2 = (-1)^2 = +1$; $(i)^5 = (i)^4 \times i = +1 \times i = +i$

$(i)^6 = (i)^4 \times (i)^2 = 1 \times -1 = -1$; $(i)^7 = (i)^6 \times i = -1 \times i = -i$

$(i)^8 = (i)^4 \times (i)^4 = 1 \times 1 = +1$; $(i)^9 = (i)^8 \times i = +1 \times i = +i$

This cycle repeats itself and the powers of i can give only 4 values ± 1 and $\pm i$.

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{a} \times \sqrt{-1} \times \sqrt{b} \times \sqrt{-1} = \sqrt{ab} \times -1 = -\sqrt{ab}$$

$$\sqrt{-a} \times \sqrt{b} = \sqrt{a} \times \sqrt{-1} \times \sqrt{b} = \sqrt{ab} \times i = i\sqrt{ab}.$$

Complex Quantity. An expression of the form $x+yi$ where x and y are real is called a **complex quantity.**

(*a*) A real quantity cannot be equal to an imaginary quantity, unless both are zero.

(*b*) If $a+bi = 0$, then $a = 0$ and $b = 0$.

(*c*) If $a+bi = c+di$, then $a = c$ and $b = d$.

If 2 complex quantities are equal, their real parts and their imaginary parts are separately equal.

Elementary operations using i.

Example 1: If $A = 5+7i$ and $B = 3-5i$, find

(i) $A+B$ (ii) $A-B$ (iii) $A \times B$.

Sol. (i) $A+B = 5+7i+3-5i = 8+2i$

(ii) $A-B = 5+7i-3+5i = 2+12i$

(iii) $A \times B = (5+7i)(3-5i)$

$= 15-25i+21i-35i^2$

$= 15-4i-35(-1)$

$= 15-4i+35 = 50-4i.$

Example 2: Simplify $(i)(2+3i)(2-3i)$.

(ii) $(3-\sqrt{-5})(3+\sqrt{-5})$. (iii) $(a+bi)(a-bi)$.

Sol. Using the formula $(p+q)(p-q) = p^2-q^2$, we have

(i) $(2+3i)(2-3i) = 4-9(i)^2 = 4-9(-1) = 4+9 = 13$.

(ii) $(3-\sqrt{-5})(3+\sqrt{-5}) = (3-\sqrt{5i})\,(3+\sqrt{5i}) = 9-5i^2$
$= 9+5 = 14$.

(iii) $(a+bi)(a-bi) = a^2-b^2i^2 = a^6-b^3(-1) = a^2+b^2$.

Example 3: Simplify $\dfrac{5+7i}{3-5i}$ making the denominator real.

Sol. $$\frac{5+7i}{3-5i} = \frac{(5+7i)(3+5i)}{(3-5i)(3+5i)}$$
$$= \frac{15+25i+21i+35i^2}{9-25i^2}$$
$$= \frac{15+46i-35}{9+25}$$
$$= \frac{-20+46i}{34} = \frac{-10+23i}{17}.$$

EXERCISE 193.

1. Multiply: (*a*) $(3+2i)(2+3i)$.
(*b*) $(5+\sqrt{-3})(2+\sqrt{-5})$.
(*c*) $(5-3i)(5+3i)$.

2. Simplify: (*a*) $(5+8i)^2$ (*b*) $(4-7i)^2$
(*c*) $(p-qi)^2$. (*d*) $(3+\sqrt{-5})^2$.
(*e*) $(5-\sqrt{-1})^2$. (*f*) $(c+\sqrt{-d})^2$.

3. Make the denominator real:

(*a*) $\dfrac{4+\sqrt{-3}}{4-\sqrt{-3}}$. (*b*) $\dfrac{13}{2+5\sqrt{-3}}$. (*c*) $\dfrac{4}{(1+\sqrt{-5})^2}$.

4. Express in the form $a+bi$:

(*a*) $\dfrac{(2+\sqrt{-5})^2}{2-\sqrt{-5}}$. (ii) $\dfrac{(1+\sqrt{-1})^2}{(1+\sqrt{-1})}$.

5. (*a*) Show that $\frac{1}{(1-i)^2}+\frac{1}{(1+i)^2}=0$.

(*b*) Simplify: $\frac{1}{(1-i)^2}-\frac{1}{(1-i)^2}$.

6. (*a*) Simplify: $\frac{p+qi}{p-qi}+\frac{p-qi}{p+qi}$.

(*b*) Divide: $\frac{a+bi}{a-bi}$ by $\frac{a-bi}{a-bi}$.

CHAPTER XXVII

THEORY OF QUADRATIC EQUATIONS

Some important properties relating to the roots and coefficients of quadratic equations will be discussed in this chapter.

1. Roots of a Quadratic Equation.

Just as a linear equation in one unknown quantity has always only one root, a *quadratic equation* in one unknown can have *two and only two roots.*

Every quadratic equation in x can, after suitable reduction, be put in the *standard form* viz. $ax^2+bx+c=0$

Transposing, $ax^2+bx=-c$

Dividing by a, $x^2+\frac{b}{a}x=-\frac{c}{a}$.

The left hand side can be made a perfect square by adding the square of half the coefficient of x.

Adding $\left[\frac{b}{2a}\right]^2$ to both sides, we have

$$x^2+\frac{b}{a}x+\left[\frac{b}{2a}\right]^2=\left[\frac{b}{2a}\right]^2-\frac{c}{a}$$

$$\text{or } x^2+2.x.\left[\frac{b}{2a}\right]+\left[\frac{b}{2a}\right]^2=\frac{b^2-4ac}{4a^2}$$

$$\left[x+\frac{b}{2a}\right]^2=\frac{b^2-4ac}{4a^2}$$

$$\therefore \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\text{or} \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

The roots of the equation are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Therefore, the value of x must be one or the other of these two quantities.

The fact that a quadratic equation cannot have more than two different roots is sometimes proved in the following way.

If possible, let the quadratic equation $ax^2+bx+c=0$ have three *different* roots, α, β and γ.

Since each of these values must satisfy the equation, we have

$a\alpha^2 + b\alpha + c = 0$ ... ... (i)
$a\beta^2 + b\beta + c = 0$ ... ... (ii)
$a\gamma^2 + b\gamma + c = 0$ ... ... (iii)

Subtracting (ii) from (i), we have

$a(\alpha^2-\beta^2)+b(\alpha-\beta) = 0$

Dividing by $\alpha-\beta$, which is not zero,

$a(\alpha+\beta)+b = 0$ ... ... (iv)

Similarly from (ii) and (iii), we have

$a(\beta+\gamma)+b = 0$ ... ... (v)

Subtracting (v) from (iv), we get

$a(\alpha-\gamma) = 0$

Since the equation is quadratic, a is not zero ; therefore $\alpha-\gamma = 0$.

This means that two of the three roots are identical, which is contrary to supposition.

Hence a quadratic equation cannot have more than two different roots.

2. Relations between the Roots and Coefficients of a Quadratic Equation.

Let the two roots of the quadratic equation be represented by α and β so that

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Then, by addition,

$$(i)\quad \alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2b}{2a} \qquad = -\frac{b}{a}$$

and by multiplication

$$(ii)\ \alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-(b^2-4ac)}{4a^2} = \frac{b^2-b^2+4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

Hence $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Note: If the equation $ax^2+bx+c=0$ is written as $x^2+\left(\frac{b}{a}\right)x+\frac{c}{a}=0$ [with the coefficient of $x^2=1$], then

(i) the sum of the roots = − coefficient of x.

(ii) the product of the roots = the constant term or term without x.

EXERCISE 194

1. Find the sum and the product of the roots of each of the following equations:

(*i*) $x^2+7x+9=0$ (*ii*) $x^2-13x+7=0$

(*iii*) $6x^2+5x-12=0$. (*iv*) $-3x^2+7x-9=0$

2. Write down (*a*) the sum, (*b*) the product of the roots of the following equations:

(*i*) $23x^2+7=8x$ (*ii*) $m^2x^2=k+lx$

(*iii*) $lx=mx^2-n$ (*iv*) $ax^2-c=0$

3. If α, β be the roots of $x^2-px+q=0$, prove that $(\alpha-\beta)^2=p^2-4q$

3. Nature of the Roots.

The roots of the quadratic equation $ax^2+bx+c=0$ are

$\alpha+\beta$ where $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

where a, b, c are real and rational quantities. The nature of the roots can be determined by considering the quantity under the radical sign. That quantity may be positive or zero or negative.

(i) If b^2-4ac is *positive*, the expression $\sqrt{b^2-4ac}$ has a real value. So α and β are both real but because there is a + sign before it in one case and − sign in another case, the roots are *real and unequal.*

(*a*) Besides being positive, if b^2-4ac is a *perfect square*, then the roots are real, unequal and *rational*.

(*b*) If b^2-4ac is *positive but not a perfect square*, then a rational value of $\sqrt{b^2-4ac}$ cannot be found. Hence the roots are real, unequal and *irrational*.

(ii) If b^2-4ac is *zero*, then $\sqrt{b^2-4ac}$ is also *zero*. Hence the roots of the equation are *real and equal*.

(iii) If b^2-4ac is *negative*, the expression $\sqrt{b^2-4ac}$ has an imaginary value, as the square root of a negative quantity is impossible. So, the roots of the equation are *imaginary*.

Since the nature of the roots depends upon the value of b^2-4ac, this quantity is called the **discriminant** of the quadratic equation $ax^2+bx+c=0$

Example 1: Examine the nature of the roots of the equation $3x^2+13x-2=0$.

Sol. The given equation is $3x^2+13x-2=0$.

Here $a=3, \quad b=13, \quad c=-2$.

Discriminant $= b^2-4ac = 13^2-4(3)(-2) = 193$, which is positive, but not a perfect square.

Hence the roots of the equation are real, unequal and irrational.

Example 2: Find the values of m, so that the roots of the equation $(4-m)x^2+2(m+2)x+(8m+1)=0$ may be equal.

Sol. The given equation is

$$(4-m)x^2+(2m+4)x+(8m+1)=0$$

Here $a=4-m, \quad b=2(m+2), \quad c=8m+1$

Since the roots of the equation are equal, $b^2-4ac=0$.

i.e. $2^2(m+2)^2-4(4-m)(8m+1)=0$

or $m^2+4m+4-(32m-8m^2+4-m)=0$

or $9m^2-27m=0$ or $9m(m-3)=0$ $\therefore \ m=0$ or 3.

EXERCISE 195

1. Examine the nature of the roots of the equations:

(*i*) $2x^2+3x+5=0$ (*ii*) $3x^2-5x-2=0$

(*iii*) $3x^2+7x+2=0$ (*iv*) $7x^2=10x-3$

(*v*) $2x^2+(2+3m)x+3m=0$.

2. Show that the roots of $(x-a)(x-b) = b^2$ are always real.

3. Prove that the equation $x^2-2x\ (m+m^{-1})+3=0$ has real roots for all real values of m.

4. Find the condition that the roots of the equation $x^2-2px+p^2-q^2+r^2=0$ may be real.

5. For what values of m are the roots of the equation $mx^2+(m+3)\,x+4=0$, equal?

6. If $x^2-15-m\ (2x-8)=0$ has equal roots, find the values of m.

7. For what value of m, will the quadratic equation $(1+m)x^2-2(1+3m)x+(1+8m)=0$ have equal roots? Also find the roots of the given equation for the values of m thus obtained.

4. Value of Symmetric Expressions containing the Roots.

Consider the following expressions involving α and β.

(i) $\alpha+\beta$ (ii) $\alpha^2+\beta^2$ (iii) $\alpha^3+\beta^3$

(iv) $\alpha^2+\alpha\beta+\beta^2$ (v) $\alpha^2-\alpha\beta+\beta^2$

(vi) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (vii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

In each case interchange α and β and compare the new form with the original one.

Are they different ? No, they are the same.

Such expressions which remain *unaltered* by *interchanging* α and β, are said to be *symmetric* in α and β.

Now we shall find the values of symmetric functions of the roots of quadratic equations in terms of the co-efficients of the equations.

Since we directly find the values of $\alpha+\beta$ and $\alpha\beta$, every symmetric function must be expressed in terms of $\alpha+\beta$ and $\alpha\beta$. For this purpose, the following results are very useful.

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

(ii) $(\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta.$

(iii) $\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta).$

(iv) $\alpha^4+\beta^4 = \{(\alpha+\beta)^2-2\alpha\beta\}^2-2(\alpha\beta)^2.$

Example 3: If α, β are the roots of the equation $2x^2+3x-7=0$, find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.

Sol. Given equation is: $2x^2+3x-7=0$.

If α, β are the roots of the equation, then $\alpha+\beta=-\frac{3}{2}$ and $\alpha\beta=-\frac{7}{2}$.

Now $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{(-\frac{3}{2})^2-2(-\frac{7}{2})}{-\frac{7}{2}}$

$=\frac{\frac{9}{4}+7}{-\frac{7}{2}}=-\frac{37}{4}\times\frac{2}{7}=-\frac{37}{14}$

$=-2\frac{9}{14}$

EXERCISE 196.

1. If α and β are the roots of the equation $x^2-2x+3=0$, find the values of

(*i*) $\alpha^2+\beta^2$ (ii) $\alpha^2\beta+\beta^2\alpha$ (iii) $\alpha^3+\beta^3$

(iv) $\alpha^3\beta+\beta^3\alpha$ (v) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ (vi) $\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}$.

2. If α, β are the roots of the equation $x^2+x+1=0$, find the value of $\alpha^2+\beta^2$.

3. If α and β are the roots of the equation $3x^2-5x+7=0$, find the value of $(\alpha-\beta)^2$.

4. If α and β be the roots of $x^2-3x+6=0$, find the value of $\alpha^3+\beta^3$.

5. If α, β are the roots of the equation $x^2-5x+6=0$, find the value of $\alpha-\beta$.

6. If α, β be the roots of the equation $x^2-px+q=0$, find the values of

(*a*) $\frac{1}{\alpha}+\frac{1}{\beta}$ (*b*) $\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}$ (*c*) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.

7. Find the values of the following expressions, if α and β are the roots of the equation $ax^2+bx+c=0$.

(i) $(\alpha+1)(\beta+1)$ (ii) $\alpha^3\beta+\alpha\beta^3$ (iii) $\alpha^2-\alpha\beta+\beta^2$.

(iv) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (v) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$ (vi) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$.

8. If α, β be the roots of $ax^3+bx+c=0$, find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\alpha^2\beta^2$.

5. To frame the quadratic equation whose roots are given.

Let, α and β be the given roots and let $ax^2+bx+c=0$ be the required equation, where a, b and c have to be determined.

The equation can be written as

$$x^2+\left(\frac{b}{a}\right)x+\frac{c}{a}=0 \quad \text{or} \quad x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}=0.$$

Since α and β are supposed to be the roots of this equation,

$$\alpha+\beta=-\frac{b}{a} \quad \text{and} \quad \alpha\beta=\frac{c}{a}$$

By substituting these values of $-\frac{b}{a}$ and $\frac{c}{a}$,

we have $x^2-(\alpha+\beta)x+(\alpha\beta)=0$.

The required equation is

x^2-(sum of the roots) $x+$(product of the roots) $=0$.

Example 4: Find the quadratic equation whose roots are -5 and 8.

Sol. Given roots are -5 and 8

Sum of the roots $=-5+8=3$

and product of the roots $=(-5)(8)=-40$

Formula is : x^2-(sum of the roots)$x+$(product of the roots) $=0$.

$\therefore$ The required equation is $x^2-3x-40=0$.

Sometimes the given roots are *not* independent, but they are expressed in terms of the roots of another equation. Here also we find the sum and product of the new roots. But the sum and product must be expressed in terms of the co-efficients of the given equation.

Example 5: If x_1 and x_2 are the roots of the equation $px^2+qx+r=0$, form an equation whose roots are $(x_1+x_2)^2$ and $(x_1-x_2)^2$.

If x_1 and x_2 are the roots of the equation, then $x_1+x_2 = -\frac{q}{p}$ and $x_1x_2 = \frac{r}{p}$.

Now we have to form an equation whose roots are $(x_1+x_2)^2$ and $(x_1-x_2)^2$.

sum of the new roots $= (x_1+x_2)^2+(x_1-x_2)^2$
$= (x_1+x_2)^2+(x_1+x_2)^2-4x_1x_2$
$= 2(x_1+x_2)^2-4x_1x_2$
$= 2\left(-\frac{q}{p}\right)^2-4\left(\frac{r}{p}\right) = \frac{2q^2}{p^2}-\frac{4r}{p}$
$= \frac{2q^2-4pr}{p^2} = \frac{2(q^2-2pr)}{p^2}$

and product of the roots $= (x_1+x_2)^2(x_1-x_2)^2$
$= (x_1+x_2)^2\{(x_1+x_2)^2-4x_1x_2\}$
$= \left(-\frac{q}{p}\right)^2\left\{\left(-\frac{q}{p}\right)^2-4\left(\frac{r}{p}\right)\right\}$
$= \frac{q^2}{p^2}\left(\frac{q^2}{p^2}-\frac{4r}{p}\right) = \frac{q^2(q^2-4pr)}{p^4}$.

Therefore the required equation is

$$x^2-\frac{2(q^2-2pr)}{p^2}x+\frac{q^2(q^2-4pr)}{p^4} = 0$$

or $$p^4x^2-2p^2(q^2-2pr)x+q^2(q^2-4pr) = 0.$$

EXERCISE 197.

1. Form the equation whose roots are

(i) 5 and 11 (ii) -7 and 16 (iii) $3a$ and $7b$.

(iv) $m+n$ and $m-n$ (v) $2+\sqrt{3}$ and $2-\sqrt{3}$.

(vi) $\frac{p}{q^2}$ and $\frac{q^2}{p}$ (vii) $\frac{p+q}{p-q}$ and $\frac{p-q}{p+q}$.

2. If α, β be the roots of the equation $x^2-2x+3=0$, form the equation whose roots are respectively,

(*a*) 3α, 3β (*b*) $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$. (*c*) $\alpha+1$, $\beta+1$.

3. If α and β are the roots of the equation $ax^2+bx+c=0$, find the equations whose roots are

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) α^2, β^2 (iii) $\frac{1}{\alpha+\beta}, \frac{1}{\alpha\beta}$.

(iv) $\alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}$ (v) $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$.

4. Form the equation whose roots are respectively greater than the roots of the equation $2x^2-3x+5=0$, by 2.

Hint. *If* α *be a root of* $2x^2-3x+5=0$. *then* $2\alpha^2-3\alpha+5=0$.
Let y *be a root of the required equation.*
$\therefore \quad y=\alpha+2 \quad$ *or* $\quad \alpha=y-2$
$\therefore \quad 2(y-2)^2-3(y-2)+5=0.$

6. To find the condition that the roots of a quadratic equation be connected by a given relation other than their sum and the product.

If α and β are the roots of the given equation, we will be having three relations now, (i) $\alpha+\beta$ (ii) $\alpha\beta$ and (iii) the new relation quoted in the question. By *eliminating* α and β from these three relations, the required condition can be obtained.

Example 6: One root of the equation $ax^2+bx+c=0$ is half of the other. Show that $2b^2=9ac$.

Sol. If α and β are the roots of the equation $ax^2+bx+c=0$,

then $\quad \alpha+\beta=-\frac{b}{a} \quad$ and $\quad \alpha\beta=\frac{c}{a}$

Also, $\quad \beta=\frac{1}{2}\alpha \quad i,e, \quad \alpha=2\beta.$

Putting 2β for α in the above relations,

we get $2\beta+\beta=-\frac{b}{a} \quad$ or $\quad 3\beta=-\frac{b}{a} \quad \ldots$ (i)

and $\quad 2\beta\times\beta=\frac{c}{a} \quad$ or $\quad 2\beta^2=\frac{c}{a} \quad \ldots$ (ii)

Squaring (i) and then dividing it by (ii),

we have $\quad \frac{9\beta^2}{2\beta^2}=\frac{b^2}{a^2}\times\frac{a}{c} \quad$ or $\quad \frac{9}{2}=\frac{b^2}{ac}.$

i.e. $\quad 2b^2=9ac$

which is the required condition.

EXERCISE 198.

1. Find the condition that the roots of the equation $px^2+qx+r=0$ may be the reciprocal of each other.

2. If the roots of $x^2-bx+c=0$ are two consecutive integers, prove that $b^2-4c=1$.

3. If one root of $px^2+qx+r=0$ be double the other, prove that $2q^2=9pr$.

4. Find the condition that one root of the equation $ax^2+bx+c=0$ may be (i) three times (ii) four times (iii) k times the other.

5. If the roots of the equation $lx^2+nx+n=0$ be in the ratio $p:q$, show that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0.$$

Hint: *If* α, β, *be the roots of the equation,* $\sqrt{\frac{\alpha}{\beta}}=\sqrt{\frac{p}{q}}$ *and* $\sqrt{\frac{\beta}{\alpha}}=\sqrt{\frac{q}{p}}$

6. Find the condition that one root of the equation $ax^2+bx+c=0$ may be the square of the other.

Hint: $\alpha+\alpha^2=-\frac{b}{a}$ *and* $\alpha^3=\frac{c}{a}$

Cube both the sides of the first relation and use the above results.

Example 7: Show that the roots of $(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0$ are real, and that they cannot be equal unless $a=b=c$.

Sol. The given equation is

$$(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0$$

or $$3x^2-2(a+b+c)x+(ab+bc+ca)=0$$

Comparing it with $Ax^2+Bx+C=0$

$A=3, \quad B=-2(a+b+c), \quad C=ab+bc+ca$

$$\begin{aligned}\text{Discriminant} &= B^2-4AC\\ &= 4(a+b+c)^2-4\times3\times(ab+bc+ca)\\ &= 4(a^2+b^2+c^2-ab-bc-ca)\\ &= 2(2a^2+2b^2+2c^2-2ab-2bc-2ca)\\ &= 2[(a-b)^2+(b-c)^2+(c-a)^2]\end{aligned}$$

This being sum of the squares of real quantities is always positive. Hence the roots are real.

The roots of the given equation will be equal, if its discriminant is zero. *i.e.* if $(a-b)^2+(b-c)^2+(c-a)^2=0$.

This is possible only when

$(a-b)^2=0$, $(b-c)^2=0$ and $(c-a)^2=0$.

i.e. $a-b=0$, $b-c=0$, $c-a=0$.

i.e. $a=b$, $b=c$, $c=a$ *i.e.* $a=b=c$

EXERCISE 199.

1. One root of $x^2+ax+8=0$ is 4, while the equation $x^2+ax+b=0$ has equal roots. Find b.

2. The roots of the equation $ax^2+8(b-a)x+4(4a-8b+c)=0$ are $4-2p$ and $4-2q$. Find the equation whose roots are p and q.

3. If the roots of $ax^2+x+b=0$ be real, show that the roots of $\frac{x^2+1}{x}=4\sqrt{ab}$ are imaginary.

4. Find the condition that the roots of the equation $ax^2+bx+c=0$ may be in the ratio 3 : 4.

5. If p, q are the roots of the equation $x^2-ax+b=0$, find the value of $(p-a)^{-4}+(q-a)^{-4}$.

Hint: $p+q=a$; *then* $p-a=-q$ *and* $q-a=-p$.

CHAPTER XXVIII

ARITHMETICAL PROGRESSION

1. Series.

Consider the following sets of real numbers :

7, 10, 13, 16,	...	...	(i)
$\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$,	...	...	(ii)
38, 31, 24, 17,	...	...	(iii)
$\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$,	...	...	(iv)
2, 9, 13, 26, $\frac{1}{2}\frac{3}{}$,	...	...	(v)
$3x$, $3x^3$, $3x^5$, $3x^7$,	...	...	(vi)
16, 13, 10, 7,	...	...	(vii)

In (i), each number is obtained by adding 3 to the preceding number. In (ii), each fraction is obtained by multiplying the preceding fraction by $\frac{1}{3}$. In (iii), each number is less than the preceding one by 7. In (iv), reciprocal of each fraction exceeds the reciprocal of the preceding one by 2. Numbers in (v) do not follow any definite law. In (vi), each number is x^2 times the preceding one. In (vii), each number is less than the preceding one by 3.

Thus we observe that all the groups except (v), follow a certain definite law; hence these groups are **sequences.**

Any set of numbers obeying a definite particular law is called a sequence. Each number of a sequence is called a **term** and thus we have the first term, the second term, the third term and so on. In sequence (iv), the first term is $\frac{1}{3}$, the second term is $\frac{1}{5}$.

In sequence (vi), the 3rd term is $3x^5$ and 4th term is $3x^7$.

NOTE : The sequences (i) and (vii) consist of the same set of numbers, but they are two different sequences—as they obey different laws of formation.

When the terms of a sequence are connected together by plus or minus signs, they are said to form a **series.**

The above sequences can be written as series thus :

$7+10+13+16+\ldots\ldots$; $3x+3x^3+3x^5+3x^7$.

A series which contains a limited number of terms is called a **finite series.** Every finite sequence or series has a last term.

A series which contains an unlimited number of terms is called an **infinite series.** Thus every infinite sequence or series is endless.

2. Arithmetical Progression.

In sequence (i), successive terms increase by a constant quantity 3. But in sequence (iii), successive terms decrease by a constant quantity 7. Thus the difference between any two consecutive terms of each sequence is the *same* throughout. Such sequences are called **arithmetical progressions.**

A sequence is said to be an arithmetical progression *if the difference between any term and its preceding term is the same throughout.* This difference is called the **common difference** of the arithmetical progression.

In sequence (*i*), common difference is $(10-7) = (16-13) = 3$.

In sequence (*iii*), common difference $= 31-38 = -7$.

An arithmetical progression is briefly written as A. P. and the common difference is denoted generally by '*d*'.

NOTE: In an A.P., any term is obtained by adding the common difference to its preceding term. The common difference is obtained by subtracting any term from its next term.

EXERCISE 200

1. Against each of the following sets, write A if the set is a sequence, B if the set is a series, or C if the set is neither a sequence nor a series.

(*i*) $8m+n,\ 7m-2n,\ 6m-5n,\ 5m-8n.$

(*ii*) $\left(1-\frac{1}{n}\right)+\left(1-\frac{2}{n}\right)+\left(1-\frac{3}{n}\right)+\ldots\ldots\ldots$

(*iii*) $18,\ 36,\ 9,\ \frac{1}{3},\ 3.$

(*iv*) $\frac{1}{7}+\frac{1}{10}+\frac{1}{13}+\frac{1}{16}+\frac{1}{19}.$

(*v*) $3x^2,\ 5x^2,\ 7x^2,\ 9x^2.$

2. Which of the following sequences are in A.P.? Find the common difference in the case of an A. P.

(*a*) 13, 7, 1, −5, −11,.....

(*b*) $\frac{3p+4q}{7}, \frac{2p+2q}{7}, \frac{p}{7}$,.....

(*c*) 22, 222, 2222,.....

(*d*) 1.7, 2.8, 3.9,.....

(*e*) $\frac{2a^2-1}{a}, \frac{4a^2-3}{a}, \frac{6a^2-5}{a}$,.....

3. General Term.

The *standard form* of an A. P. is $a, a+d, a+2d, a+3d$,..... where a is the first term and d the common difference.

Now
$$\begin{aligned} 1\text{st term} &= a &&= a+(1-1)d \\ 2\text{nd term} &= a+d &&= a+(2-1)d \\ 3\text{rd term} &= a+2d &&= a+(3-1)d \\ 4\text{th term} &= a+3d &&= a+(4-1)d \\ \cdots \quad \cdots \quad \cdots \quad \cdots & & \cdots \quad \cdots & \end{aligned}$$

We observe that (i) each term contains 'a' and 'd', (*ii*) the coefficient of 'a' is unity in all terms and (iii) coefficient of 'd' in each term is one less than the number of the term.

Hence n th *term* $= a+(n-1)d$.

This n th term is called the **general term** of the A. P. By putting $n = 1, 2, 3$, etc. in the general term successively, the 1st, 2nd, 3rd,.....terms are obtained.

The 1st, 2nd, 3rd,....., n th terms are respectively denoted by $T_1, T_2, T_3,.....T_n$, or $t_1, t_2, t_3,....., t_n$.

If l be the last term of an A. P., then

$$l = n \text{ th term} = a+(n-1)d.$$

Example 1: If the nth term of an A.P. is $3n-2$, find the series.

Sol. Given: $t_n = 3n-2$.

Putting $n = 1, 2, 3,$ ..... etc., we get $t_1 = 1$, $t_2 = 4$, $t_3 = 7$, $t_4 = 10$ etc.

The series is $1+4+7+10+$ .....

Example 2: Find the 14th term of the A.P. $(p+q)$, $2p$, $(3p-q)$, $(4p-2q)$, ...

Sol. The A. P. is $(p+q)$, $2p$, $(3p-q)$, $(4p-2q)$, ... and the 14th term is required.

By formula $T_n = a+(n-1)d$

Here $a = p+q$; $d = 2p-(p+q) = p-q$ and $n = 14$.

$\therefore$ the 14th term $= (p+q)+(14-1)\ (p-q)$

$= 14p-12q$.

EXERCISE 201

1. Write down the first five terms of the series whose nth term is $2n-1$.

2. Find the series whose general term is $3n-7$.

3. Find the 23rd term of the A. P. $7\frac{3}{4}+9\frac{1}{2}+11\frac{1}{4}+$..... .

4. Find the nth term of the series $\frac{1}{n}+\frac{n+1}{n}+\frac{2n+1}{n}+$.....

5. Find the last term of each of the following series:

(i) $3\cdot3+4\cdot6+5\cdot9+$.....to 17 terms.

(ii) $1-3-7-$ ..... to n terms.

Example 3: Which term of the series $7+4+1-2-5-$ ..... is -56 ?

Sol. Given series is $7+4+1-2-5-$ .....

Here, $a = 7$, $d = 4-7 = -3$.

Let -56 be the nth term of the A. P.

Then $tn = a+(n-1)d = 7+(n-1)(-3)$

$\therefore$ $7-3n+3 = -56$

or $-3n = -66$ $\therefore$ $n = 22$

$\therefore$ -56 is the 22nd term of the given series.

EXERCISE 202

1. Which term of the A. P. 14, 11, 8, 5, 2, ...is -40?

2. Is 74 a term of the series $-6-2+2+6+$.....?

3. What term of the series $37+33+29+$ .....is -280?

4. Which term of the series $6+5\frac{1}{4}+4\frac{1}{2}+$ .....is zero?

5. How many terms are there in the series, if the last term of 5, -1, -7, ...is -49?

6. How many terms are there in the series $3\frac{1}{2}+1-1\frac{1}{2}-4-$ ..... -14?

4. **To determine the A. P. when any two of its terms are given.**

Example 4: The fourth term of an A. P. is 3 and the 9th term -17. Find the A. P.

Sol. Let a be the first term of the A. P. and d the common difference.

The 4th term $= a+(4-1)d$ and the 9th term $= a+(9-1)d$

$\therefore\ a+3d = 3$ ..... (i) , $a+8d = -17$ .....(ii)

By subtraction, $-5d = 20$ or $d = -4$

Substituting in (i), $a = 3-3d = 3+12 = 15$

The A. P. is 15, 11, 7, 3, ...

Example 5: The pth term of an A. P. is q and the qth term is p. Find the $(p+q)$th term.

Sol. Let a be the first term and d the common difference of the A. P.

The pth term of the A. P. is $a+(p-1)d$

and the qth term is $a+(q-1)d$

$\therefore\ a+pd-d = q$ .... (1)

$a+qd-d = p$ .... (2)

Subtracting (2) from (1), $(p-q)d = q-p$

or $(p-q)d = -(p-q)$ $\therefore\ d = -1$

Putting the value of d in (1), we have

$a+(p-1)(-1) = q$ $\therefore\ a = p+q-1$

Now the $(p+q)$th term $= a+(p+q-1)d$

$= p+q-1+(p+q-1)(-1) = 0.$

EXERCISE 203.

1. The 5th term and 9th term of an A. P. are respectively 11 and 7, find the 16th term.

2. In an A. P. of 31 terms, the 5th term and the last terms are 17 and 95 respectively, find the 4th term from the beginning and from the end.

3. The 10th term of an A. P. is four times the 2nd term and exceeds the sixth term by 12. Find the series.

4. If the 15th term of an A. P. is 5 and the 5th term is 15, find the A. P. Is zero a term of the A. P.? Which term is it?

5. The mth term of an A. P. is n and the nth term is m; show that the rth term of the A. P. is $m+n-r$.

6. If p times the p th term of an A. P. is equal to q times the qth term, show that $(p+q)$th term is zero.

5. To find the sum of n terms of an A. P.

Let a be the first term and d the common difference of the A. P.

The A. P. can be written as a, $a+d$, $a+2d$, $a+3d$, ... to n terms.

Here nth term $= a+(n-1)d$ and let this be l.

If S_n denotes the sum of n terms of the A. P., then

$S_n = a+(a+d)+(a+2d)+.....+\{a+(n-1)d\}$,

or $S_n = a+(a+d)+(a+2d)+.....+(l-d)+l.$ ..... (i)

Writing the terms on the R.H.S. in the reverse order,

$S_n = l+(l-d)+(l-2d)+.....+(a+d)+a$ .....(ii)

Adding (i) and (ii), term by term, we have

$2\times S_n = (a+l)+(a+l)+(a+l)+.....$ to n groups.

$= n(a+l)$

$$\therefore \quad S_n = \frac{n}{2}(a+l) \quad ..... \quad ..... \quad ..... \quad ..... \quad \text{I}$$

Substituting $a+(n-1)d$ for l in the result I, we have

$$S_n = \frac{n}{2}\left\{ a+a+(n-1)d \right\}$$

$$\therefore \quad S_n = \frac{n}{2}\left\{ 2a+(n-1)d \right\} \quad ..... \quad ..... \quad .....\text{II}$$

In finding the sum of an A. P. to n terms, formula I is used when the first term a and last term l are given, while formula II is used when the first term a and common difference d are given.

NOTE: $S_n = T_1+T_2+T_3+.....+T_{n-1}+T_n$

$S_{n-1} = T_1+T_2+T_3+.....+T_{n-1}.$

By subtraction, $S_n - S_{n-1} = T_n$ i.e. $T_n = S_n - S_{n-1}$.

S_{n-1} is obtained by putting $n-1$ for n in the value of S_n.

Example 6: If the sum of n terms of an A. P. is $3n^2-4n$, find the A. P. and the pth term.

Sol. $S_n = 3n^2-4n$

$S_{n-1} = 3(n-1)^2-4(n-1)$

$\therefore \quad T_n = S_n - S_{n-1} = 3[n^2-(n-1)^2]-4[n-(n-1)]$

$= 3(2n-1)-4\times 1$

$= 6n-7$

$T_1 = 6\times 1-7 = -1$, $T_2 = 6\times 2-7 = 5$,

$T_3 = 6\times 3-7 = 11$, $T_4 = 6\times 4-7 = 17$,

....

$T_p = 6\times p-7 = 6p-7.$

The series is $-1+5+11+17+.....$

The pth term is $6p-7.$

EXERCISE 204.

1. Sum the following series:

(i) $18+11+4-3-.....$ ... To 22 terms.

(ii) $\cdot 4+\cdot 41+\cdot 42+\cdot 43+...$ ... to 100 terms.

(iii) $(a-b)+(2a-3b)+(3a-5b)+...$ ... to n terms.

(iv) $72+70+68+...\ +40.$

2. Find the value of

$1+6+4+9+7+12+...$ ... to 40 terms.

3. Find the sum of all even integers from 100 to 200 both inclusive and the sum of all odd integers.

4. If the sum of n terms of an A.P. is $2n^2+3n$, find the pth term.

5. If the sum of n terms of an A.P. is n^2-5n, find the 5th and 7th terms.

6. The pth term of an A.P. is $\frac{1}{q}$ and the qth term is $\frac{1}{p}$; show that the sum of pq terms is $\frac{1}{2}(1+pq)$.

6. **Inverse Cases.**

In the formula $S_n = \frac{n}{2}\left\{ 2a+(n-1)d \right\}$, four quantities a, d, n and S_n are involved. If any three of them are given,

the remaining one can be determined. So we can have the following three more cases :—

(i) given S_n, n and a, find d.
(ii) given S_n, a and d, find n.
(iii) given S_n, n and d, find a.

Example 7: In an A.P. the sum of the first 21 terms is 133 and the first term is -7. Find the common difference.

Sol. Here $a = -7$, $n = 21$, $S = 133$, $d = ?$

Using the formula $S = \frac{n}{2}\{2a+(n-1)d\}$, we have

$$133 = \frac{21}{2}\{2\times(-7)+(21-1)d\}$$

or $\frac{2}{21}\times 133 = -14+20d$ or $20d = \frac{38}{3}+14 = \frac{80}{3}$.

$$\therefore \quad d = \frac{4}{3} \quad i.e.\ 1\tfrac{1}{3}.$$

Example 8: How many terms of the A.P. $32+28+24+\ldots\ldots$ will together be equal to 132 ? Explain the double answer.

Sol. In the given series, $a = 32$, $d = 28-32 = -4$, $s = 132$, and n is required.

Using the formula $S = \frac{n}{2}\{2a+(n-1)d\}$.

we have $132 = \frac{n}{2}\{2\times 32+(n-1)(-4)\}$

or $264 = n(64-4n+4)$
or $264 = 68n-4n^2$
$\therefore$ $4n^2-68n+264 = 0$ or $n^2-17n+66 = 0$.
or $(n-6)(n-11) = 0$. $\therefore$ $n = 6$ or 11.

The number of terms is 6 or 11.

It is obvious that the first six terms give a total of 132. The sum of next five terms is zero, for some of them are positive and some negative. By adding those five terms, the total is not affected. Hence both 6 and 11 are correct solutions.

EXERCISE 205.

Find the missing quantities in the following table:

Qn.No.	First term, a =	Com. diff. d =	No. of terms n =	Sum of n terms S =
1.	10	3	24	?
2.	5	−2	?	−40
3.	−8	?	15	180
4.	?	−1	16	120

5. The sum of 12 terms of an A.P. is 72, and the common difference is −2 ; find its first term.

6. The first term of an A.P. is 1, the number of terms is 33 ; what must the common difference be, in order that the sum may be 297 ?

7. The sum of a certain number of terms of an A.P. is 36 and the first and last terms are 1 and 11 respectively. Find the number of terms and the common difference of the series.

8. How many terms of the series 9, 12, 15,..... will together make 306 ?

9. The sum of a certain number of terms of the series $25+22+19+16+$.....is 116. Find the number of terms of the series and its last term.

7. Arithmetic Mean.

A quantity 'A' is said to be the **arithmetic mean** between a and b if a, A, b are in A.P. Arithmetic mean is briefly written as A.M.

The quantities A_1, A_2, A_3,....., A_n are said to be n **arithmetic means** between a and b if a, A_1, A_2, A_3, A_4,....., A_n, b are in A.P.

8. **A single arithmetic mean between two quantities is equal to half their sum.**

Let A be the arithmetic mean between a and b.

Then a, A, b are in $A.P.$

$\therefore$ the common difference $= A-a = b-A$

Transposing, $2A = a+b$

or $$A = \frac{a+b}{2}$$

9. **To insert n arithmetic means between two given numbers a and b.**

Let $A_1, A_2, A_3, \ldots\ldots, A_n$ be the n arithmetic means between a and b.

Then $a, A_1, A_2, A_3, \ldots\ldots, A_n, b$ are in A.P.

Here the first term $= a$, last term $= b$,

number of terms $= 1+n+1 = n+2$

Let $\propto$ be the common difference of the A.P.

$T_{n+2} = a+(n+2-1)d$ $\quad\therefore\quad a+(n+1)d = b.$

or $(n+1)d = b-a$ $\quad\therefore\quad d = \frac{b-a}{n+1}$

$$A_1 = T_2 = a+(2-1)d = a+\frac{b-a}{n+1} = \frac{na+b}{n+1}$$

$$A_2 = T_3 = a+(3-1)d = a+2\left[\frac{b-a}{n+1}\right] = \frac{(n-1)a+2b}{n+1}$$

$$A_3 = T_4 = a+(4-1)d = a+3\left[\frac{b-a}{n+1}\right] = \frac{(n-2)a+3b}{n+1}$$

... ...

$$A_n = T_{n+1} = a+(n+1-1)d = a+n\left[\frac{b-a}{n+1}\right] = \frac{a+nb}{n+1}$$

Hence the required arithmetic means are

$$\frac{na+b}{n+1}, \quad \frac{(n-1)a+2b}{n+1}, \quad \frac{(n-2)a+3b}{n+1}, \ldots\ldots\ldots\ldots, \quad \frac{a+nb}{n+1}.$$

EXERCISE 206.

1. Find the arithmetic mean between.

(i) x^2-xy+y^2 and x^2+xy+y^2 (ii) 3·9 and 9·3.

2. Insert 4 arithmetic means between 5 and 35.

3. Find 7 arithmetic means between 1 and 41.

4. Insert p arithmetic means between 1 and p^2.

5. Prove that the sum of n arithmetic means between two numbers is equal to n times the single arithmetic mean between them.

10. **Properties of an A.P.**

(i) If the same number is *added* to each term of an A.P., the resulting sequence is also an A.P., having the same common difference.

(ii) If the same number is *subtracted* from each term of an A.P., the resulting sequence is also an A.P., having the same common difference.

(iii) If each term of an A.P. is *multiplied* by the same number, the resulting sequence is also an A.P., but the common difference is also the same multiple of the original difference.

(iv) If each term of an A.P. is *divided* by the same number, the resulting sequence is also an A.P., the common difference is also equal to the original difference divided by the same number.

The above properties and the following hints are very useful in solving problems in A.P.

If *three* numbers in A.P. are given, take them as $a-d$, a, $a+d$. [Their sum $= 3a$]

If *four* numbers in A.P. are given, take them as $a-3d$, $a-d$, $a+d$, $a+3d$ with $2d$ as the common difference. [Their sum $= 4a$]

Example 9: If $\frac{b+c-a}{a}$, $\frac{c+a-b}{b}$, $\frac{a+b-c}{c}$ are in A.P., show that $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are also in A.P.

Sol. Since $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.,

$\frac{b+c-a}{a}+2, \frac{c+a-b}{b}+2, \frac{a+b-c}{c}+2$ are also in A.P.

i.e. $\frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are also in A.P.

Dividing throughout by $a+b+c$, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Example 10: If the sum and product of three numbers in A.P. be 27 and 720 respectively, find them.

Sol. Let the three numbers in A.P. be $a-d$, a, $a+d$.

Then $a-d+a+a+d = 27$

or $3a = 27$ $\therefore$ $a = 9$

Also $(a-d).\ a.\ (a+d) = 720$

or $a\ (a^2-d^2) = 720$ or $9(81-d^2) = 720$

or $81-d^2 = 80$ or $d^2 = 1$ $\therefore$ $d = \pm 1$

When $d = 1$, the numbers are 8, 9, 10.

When $d = -1$, the numbers are 10, 9, 8.

Hence the required numbers are 8, 9, 10.

EXERCISE 207

1. If a, b, c are in A.P., show that

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A.P.

(ii) $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are also in A.P.

2. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.,

prove that a^2, b^2, c^2 are in A.P.

3. The sum of three consecutive terms in A.P. is 51 and the product of the two extremes is 273. Find the terms.

4. Find four numbers in A.P., whose sum is 26 and the sum of whose squares is 189.

5. The sum of five numbers in A.P. is 20 and the product of the first and the last is 15. Find the five numbers.

Example 11: The sums of n terms of two A.P.'s are in the ratio $2n-5 : n+7$; find the ratio of their 15th terms.

Sol. Let a and d be the first term and common difference of the first A.P. and let x and y be the first term and common difference of the second A.P.

S_n of I A.P. $= \frac{n}{2}\{2a+(n-1)d\}$ & T_{15} of I A.P. $= a+14d$

S_n of II A.P. $= \frac{n}{2}\{2x+(n-1)y\}$ & T_{15} of II A.P. $= x+14y$.

$$\therefore \quad \frac{\text{(I) } Sn}{\text{(II) } Sn} = \frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{(2x+(n-1)y\}} \text{ and this is given to be}$$

$$= \frac{2n-5}{n+7}$$

$$\therefore \quad \frac{2a+(n-1)d}{2x+(n-1)y} = \frac{2n-5}{n+7} \qquad \ldots \quad \text{(i)}$$

Now we have to find the value of $\frac{a+14d}{x+14y}$

which is the same as $\frac{2a+28d}{2x+28y}$.

This is obtained by putting $n-1 = 28$ *i.e.* $n = 29$ in (i).

We then have $\frac{2a+28d}{2x+28y} = \frac{2\times 29-5}{29+7}$ or $\frac{53}{36}$

The ratio of the 15th terms is 53 : 36.

Example 12: If a, b, c be respectively the sums of p, q, r terms of an A.P., show that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q) = 0$.

Sol. Let A and D be the first term and common difference of the A.P.

Then $S_p = \frac{p}{2}\{2A+(p-1)D\}$ and this is given to be $= a$.

$$\therefore \quad \frac{2a}{p} = 2A+(p-1)D = (2A-D)+pD$$

$$S_q = \frac{q}{2}\{2A+(q-1)D\} \text{ and this is } = b$$

$$\therefore \quad \frac{2b}{q} = 2A+(q-1)D = (2A-D)+qD$$

Similarly $$\frac{2c}{r} = (2A-D)+rD$$

$$\therefore \quad \frac{2a}{p}(q-r) = (2A-D)(q-r)+p\ (q-r)D$$
$$\frac{2b}{q}(r-p) = (2A-D)(r-p)+q(r-p)D$$
$$\frac{2c}{r}(p-q) = (2A-D)(p-q)+r(p-q)D$$

Adding, $2\left[\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)\right]$
$= (2A-D)(q-r+r-p+p-q)+$
$D(pq-pr+qr-pq+pr-qr) \quad = 0$

$$\therefore \quad \frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q) = 0.$$

EXERCISE 208

1. If a, b, c are in A.P., show that $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.

2. If a, b, c be respectively the xth, yth, and zth terms of an A.P., prove that $a(y-z)+b(z-x)+c(x-y) = 0$.

3. The mth term of an A.P. is $\frac{1}{n}$ and nth term is $\frac{1}{m}$; find the sum of (mn) terms.

4. If the ratio of the sum to n terms of two A.P.'s be $3n+2 : 2n+5$, find the ratio of their 9th terms.

5. The sum of p terms of an A.P. is the same as that of q terms. Show that the sum of $(p+q)$ terms is zero.

Hint : $\frac{1}{2}p\{2a+(p-1)d\} = \frac{1}{2}q\{2a+(q-1)d\}$
or $2a(p-q)+(p-q)(p+q-1)d = 0$
or $2a+(p+q-1)d = 0$ $\left[\because \quad p-q \neq 0 \text{ and } d \neq 0\right.$

6. Find n, if the sum of n terms of the series 3, 6, 9, 12, ... is equal to the sum of n terms of the series 48, 46, 44, 42 .....

7. A shop sells various sizes of tin canisters and the prices of successive sizes rise by equal amounts. The smallest costs Rs. 2·50 p. and the largest, Rs. 6·50 p. It costs Rs. 40·50 to buy one in each size. How many kinds are there ? What is the cost of one of the middle size ?

8. An assistant's monthly salary is Rs. 230 and every year he gets an increment of Rs. 10 in the monthly salary. How much does he earn in ten years ?

CHAPTER XXIX

GEOMETRICAL PROGRESSION

1. Geometrical Progression.

Examine the following sequences :

3, 9, 27, 81 ... (i)

$x, x^3, x^5, x^7,$ ... (ii)

$\frac{1}{2}, \frac{1}{10}, \frac{1}{50},$ ... (iii)

In sequence (i), each term is obtained by multiplying the preceding term by 3. In sequence (ii), each term is x^2 times the preceding one. Therefore the ratio of any two consecutive terms of each sequence is the *same* throughout. Such sequences are called **geometrical progressions.**

A sequence is said to be a geometrical progression if the ratio of any term to its preceding term is the same throughout. This ratio is called the **common ratio** of the geometrical progression.

In sequence (i), the common ratio is $\frac{9}{3}$ *i.e.* 3.

In sequence (iii), the common ratio is $\frac{1}{10} \div \frac{1}{2} = \frac{1}{5}$.

A geometrical progression is briefly written as G.P. and the common ratio is denoted generally by 'r'.

Note : In a G.P., any term is obtained by multiplying the preceding term by the common ratio. The common ratio is obtained by dividing any term by its preceding term.

EXERCISE 209

Which of the following sequences are in G.P. ? Find the common ratio in case it is a G.P.

1. $a, a^2, a^3, a^4,$ .....

2. $64p^4, 16p^2, 4,$ .....

3. $9, -6, 4, -\frac{8}{3},$ .....

4. $\cdot 3, \cdot 03, \cdot 0003,$ .....

5. $m^2-n^2, m-n, \frac{m-n}{m+n},$ .....

6. $\sqrt{2}, \sqrt{6}, 3\sqrt{2},$ .....

2. General Term.

The standard form of a G.P. is

$a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots\ldots\ldots$

where a is the first term and r the common ratio.

Now 1st term $= a = a.r^{1-1}$

2nd term $= ar = a.r^{2-1}$

3rd term $= ar^2 = a.r^{3-1}$

... ...

We observe that (i) each term contains 'a' and 'r', (ii) the index of 'a' is unity in all terms and (iii) index of 'r' in each term is one less than the number of the terms.

Hence **n th term** $= a.r^{n-1}$

This nth term is called the **general term** of the G.P. By putting $n = 1, 2, 3$, etc. in the general term successively, the 1st, 2nd, 3rd, ...terms are obtained.

If l be the last term of a G.P. then

$l = n$th term $= ar^{n-1}$.

Example 1: Find the 7th term of the G.P.

$-16, 12, -9, 6\frac{3}{4}, \ldots\ldots\ldots\ldots\ldots\ldots$

Sol. The G.P. is $-16, 12, -9, 6\frac{3}{4}, \ldots\ldots\ldots\ldots$ and the 7th term is required.

By formula $Tn = a.r^{n-1}$

Here $a = -16$, $r = -\frac{12}{16} = -\frac{3}{4}$ and $n = 7$.

$\therefore$ 7th term $= a\,r^6 = (-16)(-\frac{3}{4})^6$

$$= -16\left[\frac{729}{16\times16\times16}\right] = -\frac{729}{256}.$$

EXERCISE 210

1. Write down the first five terms of the series whose mth term is 2.3^{m-1}.

2. Find the series whose general term is $\frac{1}{2^{n-4}}$.

3. Find the 8th term of the series $1 - \frac{2}{3} + \frac{4}{9} - \ldots\ldots\ldots$

4. Find the nth term of the series.

$$\frac{n}{x^{n-1}} + \frac{n}{x^{n-2}} + \frac{n}{x^{n-3}} + \ldots\ldots\ldots$$

5. Find the last term of each of the following series :

(i) $-\frac{8}{3}+\frac{4}{3}-\frac{2}{3}+\frac{1}{3}-$ ..... to 10 terms.

(ii) $\frac{m}{n}-\frac{n}{m}+\frac{n^3}{m^3}-$ .....to p terms.

Example 2: What term of the series $50+20+8+$ ..... is $\frac{64}{125}$?

Sol. Given series is $50+20+8+$ .....

Here $a = 50$, $r = \frac{20}{50} = \frac{2}{5}$

Let $\frac{64}{125}$ be the nth term of the G.P.

nth term $= ar^{n-1} = 50\left(\frac{2}{5}\right)^{n-1}$

$\therefore \quad 50\left(\frac{2}{5}\right)^{n-1} = \frac{64}{125}$ or $\left(\frac{2}{5}\right)^{n-1} = \frac{32}{25\times125}$

or $\left(\frac{2}{5}\right)^{n-1} = \left(\frac{2}{5}\right)^5 \quad \therefore \quad n-1 = 5 \quad i.e.\ n = 6$

$\therefore \quad \frac{64}{125}$ is the 6th term of the given series.

Note: As the number denoting the position of a term is always a positive integer, any negative or fractional value of n means that it will not be a term of the series.

EXERCISE 211

1. Is $\frac{1}{27}$ a term of the series $3+\sqrt{3}+1+$ ..... ?

2. Which term of the series $5-10+20-40+$ ..... is -640 ?

3. How many terms are there in the series, if the last term of $\cdot2+\cdot4+\cdot8+1\cdot6+$ ..... is $51\cdot2$?

4. How many terms are there in the series $-\frac{2}{9}+\frac{2}{3}-2+$ ..... -162 ?

5. What term of the series $2\frac{1}{4}+1\frac{1}{2}+1+$ ..... is $\frac{128}{729}$?

3. **If any two terms of a G.P. are given, the first term and the common ratio can be determined and then the whole series can be written out.**

Example 3: Find the G.P. of which the 4th term is 54 and the 7th term is 1458.

Sol. Let a and r be the first term and common ratio respectively, of the G.P.

4th term $= ar^{4-1} = ar^3$; 7th term $= ar^{7-1} = ar^6$.

By the question, $ar^3 = 54$ ... (1)

$ar^6 = 1458$ ... (2)

Dividing (2) by (1), $r^3 = \frac{1458}{54} = 27$

$\therefore \quad r = 3$

Substituting the value of r in (1), we have

$a \times 3^3 = 54$ or $a = \frac{54}{27}$ $\therefore \quad a = 2$

The required G.P. is 2, 6, 18, 54, ...

EXERCISE 212

1. The 5th term of a G.P. is 81 and the 2nd term is 24. Find the G.P.

2. Find the G.P. of which the 3rd term is 4 and the 7th term is $\frac{64}{81}$.

3. The 5th term of a geometric series is 9 times the 3rd term and the 2nd term is 6. Find the series.

4. The 3rd term of a G.P. is the square of the first term and the 5th term is 64. Find the sequence.

5. If the $(m+n)$ the term of a G.P. is p and the $(m-n)$th term q, find the mth term.

Hint: $ar^{m+n-1} = p$, $ar^{m-n-1} = q$ $\therefore \quad (ar^{m-1})^2 = pq$.

4. To find the sum of a G.P. to n terms.

Let a be the first term and r the common ratio of the G.P. so that the nth term $= ar^{n-1}$.

Let S_n denote the sum of n terms of the G.P.

Then, $S_n = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1}$

Again, $r.\ S_n = ar + ar^2 + \ldots\ldots\ldots + ar^{n-1} + ar^n$

Subtracting, we have

$S_n\ (1-r) = a - ar^n$ or $a(1-r^n)$

$$\therefore\quad S_n = \frac{a(1-r^n)}{1-r} \qquad \text{provided } r \neq 0 \qquad \ldots\ (1)$$

The above formula may also be written as

$$S_n = \frac{a(r^n-1)}{r-1} \qquad \text{provided } r > 1 \qquad \ldots\ (2)$$

If l be the last term of the G.P. of n terms, $l = ar^{n-1}$, and $lr = ar^n$

$$\therefore\quad S_n = \frac{a-ar^n}{1-r} = \frac{a-lr}{1-r} \quad \text{or} \quad \frac{lr-a}{r-1} \quad \text{provided } r \neq 1.$$

Note: (i) The formula (1) is useful when r is a fraction between $+1$ and -1. It is convenient to use formula (1) when $r < 1$ and formula (2) when $r > 1$.

(ii) These formulae fail when $r = 1$, because in that case $S_n = \frac{a \times 0}{0}$ or $\frac{0}{0}$ which is indeterminate.

(iii) When $r = 1$, all the terms are equal to the first term and there is no series.

Example 4: Sum the sequence $-9,\ -4\frac{1}{2},\ -2\frac{1}{4},\ \ldots\ldots$ to 8 terms.

Sol. The given sequence is $-9,\ -\frac{9}{2},\ -\frac{9}{4},\ \ldots\ldots\ldots$

Here $a = -9$, $r = (-\frac{9}{2}) \div (-9) = \frac{1}{2}$ and $n = 8$.

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we have

$$S = \frac{-9[1-(\frac{1}{2})^8]}{1-\frac{1}{2}} = \frac{-9[1-\frac{1}{256}]}{\frac{1}{2}}$$

$$= -9 \times \frac{255}{256} \times 2 \qquad = -\frac{2295}{128}$$

EXERCISE 213

1. Sum the following series:

(i) $3+6+12+$ ..... to 12 terms.
(ii) $\frac{1}{8}+\frac{1}{2}+\frac{3}{4}+$ ..... to 8 terms.
(iii) $48-36+27-$ ..... to 9 terms.
(iv) $1+\sqrt{3}+3+$ ..... to 10 terms.
(v) $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+$ ..... to n terms.

2. Find the sum of 8 terms of a G. P. whose 1st term is 16 and the 2nd term 4.

3. Find the sum of the series.

(i) $3+3^2+3^3+3^4+$ ..... to n terms.
(ii) $1+2+2^2+2^3+$ ..... $+2^{n-1}$.

4. Find the sum of n terms of a series whose rth term is $(2r+1)2^r$.

5. Find the sum of 7 terms of a G.P. whose 2nd term is 3 and 6th term 243.

6. The first and last term of a G.P. are 3 and $\frac{1}{3^8}$ respectively and the common ratio is $\frac{1}{3}$. Find the number of terms and the sum.

5. Inverse Problems.

In the formula $S_n = \frac{a(1-r^n)}{1-r}$, four quantities a, r, n and S are involved. If any three of them are given, the remaining one can be obtained. So we can have the following three more cases—

(i) given S, n and r, to find a
(ii) given S, n and a, to find r
(iii) given S, a and r, to find n.

Example 5: How many terms of the series $\frac{3}{4}-\frac{1}{2}+\frac{1}{3}-$ ... must be taken to amount to $\frac{1261}{2916}$?

Sol. The given series is $\frac{3}{4}-\frac{1}{2}+\frac{1}{3}-$ .....

$a=\frac{3}{4}$, $r=-\frac{1}{2}\div\frac{3}{4}=-\frac{2}{3}$, $S=\frac{1261}{2916}$ and n is required.

Using the formula $S = \frac{a(1-r^n)}{1-r}$, we have

$\frac{1261}{2916} = \frac{\frac{3}{4}[1-(-\frac{2}{3})^n]}{1-(-\frac{2}{3})}$ or $\frac{1261}{2916} = \frac{\frac{3}{4}[1-(-\frac{2}{3})^n]}{\frac{5}{3}}$

or $\frac{1261}{2916} = \frac{3}{5} \times \frac{3}{4} [1-(-\frac{2}{3})^n]$

or $1-(-\frac{2}{3})^n = \frac{1261}{2916} \times \frac{20}{9} = \frac{6305}{6561}$

or $1-\frac{6305}{6561} = (-\frac{2}{3})^n$ or $(-\frac{2}{3})^n = \frac{256}{6561}$

or $(-\frac{2}{3})^n = (-\frac{2}{3})^8$ $\therefore$ $n = 8$

The number of terms is 8.

EXERCISE 214.

1. How many terms of the series $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - \ldots\ldots\ldots\ldots$ must be taken to amount to $\frac{55}{72}$?

2. How many terms of the sequence $\sqrt{3}, 3, 3\sqrt{3}, \ldots\ldots\ldots$ amount to $364\sqrt{3}\ (\sqrt{3}+1)$?

3. The sum of the first six terms of a G. P. is 9 times the sum of the first three terms. Find the common ratio of the sequence.

4. How many terms of the series $\frac{2}{3} - 1 + \frac{3}{2} - \ldots\ldots\ldots$ when added together are equal to $\frac{413}{96}$?

6. Geometric Mean.

A number 'G' is said to be the geometric mean between a and b, if a, G, b, are in G. P. Geometric Mean is briefly written as G. M.

The numbers $G_1, G_2, G_3, \ldots\ldots\ldots, G_n$ *are said to be n geometric means between a and b, if* $a, G_1, G_2, G_3, \ldots\ldots\ldots, G_n, b$ *are in G. P.*

7. A single geometric mean between two quantities is equal to the square root of their product.

Let G be the geometric mean between a and b.
Then a, G, b are in $G.P.$

$\therefore$ common ratio $= \frac{G}{a} = \frac{b}{G}$

Or $G^2 = ab$ or $G = \pm\sqrt{ab}$.

8. **To insert n geometric means between two given numbers a and b.**

Let $G_1, G_2, G_3, \ldots\ldots\ldots, G_n$ be the n geometric means between a and b.

then $a, G_1, G_2, G_3, \ldots\ldots\ldots, G_n, b$ are in *G.P.*

Here first term $=a$, last term $=b$, and the number of terms $=1+n+1=n+2$.

Let r be the common ratio of the *G.P.*

$T_{n+2} = ar^{n+2-1}$ and this is given to be b.

$$\therefore\ ar^{n+1} = b \quad \text{or} \quad r^{n+1} = \frac{b}{a} \qquad \therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Now
$$G_1 = T_2 = ar^{2-1} = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$
$$G_2 = T_3 = ar^{3-1} = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$
$$G_3 = T_4 = ar^{4-1} = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$G_n = T_{n+1} = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Hence the required geometric means are

$$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots\ldots\ldots\ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

EXERCISE 215

1. Find the geometric means between

(i) $\frac{p}{2q}$ and $\frac{q}{2p}$ (ii) ·6 and ·006.

2. Insert three geometric means between 4 and 324.

3. Insert three geometric means between $\frac{1}{9}$ and 9.

4. What are the four geometric means between $5\frac{1}{8}$ and $40\frac{1}{2}$?

5. If one geometric mean G and two arithmetic means p and q be inserted between two numbers a and b, show that $G^2 = (2p-q)(2q-p)$.

Hint. $G^2 = ab$; $p-a = q-p = b-q$.

6. Prove that the product of n geometric means between a and b is equal to the nth power of the single geometric mean between a and b.

9. Infinite Geometric Series.

Let r be a proper fraction, say $\frac{2}{5}$

Then $r^2 = (\frac{2}{5})^2 = \frac{2}{5}$ of $\frac{2}{5}$ $\quad\therefore\quad (\frac{2}{5})^2 < \frac{2}{5}$ i.e. $r^2 < r$

$r^3 = (\frac{2}{5})^3 = \frac{2}{5}$ of $(\frac{2}{5})^2$ $\quad\therefore\quad (\frac{2}{5})^3 < (\frac{2}{5})^2$ i.e. $r^3 < r^2$

A fraction of a fraction is less than the fraction itself. Thus, in the sequence $\frac{2}{5}$, $(\frac{2}{5})^2$, $(\frac{2}{5})^3$,.....each term is less than the preceding one. In general $(\frac{2}{5})^n$ decreases as n increases.

Hence, if r is a *proper faction*, r^n is very small and when n is *sufficiently large*, r^n is almost *zero*.

Consider the series $a+ar+ar^2+$.....to n terms. As has been already proved,

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

If r is *proper fraction* and n is *infinitely large*, r^n becomes *zero*.

Then $\frac{ar^n}{1-r}$ reduces to zero. In the limit when $n = \infty$,

$$\mathbf{S_\infty = \frac{a}{1-r}}$$ where r is numerically *less* than one.

Example 6: Sum to infinity the series $2-\frac{3}{2}+\frac{9}{8}-$ .....

Sol. The given series is $2-\frac{3}{2}+\frac{9}{8}-$.....

Here $a = 2$, $r = -\frac{3}{2}\times\frac{1}{2} = -\frac{3}{4}$ which is numerically less than one.

Using the formula $S = \frac{a}{1-r}$, we have

the required sum $= \frac{2}{1-(-\frac{3}{4})} = \frac{2}{\frac{7}{4}} = 2\times\frac{4}{7} = \frac{8}{7} = 1\ \frac{1}{7}$.

A *recurring decimal* is a good illustration of an infinite geometrical progression.

$1\cdot2\dot{8} = 1\cdot288888$.....

$= 1\cdot2+\cdot08+\cdot008+\cdot0008+$.....to infinity.

$= 1\cdot2+[\cdot08+\cdot008+\cdot0008+$.....to infinity].

$= 1\cdot2+$[an infinite G.P. in which $a = \cdot08$,

$r = \cdot008\div\cdot08 = \cdot1$]

$$= 1{\cdot}2+\frac{{\cdot}08}{1-{\cdot}1} = 1.2+\frac{{\cdot}08}{{\cdot}9} = 1\frac{2}{10}+\frac{8}{90}$$

$$= 1\frac{26}{90} \qquad = 1\frac{13}{45}.$$

EXERCISE 216.

Sum up the following series.

1. $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+$.....to ∞.
2. $\frac{1}{2}+\frac{1}{5}+\frac{2}{25}+\frac{4}{125}+$.....to ∞.
3. $4+{\cdot}8+{\cdot}16+$.....to infinity.
4. $\frac{4}{7}-\frac{5}{7^2}+\frac{4}{7^3}-\frac{5}{7^4}+\frac{4}{7^5}-\frac{5}{7^6}+$.....to ∞.
5. $\left(1+\frac{1}{2^2}\right)+\left(\frac{1}{2}+\frac{1}{2^4}\right)+\left(\frac{1}{2^2}+\frac{1}{2^6}\right)+$.....to ∞.
6. Evaluate (i) $\dot{9}\dot{7}$ (ii) $\dot{4}\dot{5}$ (iii) $3{\cdot}0\dot{4}0\dot{5}$
7. Find a geometrical series whose first term is $\frac{1}{4}$ and the sum to infinity is $\frac{1}{3}$.
8. Find the first term of a G. P. whose 2nd term is 2 and sum to infinity is 8.
9. The sum of an infinite geometrical series is 32, and the sum of its first two terms is 24; find the series.
10. If $x = 1+p+p^2+p^3+$.....to ∞ ;
$y = 1+q+q^2+q^3+$.....to ∞ ;
prove that $1+pq+p^2q^2+$.....to $\infty = \frac{xy}{x+y-1}$

10. Properties of a G. P.

If each term of a *G. P.* is (i) multiplied or (ii) divided by the *same* number, the resulting sequence is also a G. P. having the *same* common ratio.

The above properties and the following hints are very useful in solving problems in G. P.

If *three* numbers in G. P. are given, taken them as $\frac{a}{r}$, a, ar so that their *product* is a^3.

If *four* numbers in G. P. are given, take them as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ so that their *product* is a^4.

Example 7: Find three numbers in G. P. whose sum is 19 and sum of the products of the terms taken in pairs is 114.

Sol. Let the three numbers in G. P. be $\frac{a}{r}, a, ar$.

Then $\frac{a}{r}+a+ar = 19$

Or $a(1+r+r^2) = 19r$ ... (1)

Also, $\frac{a}{r}\times a + a\times ar + ar\times\frac{a}{r} = 114$ ... (given)

Or $a^2\left(\frac{1}{r}+r+1\right) = 114$

Or $a^2(1+r^2+r) = 114r$ ... (2)

Dividing (2) by (1), we get $a = \frac{114r}{19r} = 6$

Putting the value of a in (1), we have

$6(1+r+r^2) = 19r$ or $6+6r+6r^2 = 19r$.

Or $6r^2-13r+6 = 0$ or $(2r-3)(3r-2) = 0$

If $2r-3 = 0,\ 2r = 3$ $\therefore\ r = \frac{3}{2}$

If $3r-2 = 0,\ 3r = 2$ $\therefore\ r = \frac{2}{3}$

When $r = \frac{3}{2}$, the numbers are $6\times\frac{2}{3}$, 6, $6\times\frac{3}{2}$ i.e. 4, 6, 9.

When $r = \frac{2}{3}$, the numbers are $6\times\frac{3}{2}$, 6, $6\times\frac{2}{3}$ i.e. 9, 6, 4.

$\therefore$ the required numbers are 4, 6, 9.

Example 8: If a, b, c, d are in G. P., show that a^2-b^2, b^2-c^2, c^2-d^2 are also in G. P.

Sol. Since a, b, c, d are in G. P.,

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r \text{ (common ratio)}$$

$\therefore\ b = ar$

$c = br = ar^2$

$d = cr = br^2 = ar^3$.

Now $a^2-b^2 = a^2-a^2r^2 = a^2(1-r^2)$

$b^2-c^2 = a^2r^2-a^2r^4 = a^2r^2(1-r^2)$

$c^2-d^2 = a^2r^4-a^2r^6 = a^2r^4(1-r^2)$

$$\therefore \quad \frac{b^2-c^2}{a^2-b^2} = \frac{a^2r^2(1-r^2)}{a^2(1-r^2)} = r^2$$

and $$\frac{c^2-d^2}{b^2-c^2} = \frac{a^2r^4(1-r^2)}{a^2r^2(1-r^2)} = r^2$$

Hence a^2-b^2, b^2-c^2, c^2-d^2 are in *G. P.* with common ratio r^2.

EXERCISE 217.

1. If a, b, c, d are in G. P., show that
   (i) $(a-b)^2$, $(b-c)^2$, $(c-d)^2$ are also in G. P.
   (ii) a^2+b^2, b^2+c^2, c^2+d^2 are in G. P.
   (iii) $a^2+b^2+c^2$, $ab+bc+cd$, $b^2+c^2+d^2$ are in G. P.
2. Find three numbers in G. P. whose sum is 19 and whose product is 216.
3. Find four numbers in G. P. so that their sum may be 15 and the sum of their squares 85.
4. If a, b, c, d are in G. P., prove that
$(b-c)^2+(c-a)^2-(d-b)^2 = (a-d)^2$
5. Three numbers whose sum is 15 are in A. P. ; if 1, 4 and 19 be added respectively to the first, second and third number, the resulting numbers are in G.P. Find the numbers.

Example 9: Find the sum of the series $9+99+999+\ldots\ldots$ to n terms.

Sol. The given series $= (10-1)+(10^2-1)+(10^3-1)+\ldots\ldots$ to n terms.

$= [10+10^2+10^3+\ldots\ldots$to n terms$]-[1+1+1+\ldots\ldots$to n terms$]$.

$$= \frac{10(10^n-1)}{10-1} - n \qquad = \frac{10}{9}(10^n-1) - n.$$

Note: (i) The series $5+55+555+\ldots\ldots\ldots\ldots$to n terms can at once be seen to be $\frac{5}{9}$ of the series worked out above.

(ii) The sum of $\cdot 9+\cdot 99+\cdot 999+\ldots\ldots\ldots\ldots$to n terms can be worked out by splitting the various terms and rewriting them in the form

$$\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ldots\ldots\ldots\ldots\ldots \text{ to } n \text{ terms.}$$

The sum will be $n-\left[\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots\ldots\ldots\ldots\ldots\text{to } n \text{ terms}\right]$

$$= n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}} = n-\frac{1}{9}\left(\frac{10^n-1}{10^n}\right).$$

Example 10: If x, y, z be the pth, qth and rth terms of a G. P., prove that

$$x^{q-r}.\ y^{r-p}.\ z^{p-q} = 1.$$

Sol. Let A and R be the first term and common ratio of the G. P.

pth term $= AR^{p-1}$ & this is given to be x. $\therefore$ $x = AR^{p-1}$

Similarly $y = AR^{q-1}$, $z = AR^{r-1}$.

Now $x^{q-r}.\ y^{r-p}.\ z^{p-q}$

$= (AR^{p-1})^{q-r}.\ (AR^{q-1})^{r-p}.\ (AR^{r-1})^{p-q}$

$= A^{q-r}\ R^{(p-1)(q-r)}.\ A^{r-p}\ R^{(q-1)(r-p)}.\ A^{p-q}\ R^{(r-1)(p-q)}$

$= A^{q-r+r-p+p-q}.\ R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= A^0 \times R^0 = 1 \times 1 = 1.$ Q.E.D,

EXERCISE 218

1. Prove that $X^{\frac{1}{2}}.\ X^{\frac{1}{4}}.\ X^{\frac{1}{8}}$..... to infinity $= X$

2. Sum the following series to n terms :

(i) $3+33+333+3333+$.....

(ii) $8+88+888+8888+$.....

(iii) $\cdot 1+\cdot 11+\cdot 111+\cdot 11$.....

(iv) $\cdot 6+\cdot 66+\cdot 666+$.....

3. The arithmetic mean between two numbers is 15 and their geometric mean is 9. Find the numbers.

4. Find the sum, to n terms of the series $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+$.....

Hint : the series $= \frac{1}{x-y}[(x^2-y^2)+(x^3-y^3)+x^4-y^4)+$..... to n terms.]

5. If $x = a+\frac{a}{r}+\frac{a}{r^2}+$..... to ∞,

$y = b-\frac{b}{r}+\frac{b}{r^2}-$.....to ∞,

$z = c+\frac{c}{r^2}+\frac{c}{r^4}+$.....to ∞,

show that $\frac{xy}{z} = \frac{ab}{c}$.

6. If p, q, r be in A.P., show that pth, qth and rth terms of any G.P. are themselves in G.P.

7. If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{a}$ and a, b, c be in G. P.; prove that x, y, z are in A.P.

8. If a, b, c are in G. P., and x, y are respectively the arithmetic means between a, b and b, c show that

$$\frac{a}{x} + \frac{c}{y} = 2, \text{ and } \frac{1}{x} + \frac{1}{y} = \frac{2}{b}.$$

9. If P be the continued product of n terms in a G.P., S the sum, and R the sum of their reciprocals, show that $P^2 = \left(\frac{S}{R}\right)^n$.

CHAPTER XXX

HARMONICAL PROGRESSION

1. Harmonical Progression

Examine the following sequences :

(i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\ldots\ldots\ldots\ldots$

(ii) $\frac{1}{22}, \frac{1}{18}, \frac{1}{14}, \frac{1}{10}, \ldots\ldots\ldots\ldots\ldots$

In both cases the terms do not form either an A.P. or a G.P. but their reciprocals are in A.P. Such sequences are called **harmonical progressions.**

A sequence is said to be a harmonical progression if the reciprocals of its terms form an arithmetical progression.

Since $a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ldots\ldots\ldots\ldots,\ a+(n-1)d$ is the standard form of an A.P., the *standard form* of an H.P. will be $\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ldots\ldots\ldots\ldots\ldots,\ \frac{1}{a+(n-1)d}$

Thus, corresponding to every harmonical progression there always exists an arithmetical progression. The nth term of an H.P. is the reciprocal of the nth term of the corresponding A.P.

2. If a, b, c are in H.P., $\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}$ are in A.P.

$$\therefore\quad \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b} \qquad \text{or} \qquad \frac{a-b}{a}=\frac{b-c}{c}$$

By alternando, $$\frac{a-b}{b-c}=\frac{a}{c}$$

This relationship is also used to define an H.P. Accordingly three quantities a, b, c are said to be in H.P. if $\frac{a-b}{b-c}=\frac{a}{c}$.

3. All problems involving harmonical progression are dealt with by *inverting* the terms and treating them as forming the *corresponding* A.P.

If two terms of an H.P. are given, the sequence can be completely determined.

There is *no* general formula for finding the sum of a given number of terms in H.P. However, the sum of a few terms in H.P. can be obtained by actually adding them.

Example 1: Find the 7th term of the H.P. $\frac{2}{9}+\frac{4}{17}+\frac{1}{4}+\ldots\ldots$

Sol. The given series is $\frac{2}{9}+\frac{4}{47}+\frac{1}{4}\ldots\ldots$

Then the corresponding A.P. is $\frac{9}{2}+\frac{17}{4}+4+\ldots\ldots$

Here $a=\frac{9}{2}$, $d=\frac{17}{4}-\frac{9}{2}=-\frac{1}{4}$

Using the formula $t_n=a+(n-1)d$, we get

the 7th term of the A.P. $=\frac{9}{2}+(7-1)(-\frac{1}{4})$

$=\frac{9}{2}-\frac{3}{2}=3$;

$\therefore$ 7th term of the given H.P. $=\frac{1}{3}$.

Example 2: If the mth term of an H.P. is n and the nth term is m, find the $(m+n)$th term.

Sol. In the H.P., mth term $=n$, nth term $=m$

$\therefore$ in the corresponding A.P., mth term $=\frac{1}{n}$, nth term $=\frac{1}{m}$

Let a be the first term and d the common difference of the corresponding A.P.

mth term: $a+(m-1)d=\frac{1}{n}$ ... ... (1)

nth term: $a+(n-1)d=\frac{1}{m}$ ... ... (2)

Subtracting (2) from (1), $(m-n)d=\frac{1}{n}-\frac{1}{m}=\frac{m-n}{mn}$

$\therefore \quad d=\frac{1}{mn}$.

Substituting the value of d in (1), we get

$a+(m-1)\cdot\frac{1}{mn}=\frac{1}{n}$ or $a=\frac{1}{n}-\frac{m-1}{mn}$

or $a=\frac{m-m+1}{mn}$ $\therefore \quad a=\frac{1}{mn}$.

Now the $(m+n)$th term of the A.P. $=a+(m+n-1)d$

$=\frac{1}{mn}+(m+n-1)\left(\frac{1}{mn}\right)=\frac{m+n}{mn}$

$\therefore$ the $(m+n)$th term of the given H.P. $=\frac{mn}{m+n}$.

EXERCISE 219

1. Find the 9th term of the H.P. 6, 4, 3 .....

2. Find the 7th term of the H.P. 3, 2, $\frac{3}{2}$, ...

3. Continue the sequence 4, 2, $1\frac{1}{3}$, ...for three more terms.

4. Find the nth term of the following series :

(i) $2+1\frac{3}{4}+1\frac{5}{9}+$ .....

(ii) $\frac{1}{8}+\frac{3}{10}+\frac{3}{11}+\frac{1}{4}+$ .....

5. The first term of an H.P. is 5 and the 17th term is $-\frac{1}{4}$. Find the 22nd term.

6. The 7th term of an H.P. is $\frac{1}{10}$ and the 12th term is $\frac{1}{25}$. What is the 19th term ?

7. Find the first three terms of an H.P. whose 5th term and 9th term are $\frac{3}{5}$ and $\frac{1}{2}$ respectively.

8. If the mth term of an H.P. be n and the nth term m, show that the (mn)th term is 1.

4. Harmonic Mean.

A number 'H' is said to be the harmonic mean between a and b, if a, H, b are in H.P. Harmonic mean is briefly written as H.M.

The numbers $H_1, H_2, H_3, \ldots\ldots, H_n$ *are said to be n harmonic means between a and b if* $a, H_1, H_2, H_3, \ldots\ldots, H_n, b$ *are in H.P.*

5. To find the Harmonic Mean between a and b.

Let H be the harmonic mean between a and b.

Then a, H, b are in H. P.

$\therefore$ $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$\therefore$ the common difference $= \frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}$.

Transposing, $\frac{2}{H}=\frac{1}{a}+\frac{1}{b}$ or $\frac{2}{H}=\frac{b+a}{ab}$

or $\frac{H}{2}=\frac{ab}{a+b}$ $\therefore$ $\boldsymbol{H=\frac{2ab}{a+b}}$.

6. **To insert n harmonic means between two given numbers a and b.**

Let $H_1, H_2, H_3, \ldots\ldots\ldots, H_n$ be the n harmonic means between a and b.

Then $a, H_1, H_2, H_3, \ldots\ldots\ldots, H_n, b$ are in H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots\ldots\ldots, \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

Here, the first term $= \frac{1}{a}$, last term $= \frac{1}{b}$, and the number of terms $= 1+n+1 = n+2$.

Let d be the common difference of the corresponding A.P.

$$T_{n+2} = \frac{1}{a}+(n+2-1)d \qquad \therefore \quad \frac{1}{a}+(n+1)d = \frac{1}{b}$$

$$\text{or} \quad (n+1)d = \frac{1}{b}-\frac{1}{a} \qquad \therefore \quad d = \frac{a-b}{(n+1)ab}$$

$$\text{Now} \quad \frac{1}{H_1} = \frac{1}{a}+d \qquad = \frac{1}{a}+\frac{a-b}{(n+1)ab} \qquad = \frac{a+nb}{(n+1)ab}$$

$$\frac{1}{H_2} = \frac{1}{a}+2d \qquad = \frac{1}{a}+\frac{2(a-b)}{(n+1)ab} \qquad = \frac{2a+(n-1)b}{(n+1)ab}$$

$$\frac{1}{H_3} = \frac{1}{a}+3d \qquad = \frac{1}{a}+\frac{3(a-b)}{(n+1)ab} \qquad = \frac{3a+(n-2)b}{(n+1)ab}$$

$$\ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots \qquad \ldots \quad \ldots$$

$$\frac{1}{H_n} = \frac{1}{a}+nd \qquad = \frac{1}{a}+\frac{n(a-b)}{(n+1)ab} \qquad = \frac{na+b}{(n+1)ab}.$$

Therefore $H_1 = \frac{(n+1)ab}{a+nb}, \quad H_2 = \frac{(n+1)ab}{2a+(n-1)b},$

$$H_3 = \frac{(n+1)ab}{3a+(n-2)b}, \qquad H_n = \frac{(n+1)ab}{na+b}$$

which are the required harmonic means.

EXERCISE 220.

1. Find the harmonic mean between

(*a*) 10 and 15 (*b*) $\frac{2}{5}$ and $\frac{2}{11}$

(*c*) $\frac{m-n}{m+n}$ and $\frac{m+n}{m-n}$ (*d*) $a+b$ and $a-b$.

2. Insert two harmonic means between 2 and $\frac{1}{5}$.

3. Insert three harmonic means between 2 and 4.

4. Insert four harmonic means between $\frac{2}{3}$ and $\frac{6}{19}$.

5. Insert six harmonic means between 3 and $\frac{6}{23}$.

6. If a, b, c are in H.P., show that $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$.

7. Relation between the Arithmetic, Geometric and Harmonic Means.

Let A, G, H be the arithmetic, geometric and harmonic means respectively between two positive numbers, a and b.

Then, $A=\frac{a+b}{2}$, $G=\sqrt{ab}$, $H=\frac{2ab}{a+b}$.

Now $A\times H=\frac{a+b}{2}\times\frac{2ab}{a+b}=ab=(\sqrt{ab})^2=G^2$

i.e. $\frac{G}{A}=\frac{H}{G}$.

Hence A, G, H are in geometric progression.

Also, $A-G=\frac{a+b}{2}-\sqrt{ab}=\frac{a+b-2\sqrt{ab}}{2}$

$=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2$, which is positive, as a and b are positive quantities.

$\therefore$ $A>G$.

We know that $AH=G^2$ *i.e.* $\frac{A}{G}=\frac{G}{H}$.

Since $A>G$ (proved), $G>H$.

Hence, $\mathbf{A>G>H}$.

Note: A, G, H are in descending order of magnitude, when only the positive value of G is taken into account.

Example 3: If a, b, c are in A.P., b, c, d in G. P. and c, d, e in H.P., prove that a, c, e are in G.P.

Sol. Since a, b, c are in A.P., $b = \frac{a+c}{2}$ ... (1)

Since b, c, d are in G.P., $c^2 = b.d$ ... (2)

& since c, d, e are in H.P., $d = \frac{2ce}{c+e}$ ... (3)

Substituting the values of b and d in (2),

we have $c^2 = \frac{a+c}{2} \times \frac{2ce}{c+e} = \frac{ce(a+c)}{c+e}$

or $c^2(c+e) = ce(a+c)$

or $c(c+e) = e(a+c)$

or $c^2+ce = ae+ce$

or $c^2 = ae$ *i.e.* $\frac{c}{a} = \frac{e}{c}$.

$\therefore$ a, c, e are in G.P.

Example 4: If a, b, c be in H.P., show that

$\frac{1}{a(b+c)}, \frac{1}{b(c+a)}, \frac{1}{c(a+b)}$ are also in H.P.

Sol. Since a, b, c are in H.P., $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Multiplying throughout by $-abc$,

$-bc, -ca, -ab$ are also in A.P.

Adding $ab+bc+ca$ to each term, we have

$ab+ca, ab+bc, bc+ca$ are in A.P.

or $a(b+c), b(c+a), c(a+b)$ are in A.P.

i.e. $\frac{1}{a(b+c)}, \frac{1}{b(c+a)}, \frac{1}{c(a+b)}$ are in H.P.

Example 5: If $a^x = b^y = c^z$, and a, b, c are in G.P., prove that x, y, z are in H.P.

Sol. Let $a^x = b^y = c^z = k$.

Then $a = k^{\frac{1}{x}}; \; b = k^{\frac{1}{y}}; \; c = k^{\frac{1}{z}}$

Since a, b, c are in G.P., $b^2 = ac$.

Substituting the values of a, b & c, in the above relation, we have

$$\left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}} \cdot k^{\frac{1}{z}}$$

or $$k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}}$$

or $$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$\therefore$ $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

i.e. x, y, z are in H.P.

EXERCISE 221.

1. If the harmonic mean between two numbers is 3 and their arithmetic mean is 4, find the numbers and their geometric mean.

2. If a, b, c are in H.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H.P.

3. If $b+c, c+a, a+b$ be in H.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

4. If $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in H.P., show that $b+c, c+a, a+b$ are also in H.P.

5. If a, b, c are in A.P. and b, c, a are in G.P., prove that c, a, b are in H.P.

6. If a, b, c are in A.P., and a, mb, c are in G.P., show that $a, m^2 b, c$ are in H.P.

7. If $\frac{a+b}{2}, b, \frac{b+c}{2}$ are in H.P., prove that a, b, c are in G.P.

8. If a, b, c are in A.P., p, q, r are in H.P. and ap, bq, cr in G.P., prove that $\frac{p}{r} + \frac{r}{p} = \frac{a}{c} + \frac{c}{a}$.

9. If, between any two quantities, there be inserted two Arithmetic Means A_1, A_2, two Geometric Means G_1, G_2 and two Harmonic Means H_1, H_2, show that $G_1.G_2 : H_1.H_2 = A_1+A_2 : H_1+H_2$.

10. If x, y, z be the pth, qth and rth terms of an H.P., prove that $yz(q-r)+zx(r-p)+xy(p-q) = 0$.

SECTIONAL REVISION VI

TEST PAPERS

Paper 1

1. Solve $2x-1 = \frac{15}{x}$ by factors and verify the solution.

2. One roof of the equation $3x^2-2kx+(k^2-3) = 0$ is three times the other. Find the possible values of k.

3. Find the sum of the first ten terms of the sequence whose nth term is $3-\frac{1}{2}n$.

4. The fourth term of a G.P. exceeds the third term by 25, and the sum of these two terms is 225. Find the first term and the common ratio.

5. Continue the H.P. $-\frac{2}{7}$, -1, $\frac{2}{3}$,.....to 7 terms.

6. The A.M. of two quantities is $1\frac{2}{3}$ and the G.M. is $1\frac{1}{3}$. Find the H.M.

Paper 2.

1. Solve the equation $2x^2-x-6 = 0$ by completing the square and verify your solution.

2. How many terms of the A.P. 3, 7, 11, 15, ... are needed to make a total of 820 ?

3. The second and third terms of a G.P. are 24 and $12(b+1)$ respectively. What is the first term ?

If these three terms have a sum of 76, what are the possible numerical values of the third term ?

4. Sum the series $4+44+444+$ ..... to n terms.

5. If α and β are the roots of the equation $7x^2-8x+2 = 0$, find, without solving this equation, the quadratic equation whose roots are $2\alpha+1$ and $2\beta+1$.

6. Simplify : $\frac{(4+i)^2}{4-i}$.

Paper 3.

1. If $x^2 = 3-3x$, find the values of x correct to 3 significant figures.

2. For all values of r the sum of the first r terms of a series is $3r^2-2r$. Find the first three terms.

3. Find the two values of k for which the equation $4x^2+(5k+3)x+(2k^2-1) = 0$ has equal roots for x.

4. Show that the sum of the first 50 terms of the G.P. whose second term is 6 and fifth term is 162 is 1 less than a power of 3.

5. If a, b, c be in A.P. and b, c, d be in H.P., prove that $\frac{a}{b} = \frac{c}{d}$.

6. Show that the geometric mean between two positive numbers is equal to the geometric mean between their arithmetic and harmonic means.

Paper 4.

1. Show that the sum to n terms of the arithmetical progression a, $a+2d$, $a+4d$, ...is $na+(n^2-n)d$.

2. The third term of a G.P. exceeds the second by $\frac{9}{14}$, and the second exceeds the first by $\frac{3}{7}$. Find the common ratio and the first term.

3. If α and β are the roots of the equation $x^2+px+q = 0$, find the equation whose roots are $\alpha+\frac{1}{\alpha}$ and $\beta+\frac{1}{\beta}$.

4. Find the nth term of the series

$$4+4\tfrac{2}{7}+4\tfrac{8}{13}+5+ \ldots\ldots\ldots$$

5. If a, b, c, d, e be five quantities such that a, b, c are in A.P. ; b, c, d in G.P. ; c, d, e in H.P.; show that a, c, e are in G.P.

6. Solve the equation $2x^2-8x+3 = 0$, giving the roots correct to two places of decimals.

Paper 5.

1. The fifth term of an arithmetical progression is 9, and the sum of the first eleven terms is $115\frac{1}{2}$. Find the first term and the common difference.

2. If the roots of the equation $px^2-4x+q=0$ are -2 and $3\frac{1}{3}$, find the value of the coefficients, p and q.

3. The sum of the first 8 terms of a G.P. is 17 times the sum of the first 4 terms; and the common ratio is positive. Find the common ratio.

4. Insert four harmonic means between 1 and 30.

5. Show that the arithmetic mean, geometric mean and the harmonic mean between two positive numbers are in geometrical progression.

6. Solve the equation $2x^2-3x-5=0$, and check by finding the sum and product of the roots.

OBJECTIVE TYPE ITEMS—VI

1. Complete each of the following statements by supplying the proper letters/words/phrases.

(*i*) A quadratic equation cannot have more than..... roots.

(ii) The roots of a quadratic equation are equal, if the discriminant is equal to.....

(iii) If the same number is added to each term of an *A.P.*, the resulting sequence is.....

(iv) The geometric mean between m and n is.....

(v) The common ratio in an infinite geometric series is always.....

(vi) The arithmetic, geometric and harmonic means between two positive numbers are in.....

2. For each part below, choose the right answer and indicate it by the capital letter printed alongside.

(i) The discriminant of the equation $x-1=x^2$ is

A. Zero B. -3
C. -1 D. $+5$.

(ii) The quadratic equation formed by the two roots 3 and 7 is given by

A. $x^2+10x+21=0$ B. $x^2-10x+21=0$
C. $x^2+10x-21=0$ D. $x^2-10x-21=0$.

(iii) The roots of the quadratic equation $x^2+px+\frac{1}{4}q^2=0$ are equal, if

A. $4p^2=q^2$ B. $4p^2+q^2=0$
C. $p^2=q^2$ D. $p^2+q^2=0$.

(iv) The roots of the quadratic equation $px^2+qx+r=0$, where p, q, r are real and rational numbers, are real, rational and unequal, if

A. q^2-4pr is positive
B. q^2-4pr is positive and a perfect square
C. q^2-4pr is equal to zero
D. q^2-4pr is negative.

3. Put a tick mark ($\surd$) against the correct answer in the space provided.

(a) If A, G, H be the arithmetic, geometric and harmonic means of two positive and unequal numbers m and n, then

..... (i) $A<G<H$ .....(ii) $A>G<H$
.....(iii) $A<G>H$ .....(iv) $A>G>H$.

(b) The harmonic mean between two numbers p and q is

..... (i) $\frac{p+q}{2}$ .....(ii) $\frac{2}{p+q}$
.....(iii) $\frac{p+q}{2pq}$ .....(iv) $\frac{2pq}{p+q}$.

(c) If a, b, c be three quantities in H.P., then

..... (i) $\frac{a-b}{b-c}=\frac{a}{c}$ .....(ii) $\frac{a-b}{b-c}=\frac{a}{b}$
.....(iii) $\frac{a-b}{b-c}=\frac{b}{c}$ .....(iv) $\frac{a-b}{b-c}=\frac{c}{a}$.

(d) The geometric mean between two numbers m and n is

..... (i) $\sqrt{m+n}$ .....(ii) $\sqrt{m-n}$
.....(iii) $\sqrt{mn}$ .....(iv) $\sqrt{m\div n}$.

(e) The value of i^5 is

..... (i) -1 .....(ii) $+1$
.....(iii) $+i$ .....(iv) $-i$.

(f) If $\frac{a}{b}=\frac{b}{c}$, which of the following statements is correct?

..... (i) b, c, a are in G.P.
..... (ii) c, a, b are in G.P.
.....(iii) c, b, a are in G.P.

(g) If $a-b=b-c$, which of the following statements is correct ?

..... (i) b, c, a are in A.P. ..... (ii) a, b, c are in G.P.
.....(iii) c, b, a are in A.P. .....(iv) c, a, b are in G.P.

ANSWERS

EXERCISE 1. (page 2)

1. (i) $x+5$ (ii) $x+6$ (iii) $x+y$ (iv) $x+5+y$.

2. (i) Seven added to x. (ii) p increased by q.
(iii) x increased by 11 and y added to the result.
(iv) b added to a and c added to the result, or the sum of three quantities a, b and c.

3. (i) $15-9$ (ii) $m-9$ (iii) $11-x$ (iv) $b-a$.
(v) $m-n$ (also $n-m$) (vi) $11+b-c$ (vii) $x-y+z$.

4. (i) p diminished by 7 (ii) The difference between m and n.
(iii) The sum of l and m diminished by n.
(iv) p diminished by q and r added to the result.
(v) The excess of x over y diminished by z.
(vi) The excess of the sum of a and b over c diminished by d.

5. (i) 600 cm (ii) 100 x paise (iii) 60 t minutes
(iv) 1000 l metres (v) 1000 w mg.

6. Rs. $n \times r$. **7.** $t \times m$ km.

8. (i) $5x$ (ii) $m \times 5$ (iii) $m \times n$ (iv) $7ab$.

9. (i) $\frac{15}{7}$ (ii) $\frac{15}{x}$ (iii) $\frac{m}{7}$ (iv) $\frac{p}{q}$
(v) $\frac{p+q}{m}$ (vi) $\frac{a-b}{c}$.

10. Rs. $4m$; Rs. $4mx$.

11. (i) $6+5=11$ (ii) $4+5=9$ (iii) $2+2=4$
(iv) $4 \times 3 = 3 \times 4$ (v) $20 \div 5 = 80 \div 20$.
(vi) $\frac{1}{2}p = q$ (vii) $\because\ 5x = 40$, $\therefore\ x = 8$ (viii) $\because\ \frac{1}{3}y = 15$, $\therefore\ y = 45$.
(ix) $\because\ 7m > 35$, $\therefore\ m > 5$ (x) $\because\ 6p < 18$ $\therefore\ p < 3$.

12. (i) The sum of x and 7 equals 16.
(ii) x diminished by 11 equals 5.
(iii) Five times x is equal to 30.
(iv) Two-ninths of x is equal to 6.
(v) The sum of twice a and 3 times $b = 5$.
(vi) The sum of $2a$ and $3b$ divided by 7 gives 10.
(vii) $\frac{1}{5}$ of the excess of $3a$ over $2b$ equals 12.
(viii) Since five times x diminished by 3 gives 12, therefore x is equal to 3.
(ix) Since $7x$ minus 2 is greater than 12, therefore x is greater than 2.
(x) Since twice x plus 5 is less than 15, therefore x is less than 5.

EXERCISE 2. (page 5)

1. (i) 7^4 (ii) a^5 (iii) ap^3 (iv) ab^2 (v) $(\cdot 01)^3$ (vi) $\left(\frac{1}{a}\right)^4$
(vii) $3m^2+4n^2$ (viii) a^2n^3 (ix) p^4+p^3 (x) ax^3+by^3.

2. (i) p cubed (ii) x raised to the power of 4 (iii) r raised to the power of 6 (iv) n raised to the power of 15.

3. (i) 5 raised to the 4th power (ii) 7 squared (iii) 2 raised to the power of 5 (iv) 3 raised to the power of 4.

4. (i) $5a = 5$ times $a = a+a+a+a+a$.
$a^5 = a \times a \times a \times a \times a$.
(ii) $p^4 = p \times p \times p \times p$. $4p = 4 \times p$ or $p+p+p+p$.
(iii) 3 times $5 = 5+5+5 = 15$; 5 cubed $= 5 \times 5 \times 5 = 125$.

5. (i) 9 (ii) $1\frac{7}{12}$ (iii) $\cdot 55$ (iv) 18.

6. (i) 5 (ii) $\frac{5}{6}$ (iii) $\cdot 47$

7. (i) 72 (ii) $1 \cdot 76$ (iii) $3\frac{5}{12}$ (iv) 35.

8. (i) $2\frac{7}{8}$ (ii) $1 \cdot 1$ (iii) $6\frac{1}{4}$.

9. (i) 129 (ii) 359 (iii) 1113 (iv) 1.114.
(v) 14 (vi) 67.

EXERCISE 3. (page 7)

1. (See Laws of Indices).

2. (i) 5^7 or 78125 (ii) 7^6 or 117649 (iii) a^{13}
(iv) x^{13}.

3. (i) $6abcd$ (ii) $12x^4y^3$ (iii) $a^3b^3c^2$
(iv) $a^4b^2c^2$ (v) $15a^5b^7c^3$ (vi) $6p^5q^4$.

4. (Refer to Laws of Indices).

5. (i) a^5 (ii) p^9 (iii) m^6 (iv) x^{12}.

6. (i) $2x^2y^4$ (ii) $5m^3n$ (iii) $2p^2q^2$ (iv) $2m^4p$
(v) $3l^2n^7$.

7. (i) 64 (ii) 729 (iii) a^{3m} (iv) x^{3m}
(v) p^{4n} (vi) y^{15}.

EXERCISE 4. (page 9)

1. (Refer to Laws of Indices).

2. (i) 5^2 (ii) 4^3 (iii) x^{11} (iv) $2x^3$
(v) $3y^4$ (vi) $5ab^2$.

3. (i) $4a^3$ (ii) $6x^4$ (iii) $5y^9$.

4. (i) $2a^2$ (ii) $3b^3$ (iii) $4c^5$.

EXERCISE 5 (a). (page 10)

1. $11a^2$ **2.** $9ab$ **3.** $15p$
4. $28xy$ **5.** $6ab$ **6.** $5abc$
7. $5a^2$ **8.** $4x^2$ **9.** $2x+3y$
10. $2a+2b$ **11.** $3x+2y$ **12.** $3x+7$
13. $3x+7y+4$ **14.** $2a$ **15.** $3a+2b-2c$
16. $2xy+3xz$ **17.** $6a^2-3a$ **18.** x^3+x^2+4x
19. $2p^2+p+3$ **20.** $7x^3-3x^2-2x+4$.

EXERCISE 5(b). (page 10)

1. (i) $7x^4+x^3+5x^2+3x+4$ (ii) $4+3x+5x^2+x^3+7x^4$.

2. (i) $a^7+4a^6+5a^3+a^2+1$ (ii) $1+a^2+5a^3+4a^6+a^7$.

3. $5+3x+2x^2+2x^3+x^4$.

4. $3x^4-4x^3+8x^2+4x+9$.

5. $3\times10^2+7\times10+4$; $3x^2+7x+4$.

6. $5\times10^3+7\times10^2+4$; $5x^3+7x^2+4$.

EXERCISE 6. (page 13)

1. (i) $3a$ (ii) $3a$, ab (iii) none (iv) $12a^3b^3$, $12a^2b^2$, $16a^4$
(v) $6ab$, $12a^3b^3$, $12a^2b^2$, $6a^2b$, $3a^2b$, $6abc^2$
(vi) $12a^3b^3$, $12a^2b^2$, $6a^2b$.

2. (i) $2a^2b^2$, $12a^3b^3$ (ii) $6ab$, $36a^3b^3$
(iii) $3ab$, $12a^2b$ (iv) $13a^2b^4$, $156a^3b^5$
(v) ab, $12a^3b^3c^2$ (vi) $2ab$, $72a^2b^2c^2$.

3. (i) 98, 2744. (ii) 225, 3375.

4. (i) $\frac{3bc}{5a}$ (ii) $\frac{3ax}{y}$ (iii) $\frac{4y}{x}$
(iv) $\frac{1}{3abc}$ (v) $3xyz$ (vi) $\frac{3x}{ay^2}$.

5. (i) $35xy$ (ii) $12a^3b$, $2a$ (iii) a^3b^2
(iv) ab^2 (v) amn^3.

6. (i) $\frac{5x}{6ax}$, $\frac{8a}{6ax}$, $\frac{18c}{6ax}$ (ii) $\frac{9xy^2}{12x^2y^2}$, $\frac{2x^3}{12x^2y^2}$, $\frac{4y^3}{12x^2y^2}$
(iii) $\frac{6cx^2}{12abc}$, $\frac{4by^2}{12abc}$, $\frac{3az^2}{12abc}$
(iv) $\frac{60b^3x^2y}{60a^3b^3}$, $\frac{40ab^2x^3}{60a^3b^3}$, $\frac{45a^2by^3}{60a^3b^3}$, $\frac{48xy^2a^3}{60a^3b^3}$.

7. (i) $\frac{17x}{12}$ (ii) $\frac{3a+2b}{2x}$ (iii) $\frac{5}{6x}$ (iv) $\frac{x^2+y^2}{xy}$
(v) $\frac{y+x}{y}$ (vi) $\frac{1+3x}{x}$ (vii) $\frac{2}{3x}$ (viii) $\frac{a}{3b}$
(ix) $\frac{b-a}{b}$ (x) $\frac{xy-1}{xy}$ (xi) $\frac{2z-x^2y^2}{xyz}$ (xii) $\frac{13x}{12}$
(xiii) $\frac{bcx+acy+abz}{abc}$ (xiv) $\frac{61x}{60}$ (xv) $\frac{13a}{12x}$
(xvi) $\frac{8ayz+5bxz+9cxy}{12xyz}$ (xvii) $\frac{15x^2-36xy+20y^2}{180}$
(xviii) $\frac{a^2+b^2+c^2}{abc}$ (xix) 0.

8. (i) $\frac{ab}{c^2}$ (ii) z^2 (iii) 1 (iv) abc (v) $\frac{6c}{25a}$ (vi) $\frac{8x}{9z}$.

9. $\frac{1}{25}$ **10.** (i) $\frac{x}{x+y}$ (ii) $\frac{x+y}{y}$.

EXERCISE 7. (page 15)

1. $x+y=y+x$.
2. $x^2 \times x = x^3$.
3. $2n+1$ is an odd number
4. The sum of the first n odd numbers is n^2.
5. $4x+x=5x$.
6. $A = l \times b$.
7. $10n+5$
8. $10n+3$.
9. $A = \pi r^2$.
10. $\frac{1}{2}(a+b) \times h$.

EXERCISE 8. (page 17)

1. $100x+y$ paise.
2. (i) $\frac{x}{5}$ hrs. (ii) $\frac{x}{y}$ hrs.
3. (i) $\frac{m}{t}$ km.p.h. (ii) $\frac{250m}{t}$ metres.
4. $\frac{50}{3}m$ metres per minute.
5. $\frac{60x}{s}$ days.
6. $\frac{4x}{3}$ km.
7. $\frac{ax}{15}$ metres.
8. $\frac{l+m+n}{a+b+c}$ km per hour.
9. $\frac{29x}{25}$.
10. $\frac{100x}{y}\%$
11. (i) $\frac{100}{a}\%$ (ii) $\frac{100a}{b}\%$
12. Rs. $15r$.
13. Rs. $\frac{5pr}{100}$.
14. Rs. $\frac{plr}{100}$.
15. $\frac{A}{8}$ metres.

EXERCISE 9. (page 19)

1. Re. $\frac{n-m}{m}$
2. Rs. $\frac{19r}{20}$
3. x paise.
4. $\frac{200}{3(100-p)}$ kg/Re.
5. Rs. $3xy$.
6. $\frac{xy}{10}$.
7. abc *cu. cm.*; $2(ab+bc+ca)$ sq. cm.
8. $\frac{x}{lb}$ cm.
9. hk.
10. t cm.

EXERCISE 10. (page 22)

1. Rs. 4 gain & Rs. 4 loss; Rs. 10 credit & Rs. 10 debit; 3 boys joining the class and 3 boys leaving.

2. Going 2 km east and 2 km west ; rising $3m$ above the ground and going $3m$ down a pit ; going $1m$ away from and towards a fixed point.

3. Rs. 12 given.

4. 500 metres below sea-level.

5. Departure of 5 men.

6. (i) +Rs. 50. (ii) −Rs. 20.
(iii) −Rs. 100 (iv) −Rs. 40 (v) +Rs. 15
(vi) +25 metres (vii) +Rs. 5 (viii) −200 m
(ix) +13 km (x) +15 km (xi) −17 km
(xii) −20 km (xiii) +8 men (xiv) −10 men
(xv) −16 metres (xvi) +24 metres.

7. (i) +Rs. 12 (ii) −Rs. 9 (iii) +Rs. 25 (iv) −Rs. 15.

8. (i) −Rs. 12 (ii) +Rs. 9 (iii) −Rs. 25 (iv) +Rs. 15.

9. (i) +5 km (ii) −7 km (iii) +3 km (iv) −8 km.

10. (i) −5 km (ii) +7 km (iii) −3 km (iv) +8 km.

EXERCISE 11. (page 24)

2. (i) 6 divisions (ii) 13 divisions (iii) 20 divisions
(iv) 3 divisions (v) 10 divisions (vi) 7 divisions
(vii) 9 divisions (viii) 8 divisions.

3. 6 divisions towards west.

4. 3 divisions towards east.

5. 3 divisions towards east.

6. 13 divisions towards east.

7. 5 divisions towards west, (x^1).

8. 14 divisions towards west.

9. 12 divisions towards east.

10. 6 divisions towards west.

11. $+60-25-15+12-16+24+40$

12. $(+850m)+(-350m)+(+260m)+(-145m)$

13. $(+8x$ metres$)+(-5x$ m.$)+(+11x$ m.$)+(-6x$ m.$)+(+13x$ m.$)+(-7x$ m.$)+(+15x$ m.$)$

14. $(-10a$ m.$)+(+14a$ m.$)+(-8a$ m.$)+(+16a$ m.$)$

15. $-5°$, $-2°$, $+4°$, $+8°$.

16. (i) +6 (ii) +2 (iii) −2-30 (iv) +5-20.

17. (i) $+46°$ (ii) $-8°$ (iii) $0°$ (iv) $-32°$.

EXERCISE 12. (page 28)

1. $+19, -19, -5, +5, -3, +3, 0, 0, +4, -4.$
2. $-3x.$ **3.** $-2ax.$ **4.** $+10axy.$ **5.** $+6ab.$ **6.** $-10xyz.$
7. $16x^2y.$ **8.** $-4y^2z.$ **9.** $10m^2n.$ **10.** 0. **11.** $-2ab.$
12. $14xyz.$ **13.** $-4abx^2.$ **14.** $-18pq^2r.$ **15.** $-a+4b-2c.$
16. $-3x-z.$ **17.** $-4p^2+2pq+9q^2.$ **18.** $3ab+4bc+4ca.$
19. $2x^2yz+4xy^2z+2xyz^2.$ **20.** $12ax^2y-3a^2xy-11axy^2.$

EXERCISE 13. (page 29)

1. $x^2+xy+y^2.$ **2.** $x^2+y^2.$ **3.** $a^2-b^2.$ **4.** $-4p^2+3pq+3q^2.$
5. $9a^3-4a-4.$ **6.** $-6x^2+4xy-5y^2.$ **7.** $-6x-2y-5z.$
8. $-3b^2+b^2c-4c^2b.$ **9.** $a^2x^2-2ax-7.$ **10.** $3a^2-8b^2.$
11. $-2p^2-4pq+11q^2.$ **12.** $4x^3+x^2-2.$
13. $-a+4c-19d+5e+6f.$ **14.** $\frac{4}{3}a^2+\frac{7}{3}ab+\frac{1}{6}b^2.$
15. $-x+12.$ **16.** $-a.$

EXERCISE 14. (page 32)

1. $+5.$ **2.** $+19.$ **3.** $-19.$ **4.** $-5.$ **5.** $-5.$ **6.** 19.
7. $-19.$ **8.** $+5.$ **9.** (i) $-5a.$ (ii) $19a.$ (iii) $-19a.$
(iv) $5a.$ (v) $5a.$ (vi) $19a.$ (vii) $-19a.$ (viii) $-5a.$
10. (i) $-10x.$ (ii) $14m.$ (iii) 0. (iv) 0.
11. (i) $m.$ (ii) $-m.$ (iii) $-m.$ (iv) $m.$
12. (i) $2y.$ (ii) 0. (iii) 0. (iv) $2(x-y).$
13. (i) $2x+2z.$ (ii) $-3ab+7bc-ca.$
(iii) $-6y^2+4xy.$ (iv) $-18+4y^2-3x^2.$
14. $2-m^2-m^3-m^4.$ **15.** $3x^2-4y^2-1.$
16. $-7a^2+3ab-12b^2.$ **17.** $-4x^2y^3-y^2z^2+8z^2x^2.$
18. $p^3-5p^2q+4q^3-6.$ **19.** $a^2+b^2.$
20. $x^2-2xy+y^2.$ **21.** $3a^2-2b^2-5.$
22. $3b^2-a^2-3.$ **23.** $-2a^2-b^2+8.$
24. $2a^2+3b^2-4.$ **25.** $4a^2-3b^2+2.$
26. $-2a^2+b^2+12.$

EXERCISE 15. (page 35)

1. (i) $+5.$ (ii) $-3.$ (iii) $-7.$ (iv) $+6.$
2. (i) $+3m.$ (ii) $-5q.$ (iii) $-2.$ (iv) 0.

3. (i) 0. (ii) 0.
4. (i) -3. (ii) -9. (iii) 9.
5. $-3q-p$. **6.** $2a-b$. **7.** $2p+q$. **8.** $3q-p$.
9. $11k+2l$. **10.** $5a-9b$. **11.** $6l-9m+5n$.
12. $-2c$. **13.** $-4p+3q+1$. **14.** $p+2q+1$.
15. $4a-2b$. **16.** $2a-6$. **17.** $-3x+6y-6z$.
18. 1. **19.** 1. **20.** $2b$.
21. $2y-2z-2x$. **22.** 0. **23.** 0.
24. $-2b$.

EXERCISE 16. (page 36)

1. (i) $+(2a+3b-4c)$ (ii) $-(-2a-3b+4c)$.
2. (i) $+(2a-3b+4c)$. (ii) $-(-2a+3b-4c)$.
3. (i) $+(3x+4y+5z)$. (ii) $-(-3x-4y-5z)$.
4. (i) $+(3x-4y-2z)$. (ii) $-(-3x+4y+2z)$.
5. (i) $ax+(by-c)$. (ii) $ax-(-by+c)$.
6. (i) $x+(-2y+z)$. (ii) $x-(2y-z)$.
7. (i) $3p+(4q-5n)$. (ii) $3p-(-4q+5n)$.
8. (i) $5l+(-4m-n)$. (ii) $5l-(4m+n)$.
9. (i) $-a^2+(m^2+b^2-n^2)$. (ii) $-a^2-(-m^2-b^2+n^2)$.
10. (i) $x^2+(-a^2+bc-b^2-ca)$. (ii) $x^2-(a^2-bc+b^2+ca)$.
11. (i) $(2x^4-5x^4)+(x^3-4x^3)+(3x^2+x^2)$; $-(-2x^4+5x^4)-(-x^3+4x^3)-(-3x^2-x^2)$.
12. $(a^2-ab)+(-b^2+bc)+(c^2-ca)$; $+(a^2-b^2+c^2)-(-bc+ca+ab)$.

EXERCISE 17. (page 38)

1. (i) 35. (ii) -14. (iii) -45. (iv) 32.
(v) -60. (vi) -120. (vii) $40a$. (viii) $-28m$.
(ix) $-32m$. (x) $56m$. (xi) $-72a$. (xii) $-24m$.
2. (i) 0. (ii) 0. (iii) 0. (iv) 0.
(v) 25. (vi) 25. (vii) a^2. (viii) $-b^3$.
(ix) c^4. (x) $-p^5$. (xi) 1. (xii) -1.
4. (i) $+1$. (ii) -1. (iii) $-$, $+$, $+$, $+$.
5. (i) 0. (ii) 0.
6. -49, 63, -21, 3, -3, -1.

7. (i) -59, 121, -11, 1.
(ii) -42, -197, 3, 4.
(iii) 3, -117, -13, -9.

8. (i) $648y^{12}$. (ii) $-4a^{14}b^{7}$.

EXERCISE 18. (page 42)

1. (i) $12a+15b$ (ii) $-15x-21y$
(iii) $14m-21n$ (iv) $-21m+35n$
(v) $12p^2-8pq$ (vi) $-15pq+10q^2$
(vii) $8ac+4bc-12c^2$ (viii) $-10ac+5bc+10c^2$.

2. x^2+5x+6. **3.** $x^2+7x+10$.

4. $ab+b^2+ac+bc$. **5.** $a^2+2ab+b^2$.

6. $6a^2+7ab+2b^2$. **7.** $2x^2+5xy+3y^2$.

8. $a^3b-2a^2b^2+ab^3$. **9.** $-x^6+20x^3-100$.

10. $-3x^3y+7x^2y^2-2xy^3$. **11.** $-x^3-2x^2y-xy^2$.

12. $4x^4-25a^2b^2$. **13.** $9m^2n^2-9m^3n-2mn^3$.

14. $\dfrac{10x^2-121x+12}{8}$. **15.** $-10a^3-\frac{2}{3}a^2-\frac{5}{4}a+1\frac{1}{2}$.

EXERCISE 19. (page 43)

1. $1-a^3$. **2.** $1+a^3$.

3. $a^3+2a^2b+2ab^2+b^3$. **4.** $a^3-2a^2b+2ab^2-b^3$.

5. a^6-b^6. **6.** a^6+b^6.

7. $21a^4-36a^3b-a^2b^2+48ab^3-36b^4$.

8. $12x^4+24x^3y-25x^2y^2+29xy^3-12y^4$.

9. $21x^5-34x^4y+18x^3y^2-18x^2y^3+17xy^4-4y^5$.

10. $x^4+x^2y^2+y^4$.

11. $x^4+y^4+z^4+xy^3+yz^3+zx^3-3x^2yz-3xy^2z-3xyz^2$.

12. $x^4+y^4+z^4+xy^3-yz^3-zx^3+3x^2yz+3xy^2z-3xyz^2$.

13. $1-a^5$. **14.** $a^3+b^3+c^3-3abc$.

15. $x^3-y^3-z^3-3xyz$. **16.** $\dfrac{61a^2b^2+6ab^3+6a^3b-12a^4-12b^4}{144}$.

17. $2a^{11}-a^9-a^8+6a^7-10a^6-4a^5+8a^4-5a^3-2a^2+5a-1$.

18. $a^5-5a^4+10a^3-10a^2+5a-1$; 1.

19. $4a^4-5a^2+1$; 280. **20.** $x^3+y^3+3xy-1$; 0.

EXERCISE 20. (page 44)

1. $1-a^4$. **2.** $a^3+6a^2+11a+6$. **3.** a^3+6a^2-a-30.

4. $x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$.

5. $x^3+6x^2y+11xy^2+6y^3$. **6.** $a^3+3a^2b+3ab^2+b^3$.

7. $a^3+3ab^2-3a^2b-b^3$. **8.** x^6-y^6.

9. $a^8+a^4b^4+b^8$. **10.** $a^{12}-b^{12}$.

11. $2c^2a^2+2b^2c^2+2a^2b^2-a^4-b^4-c^4$.

12. $a^2+b^2+c^2+2ab+2bc+2ca$.

13. $a^2+b^2+c^2-2ab-2bc+2ca$.

14. $4a^2+9b^2+16c^2-12ab-24bc+16ca$.

15. $a^3+b^3+c^3+3ab^2+3ac^2+3bc^2+3ba^2+3ca^2+3cb^2+6abc$.

16. $a^3-b^3+c^3+3ab^2+3ac^2-3bc^2-3a^2b+3a^2c+3b^2c-6abc$.

EXERCISE 21. (page 45)

1. -1. **2.** $+2$. **3.** -4. **4.** $-3y^2$.

5. -1. **6.** 39. **7.** -23. **8.** $-2x^2$.

9. $-47x$. **10.** $-3x$. **11.** $-3y$.

12. $12x^2+4y^2-4xy$.

13. $3(x^2-y^2)$.

14. $4(x^4+2x^2a^2+a^4)$.

15. (i) $a^3+b^3+c^3-3abc$.
(ii) 0. (iii) 0.

16. $6x^4+11x^3+10x^2+7x+2$; 72072.

17. $3x^4+8x^3+14x^2+8x+3$; 39483.

18. $2x^6+x^5+6x^4+14x^3+4x^2+9x+12$; 2174502.

EXERCISE 22. (page 48)

1. $3x^2$. **2.** $-3ab^2$. **3.** $3ab^2$.

4. $\frac{-3a}{2}$. **5.** $3by$. **6.** $-m^2np^2q$.

7. $-7ab^3c^2$. **8.** $\frac{4pq}{3}$. **9.** $bd-xm$.

10. $\frac{3by-2cz+bcyz}{6}$.

11. $ab+bc+ca$.

12. $5a^2x^3-4ax^2-7x$.

13. $-3a^2b^2+2x-abx^2$.

14. $-2a^3+3ab^2+6c^3$.

EXERCISE 23. (page 49)

1. $x+3$.
2. $x+5$.
3. $x-3$.
4. $x-7$.
5. $x-2$.
6. $5x+3$.
7. $3m-2$.
8. $a+3$.
9. $2a-3$.
10. $2p+1$.
11. $m-3$.
12. $3a+2$.
13. $1-x$.
14. a^2-2a+1.
15. $x^2-3xy+y^2$.
16. a^2-ab-b^2.
17. a^2-2a+3.
18. $-2x^2+3x-1$.
19. $1+a$.
20. x^2+x+1.

EXERCISE 24. (page 51)

1. x^2-xy+y^2.
2. $x^4-x^3y+x^2y^2-xy^3+y^4$.
3. $x^4+x^3y+x^2y^2+xy^3+y^4$.
4. $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.
5. x^3-2x^2+2x-1.
6. $x^4+2x^3+3x^2+2x+1$.
7. $x^6-x^3y^3+y^6$.
8. $x^2+y^2+1-xy+x+y$.
9. $x^3-2x^2y+2xy^2-y^3$
10. $x^8+x^6y^2-3x^2y^6-3y^8$.
11. $a^3+\frac{1}{3}a^2b+\frac{1}{9}ab^2+\frac{1}{27}b^3$.
12. $\frac{1}{2}x+\frac{1}{3}y-\frac{1}{6}z$.
13. $x^2+y^2+z^2+1$.
14. $a-b-c$.
15. $x^2+4y^2+9z^2-2xy+6yz+3zx$.
16. $a+2b+3c$.

EXERCISE 25. (page 53)

1. $a+b-1+\dfrac{4}{a+b+1}$.
2. $a+2+\dfrac{5(a+1)}{a^2+5a+6}$.
3. $a^2+4a+1+\dfrac{2a+3}{a^2+3a+1}$
4. $2a^2-ab+b^2+\dfrac{-3b^4}{a^2+ab-2b^2}$.
5. Remainder $= x^4$; complete quotient $1-x+x^2-x^3+\dfrac{x^4}{1+x}$.
6. Remainder $= 135\,a^4$; complete quotient
$$1-5a+15a^2-45a^3+\frac{135a^4}{1+3a}.$$
7. Remainder $= -4a^5+8a^6$; complete quotient
$$1+a-2a^3-4a^4+\frac{-4a^5+8a^6}{1-2a+2a^2}.$$

8. Remainder $= x^4$; complete quotient $1+x+x^2+x^3+\frac{x^4}{1-x}$.

13. $x^4-4x^3+6x^2-4x+1$. **14.** x^2-3x+2.

15. -2. **16.** -2. **17.** $ba-a+4$.

18. $m+4=0$. **19.** -36. **20.** $\frac{5}{m}$

EXERCISE 26. (page 58)

1. 6.	**2.** 7.	**3.** 3.	**4.** 4.
5. 28.	**6.** 18.	**7.** 40.	**8.** 48.
9. 18.	**10.** 10.	**11.** 18.	**12.** 15.
13. 13.	**14.** 17.	**15.** 13.	**16.** 11.
17. 7.	**18.** 9.	**19.** 9.	**20.** 11.
21. 6.	**22.** 6.	**23.** 5.	**24.** 4.
25. 3.	**26.** 3.	**27.** 6.	**28.** 9.
29. 9.	**30.** 7.	**31.** 1.	**32.** $2\frac{2}{5}$.
33. 3.	**34.** $2\frac{1}{2}$.	**35.** $3\frac{1}{3}$.	**36.** $3\frac{1}{3}$.
37. 6.	**38.** 6.	**39.** 4.	**40.** 3.
41. 9.	**42.** $\frac{3}{4}$.	**43.** $1\frac{2}{3}$.	**44.** 3.
45. 13.	**46.** 10.	**47.** 2.	**48.** 3.
49. 3.	**50.** $\frac{2}{3}$.		

EXERCISE 27. (page 61)

1. 2.	**2.** -3.	**3.** -11.	**4.** $-1\frac{1}{3}$.
5. -4.	**6.** 1.	**7.** -7.	**8.** $-\frac{1}{2}$.
9. 2.	**10.** $2\frac{5}{6}$.	**11.** $1\frac{1}{2}$.	**12.** $-\frac{2}{3}$.
13. $-\frac{10}{19}$.	**14.** 5.	**15.** 0.	**16.** $4\frac{2}{5}$.
17. $2\frac{4}{9}$.	**18.** 2.	**19.** $2\frac{1}{4}$.	**20.** 7.
21. $-3\frac{3}{11}$.	**22.** $1\frac{9}{13}$.	**23.** $4\frac{2}{3}$.	**24.** $-5\frac{7}{8}$.
25. $\frac{17}{18}$.	**26.** -2.	**27.** $\frac{2}{7}$.	**28.** $\frac{5}{13}$.
29. $-\frac{1}{19}$.	**30.** $2\frac{3}{7}$.	**31.** 10.	**32.** 1.
33. $8\frac{1}{3}$.	**34.** -1.	**35.** 1.	**36.** $-3\frac{1}{10}$.
37. $3\frac{16}{25}$.	**38.** 14.	**39.** $1\frac{2}{5}$.	

40. (i) -1. (ii) any value of x. (iii) no value of x.

EXERCISE 28. (page 65)

1. 6. **2.** 36. **3.** 3.
4. 24. **5.** 9. **6.** $\frac{5}{6}$.
7. $1\frac{1}{2}$. **8.** $3\frac{3}{7}$. **9.** $-\frac{16}{35}$.
10. $4\frac{1}{4}$. **11.** 3. **12.** $17\frac{1}{7}$.
13. 10. **14.** -11. **15.** 11.
16. 24. **17.** 24. **18.** $54\frac{2}{5}$.
19. $\frac{1}{2}$. **20.** 2. **21.** $3\frac{1}{2}$.
22. $1\frac{2}{5}$. **23.** $-1\frac{31}{95}$. **24.** $3\frac{3}{19}$.
25. $-\frac{5}{24}$. **26.** $1\frac{5}{7}$. **27.** $2\frac{3}{5}$.
28. $1\frac{43}{49}$. **29.** $\frac{15}{184}$. **30.** 24.

EXERCISE 29. (page 67)

1. (i) $b = \frac{2A}{h}$. (ii) $h = \frac{2A}{b}$. **2.** $r = \frac{C}{2\pi}$.

3. (i) $h = \frac{2A}{a+b}$. (ii) $a+b = \frac{2A}{h}$. **4.** $r = \sqrt{\frac{A}{\pi}}$.

5. (i) $r = 100\left(\frac{y}{x}-1\right)$. (ii) 5.

6. (i) $r = \sqrt{\frac{V}{\pi h}}$. (ii) $h = \frac{V}{r^2\pi}$.

7. (i) $f = \frac{v-u}{t}$. (ii) $t = \frac{v-u}{f}$.

8. $a = \sqrt{c^2-b^2}$. **9.** $C = \frac{5}{9}(F-32)$; 10.

10. (i) $h = \frac{3V}{\pi r^2}$. (ii) $r = \sqrt{\frac{3v}{\pi h}}$.

11. $f = \frac{2(s-ut)}{t^2}$.

12. $t = \frac{pv}{10}-273$; 42.

EXERCISE 30. (page 68)

1. $7+a$. **2.** $m+n$. **3.** $q-15$.
4. $p-q$. **5.** $x-7$. **6.** $2m-n$.
7. m. **8.** $m-17$. **9.** $20-m$.

10. $m-n$. **11.** $x+14$. **12.** $25-d$.
13. Rs. $160+x$. **14.** $53-n$. **15.** $m-34$.
16. $p-q$. **17.** 4. **18.** $40-m$.
19. $y-x$. **20.** $60-m$. **21.** $2m-n$.
22. $21-n$. **23.** $2p-q$. **24.** $\frac{28}{n}$.
25. $\frac{n}{m}$. **26.** $\frac{m}{5}$. **27.** $2p$.
28. 6. **29.** $\frac{48}{a}$. **30.** $\frac{n}{m}$.
31. a^3 **32.** $5x+15$. **33.** $7n-12$.
34. $3p+q$. **35.** $6m-n$. **36.** $4x+13$.
37. $2y+5$. **38.** $\frac{1}{6}m+7$. **39.** $\frac{1}{8}x-9$.
40. $\frac{3}{4}y+11$. **41.** $\frac{5}{6}m-13$.
42. (i) 14, 15, 16. (ii) $x-1$, x, $x+1$.
43. (i) 11, 12, 13. (ii) k, $k+1$, $k+2$.
44. (i) 25, 26, 27. (ii) $p-2$, $p-1$, p.
45. $2m+2$, $2m+3$, $2m+4$.
46. $2n+2$, $2n+4$, $2n+6$.
47. $2x-6$, $2x-4$, $2x-2$.
48. $2x-2$, $2x$, $2x+2$.
49. $2x-3$, $2x-1$, $2x+1$.
50. $2m-1$, $2m+1$, $2m+3$.
51. $2n+1$, $2n+3$, $2n+5$.
52. $80+n$. **53.** $10k+4$.
54. $10m+n$. **55.** $10n+m$.
56. $30p+2q$. **57.** (i) $(m-16)$ years. (ii) $(m-n)$ years.
58. (i) 47 years. (ii) $(35+x)$ years. 59. $(y+k)$ years.
60. (i) 30 years. (ii) $17+x$ years.
61. (i) $p+18$ years. (ii) $(p+n)$ years.
62. $(60-k)$ years. **63.** $(n-m)$ years.
64. (i) $(m-5)-k$ years. (ii) $(m-5)+p$ years.
65. $m+34$ years. **66.** (i) $3x$ km. (ii) hx km.
67. (i) $4x$ km. (ii) tx km. **68.** (i) 4 km per hour.
(ii) $\frac{32}{x}$ *km* per hour. (iii) $\frac{m}{t}$ *km* per hour.

69. (i) 10 hours. (ii) $\frac{m}{28}$ hours.

70. (i) $\frac{250}{s}$ hours. (ii) $\frac{m}{s}$ hours.

71. Rs. $(3m+5)$. **72.** Rs. $(7x-9)$.

73. Rs. $(2n+6)$. **74.** (i) Rs. 50 (ii) Rs. $50x$.

75. $\frac{r}{k}$ cows. **76.** 5; $\frac{x}{8}$.

77. Rs. $\frac{5m}{2}$. (ii) Rs. $\frac{km}{100}$. **78.** Rs. $\frac{107}{100}x$.

79. Rs. $\frac{88}{100}\times 3m$. **80.** Rs. $c\left(1+\frac{p}{100}\right)$.

EXERCISE 31. (page) 74

1. 50, 33 **2.** 10.
3. 19. **4.** 30.
5. 170. **6.** 28.
7. 24. **8.** 25.
9. 48 years ; 24 years. **10.** 45 years ; 15 years.
11. 69°, 37°. **12.** 72°, 72°, 36°.
13. 5°, 25°. **14.** *A* Rs. 43; *B* Rs. 53.
15. *A* Rs. 81; *B* Rs. 69. **16.** *A* Rs. 20; *B* Rs. 27; *C* Rs. 34.
17. Rs. 90, Rs. 120, Rs. 130. **18.** Rs. 200, Rs. 160, Rs. 140

EXERCISE 32. (page) 76

1. 60. **2.** 420. 3. 60.
4. 36, 45. **5.** 6 metres. **6.** 120 km.
7. 56 litres. **8.** Rs. 7200 **9.** 12, 28 or 108, 92
10. 566, 112. **11.** 854, 210. **12.** 645, 214.

EXERCISE 33. (page 77)

1. 9 years, 24 years. **2.** 16 years, 36 years.
3. 21 years, 45 years. **4.** 48 years.
5. 28 years, 56 years. **6.** 54 years, 44 years.
7. 8 years, 34 years.

EXERCISE 34. (page 78)

1. 28, 29, 30.
2. 14, 16, 18.
3. 25, 27, 29.
4. 50, 51.
5. 110, 112.
6. 75, 77.
7. 123, 124.
8. 115, 117.

EXERCISE 35. (page 79)

1. 48.
2. 25.
3. 71.
4. 53.
5. 62.
6. 21.
7. 72.
8. 24.

EXERCISE 36. (page 81)

1. $27\frac{6}{7}$ minutes past 8 P.M.
2. $12\frac{1}{2}$ days after the start of first steamer ; 3000 km from P.
3. $1\frac{10}{11}$ hours.
4. 3 km per hour.
5. 20 km per hour.
6. $5\frac{5}{6}$ km.
7. 48 km.
8. *A* meets after walking $21\frac{2}{3}$ km.
9. 30 km per hour, 240 km.
10. *B* overtakes *A* in $10\frac{10}{11}$ minutes and after travelling $10\frac{10}{11}$ km.

EXERCISE 37. (page 83)

1. 32 metres, 48 metres.
2. Tea at Rs. 8 per kg, coffee at Rs. 7.50 per kg.
3. 20 kg. superior, 30 kg. inferior.
4. Rs. 5400.
5. Rs. 1200, Rs. 2000.
6. Rs. 2400.
7. No. of males 3400, No. of females 3200.
8. 40 goats of each kind.

EXERCISE 38. (page 86)

1. 255 metres.
2. 49 metres, 40 metres.
3. 16 metres, 10 metres.
4. 27 metres, 15 metres.
5. 20, 30, 5 and 125.
6. 40, 31, 72, 12.
7. 19, 13, 7, 42.
8. Rupee coins—170, 50 paise coins—310.
9. 25 paise coins—60, 10 paise coins—20.
10. 28 boys, Rs. 125.

SECTIONAL REVISION 1. (page 88)

PAPER 1.

1. $4x-8y$.
2. $-35,\ -11,\ -5$.
3. $x^2+xy-xz-yz$.
5. $\frac{3}{b}$.
6. 48.

PAPER 2.

1. $a^2+b^2+c^2$.
2. 3.
3. $\frac{6a-5b}{3}$.
4. a^6-b^6.
5. $-4\frac{2}{9}$.
6. 15 years, 36 years.

PAPER 3.

1. $6a-2b$ km.
2. $2x+y$.
3. Quotient is $1+3x+4x^2+2x^3$; remainder, $-4x^4-4x^5$.
4. $20a^2-5ab$.
5. $\frac{1}{2}$.
6. $5\frac{1}{8}$ hours after the departure of B; 24 km from the starting point.

PAPER 4.

1. $-a+b+4c$.
2. x^2+3x-1.
3. x^6-1.
4. 1.
6. $\frac{5}{12}$

PAPER 5.

2. (i) $+1$ (ii) $+1$ (iii) $+1$ (iv) -1.
3. Quotient $1-a+a^3-a^4$; remainder, a^6.
4. 5.
5. a^8+a^4+1.
6. 30 km.

PAPER 6.

1. $2ab+4bc+2ca$.
2. $-56+68x-158x^2+107x^3-98x^4+47x^5-15x^6+x^7$.
3. $11am^2$.
4. x^3.
5. 2.
6. 48 km.

PAPER 7.

1. (i) 20, (ii) $9a^2+3a-5$.
2. $8a^2+8b^2-22c^2+86bc-86ca-16ab$.
3. $18x^2+40xy$.
4. $10a^2+2$.
5. -1.
6. Rs. 2300.

PAPER 8.

2. $2p$, $2q$. **3.** $a^4-2a^3+3a^2+5a+11$.
4. 4 km North. **5.** $-7\frac{13}{17}$. **6.** 24.

PAPER 9.

1. $11+7x-10x^2+x^3+2x^4-4x^5$. **2.** 25, 121, 361, 841 etc.
3. (i) $r=\sqrt{\frac{3v}{\pi h}}$ (ii) $h=\frac{3v}{\pi r^2}$. **4.** $13x^2$.
5. $-\frac{5}{6}$. **6.** 31m, 20m.

OBJECTIVE TYPE ITEMS 1. (page 91)

1. (i) terms. (ii) -7. (iii) the *index*. (iv) Root.
(v) Conditional. (vi) Ascending. (vii) Odd.

2. (i) *A*. variables (ii) *C*. $\frac{2}{2-m}$.

(iii) *B* Unity in all cases.

3. *B*. $a-1$, a, $a+1$.

EXERCISE 39. (page 93)

1. $4x^2+20xy+25y^2$. **2.** $9a^2+42ab+49b^2$.
3. $25x^2+30x+9$. **4.** $9+24p+16p^2$.
5. $9a^2b^2+24ab^3+16b^4$. **6.** $64a^4+48a^3b+9a^2b^2$.
7. $81a^2x^2+72axy^2+16y^4$. **8.** $m^4+2m^2n^2+n^4$.
9. $p^2x^2+2pqxy+q^2y^2$. **10.** $p^6+2p^4q^2+p^2q^4$.
11. $4a^4b^2+12a^3b^3+9a^2b^4$. **12.** $a^2-2ab+b^2$.
13. $56xy$. **14.** $25n^2$. **15.** $49q^2$.
16. $198a$. **17.** 9. **18.** $16p^2$.

EXERCISE 40. (page 94)

1. $9a^2-30ab+25b^2$. **2.** $25m^2-30m+9$.
3. $36p^2-60p+25$. **4.** $49x^2-14x+1$.
5. $16-56p+49p^2$. **6.** $a^2+2ab+b^2$.
7. $16m^4-40m^2n^2+25n^4$. **8.** $a^2x^2-2abxy+b^2y^2$.
9. $4a^4x^2-12a^2b^2xy+9b^4y^2$. **10.** $49a^4-56a^2b^2+16b^4$.

11. $4a^6b^2-4a^3b^4+b^6$.
12. $49p^6q^4-112p^5q^5+64p^4q^6$.
13. $24ab$.
14. $16y^2$.
15. $16q^2$.
16. $154m$.
17. $36q^2$.
18. $25x^2$.

EXERCISE 41. (page 95)

1. 91204.
2. 164025.
3. 494209.
4. 649636.
5. 373321.
6. 1002001.
7. 39204.
8. 60025.
9. 156816.
10. 19044.
11. 247009.
12. 358801.

EXERCISE 42. (page 96)

1. $9x^2-25$.
2. $a^2m^2-b^2n^2$.
3. $a^2x^2-b^4$.
4. $p^2x^2-q^2y^2$.
5. $16a^2-25b^2$.
6. $9x^2y^2-49x^2z^2$.
7. a^4-b^4.
8. a^4-1.
9. a^8-b^8.
10. $x^2-y^2-z^2+2yz$.
11. $x^2-6xy+9y^2-16z^2$.
12. $x^4+x^2y^2+y^4$.
13. x^4+x^2+1.
14. x^8+x^4+1.
15. $3a^2-6ab+8ac+2bc-3c^2$.
16. $4a^3b+4ab^3$.
17. $4ab-4ac+4ad$.
18. $24ab-32ac+40ad$.

EXERCISE 43. (page 97)

1. 369.
2. 185.
3. 289.
4. 305.
5. 5445.
6. 7815.
7. 6460.
8. 8547.
9. $4ab$.
10. $13a^2-24ab+13b^2$.
11. $4pqxy$.
12. $3m^2x^2-10mnxy-8n^2y^2$.
13. $17a^2b^2-4a^4-12ab^3$.

EXERCISE 44. (page 98)

1. $x^2+8x+15$.
2. $x^2+10x+24$.
3. $x^2+12x+35$.
4. $x^2+16x+63$.
5. $m^2-3m-40$.
6. $m^2-4m-77$.
7. $p^2-5p-126$.
8. $a^2-9a-220$.
9. $x^2+8x-105$.
10. $x^2+4x-96$.
11. $x^2+11x-102$.
12. $x^2+8x-240$.
13. $x^2-x-110$.
14. $x^2+5x-104$.
15. $x^2+7x-144$.
16. $x^2+5x-84$.

17. $x^2-10x-119$.
18. $x^2-12x-45$.
19. $x^2-23x+90$.
20. $x^2-12x-133$.
21. $x^2-16x+63$.
22. $x^2-27x+180$.
23. $x^2-23x+130$.
24. $x^2-23x+126$.
25. $4x^2+16x+15$.
26. $9m^2+9m-10$.
27. $1-x-6x^2$.
28. $9-3x-20x^2$.

EXERCISE 45. (page 98)

1. $x^3+6x^2+11x+6$.
2. $x^3+9x^2+26x+24$.
3. $x^3+15x^2+74x+120$.
4. $x^3+2x^2-23x-60$.
5. $x^3+2x^2-43x-140$.
6. $x^3+3x^2-18x-40$.
7. $x^3-6x^2-7x+60$.
8. $x^3-5x^2-18x+72$.
9. $x^3-12x^2+47x-60$.
10. $x^3-12x^2+39x-28$.
11. $x^3-10x^2+31x-30$.
12. $x^3-18x^2+104x-192$.
13. $8x^3+36x^2+46x+15$.
14. $27x^3-63x^2+42x-8$.
15. $a^3x^3+11a^2x^2+36ax+36$.
16. $m^3x^3-8m^2x^2+11mx+20$.
17. $(x+p)(x+q)(x+r)=x^3+(p+q+r)x^2+(pq+qr+rp)x+pqr$.
18. $(p+a)(p+2b)(p-3c)=p^3+(a+2b-3c)p^2+(2ab-6bc-3ac)p-6abc$.
19. $(m+3)(m-4)(m+5)=m^3+4m^2-17m-60$.

EXERCISE 46. (page 100)

1. $8x^3+12x^2y+6xy^2+y^3$.
2. $x^3+9x^2y+27xy^2+27y^3$.
3. $8a^3+36a^2b+54ab^2+27b^3$.
4. $a^3x^3+3a^2x^2by+3axb^2y^2+b^3y^3$.
5. $1+9x+27x^2+27x^3$.
6. $8+60x+150x^2+125x^3$.
7. $27x^6+54x^4+36x^2+8$.
8. $a^6+3a^4b^2+3a^2b^4+b^6$.
9. $27a^6+54a^4b^2+36a^2b^4+8b^6$.
10. $(4m+2n)^3 = (4m)^3+3(4m)^2(2n)+3(4m)(2n)^2+(2n)^3$.
11. $(2p+3q)^3 = (2p)^3+3(2p)^2(3q)+3(2p)(3q)^2+(3q)^3$.
12. $(5p+2q)^3 = (5p)^3+3(5p)^2(2q)+3(5p)(2q)^2+(2q)^3$.
13. $(2a+3b)^3 = (2a)^3+(3b)^3+3(6ab)(2a+3b)$.

EXERCISE 47. (page 101)

1. $8x^3-12x^2+6x-1$.
2. $27-54x+36x^2-8x^3$.
3. $27a^3-108a^2b+144ab^2-64b^3$.
4. $64x^3-144x^2y+108xy^2-27y^3$.
5. $125x^3-300x^2y+240xy^2-64y^3$.
6. $64m^6-144m^4n+108m^2n^2-27n^3$.
7. $8a^3x^3-36a^2x^2by+54axb^2y^2-27b^3y^3$.

8. $1-3a^2+3a^4-a^6$.
9. $(2x-3y)^3 = (2x)^3-3(2x)^2(3y)+3(2x)(3y)^2-(3y)^3$.
10. $(4m-5x)^3 = (4m)^3-3(4m)^2(5x)+3(4m)(5x)^2-(5x)^3$.
11. $(4a-6b)^3 = (4a)^3-3(4a)^2(6b)+3(4a)(6b)^2-(6b)^3$.
12. $(2p-3q)^3 = (2p)^3-(3q)^3-3(2p)(3q)(2p-3q)$.
13. $27(m^3-12m^2n+48mn^2-64n^3)$.
14. $p^3+15p^2q+75pq^2+125q^3$.
15. $12x^2+24x+28$. **16.** $96x^2-96x+152$.

EXERCISE 48. (page 102)

1. 35. **2.** 91. **3.** 243.
4. $343a^3+294a^2b+84ab^2+8b^3$.
5. $125(x^3+3x^2y+3xy^2+y^3)$.
6. $2a^3+9a^2+15a+9$.
7. $24a^2+48a+26$.
8. 189. **9.** 335. **10.** 448.

EXERCISE 49. (page 102)

1. x^3+27. **2.** x^3+125. **3.** x^3+343.
4. x^3+1331. **5.** $125a^3+27$. **6.** $27m^3+125n^3$.
7. $x^3y^3+64z^3$. **8.** $64x^3+729y^3$. **9.** $a^3+b^3c^3$.
10. a^6+64b^6. **11.** x^6+1. **12.** 98.
13. $9(x^3+y^3)$. **14.** $37(x^3-y^3)$. **15.** 56.

EXERCISE 50. (page 103)

1. x^3-8. **2.** x^3-216.
3. x^3-512. **4.** x^3-1728.
5. $27x^3-8$. **6.** $125x^3-27y^3$.
7. $p^3q^3-8r^3$. **8.** $a^6-b^3c^3$.
9. $19m^3-37n^3$. **10.** $37(p^3+q^3)$. **11.** -189.
12. $(a)^3+(3b)^3 = (a+3b)\{(a)^2-(a)(3b)+(3b)^2\}$.
13. $(2x)^3-(5y)^3 = (2x-5y)\{(2x)^2+(2x)(5y)+(5y)^2\}$.
14. $(4p)^3+(3q)^3 = (4p+3q)\{(4p)^2-(4p)(3q)+(3q)^2\}$.
15. $(7a)^3-(6b)^3 = (7a-6b)\{49a^2+(7a)(6b)+36b^2\}$.

EXERCISE 51. (page 105)

1. $a^2+b^2+c^2+2ab-2bc-2ca$.
2. $a^2+b^2+c^2-2ab+2bc-2ca$.
3. $4a^2+b^2+c^2-4ab+2bc-4ca$.

4. $4a^2+b^2+c^2+4ab-2bc-4ca$.
5. $4a^2+b^2+9c^2-4ab-6bc+12ca$.
6. $25x^2+4y^2+9z^2-20xy-12yz+30zx$.
7. $4x^2+9y^2+1-12xy+4x-6y$.
8. $x^4+3x^2y^2+y^4-2x^3y-2xy^3$.
9. $p^2x^4+q^2x^2y^2+r^2y^4-2pqx^3y-2qrxy^3+2prx^2y^2$.
10. $4a^2+b^2+9c^2+d^2+4ab+12ac+4ad+6bc+2bd+6cd$.
11. $a^2+4b^2+9c^2+16d^2-4ab-6ac+8ad+12bc-16bd-24cd$.
12. $4p^2+9q^2+r^2+4s^2+t^2-12pq+4pr-8ps+4pt-6qr+12qs-6qt-4rs+2rt-4st$.
13. $9x^2+16y^2+4z^2+w^2-24xy+12xz-6xw-16yz+8yw-4zw$.
14. $3a^2-6ab+8ac+2bc-3c^2$.
15. $2a^2+4ab+8ac+2bc-b^2-c^2$.
16. $4a^2+4b^2-2c^2+10ab+6bc-6ca$.
17. $2(a^2+b^2+c^2+d^2+2ac+2bd)$.
18. $2(a^2+b^2+c^2+d^2-2ac-2ad+2cd)$.
19. $2a^2-b^2-c^2-d^2-4ab+4ac-4ad+2bc-2bd+2cd$.
20. 361. **21.** 841. **22.** 20.
23. 52. **24.** 20.

EXERCISE 52. (page 106)

1. 169. **2.** 324. **3.** 4.
4. 81. **5.** 28. **6.** 105.
7. 3. **8.** 61. **9.** 3.
10. 3.

EXERCISE 53. (page 107)

1. 23; 527. **2.** 11; 119. **3.** p^4+4p^2+2.
4. 110. **5.** 14. **6.** m^3-3m.
7. a^3+3a.

EXERCISE 54. (page 108)

1. $x^3+y^3-1+3xy$.
2. $x^3-y^3-1-3xy$.
3. $8x^3+y^3+1-6xy$.
4. $x^3+8y^3-z^3+6xyz$.
5. $x^3+8y^3-27z^3+18xyz$.
6. $8x^3-y^3-27z^3-18xyz$.
7. $(9x^2+y^2+z^2+3xy-yz+3zx)$.
8. $(5x+3y-z)$.
9. -572.
10. 63.
11. 740.

EXERCISE 55. (page 109)

1. $(3x+3y-z)^2$.
2. $(2x+5y+z)^2$.
3. $(x-y+z)^2$.
4. $(x-2y+3z)^2$.
7. 124.
8. 35,984.

EXERCISE 56. (page 111)

1. $m(a+b)$.
2. $x(x+2)$.
3. $mn(m+n)$.
4. $3(x^2-2)$.
5. $bc(a-d)$.
6. $a(p+q-r)$.
7. $x(ax^2+bx+c)$.
8. $a^2x^2(x-2a+1)$.
9. $2a^3b(2a^2-3a+4b^2)$.
10. $axy(x^2y+ay^2+a^2x)$.
11. $(a+b)(c+d)$.
12. $(x-1)(7m+2)$.
13. $(x^2+x+1)(2a+3b)$.
14. $(ax+b)(3p+q+2r)$.
15. $(pq+r)(m^2+mn+n^2)$.
16. $(x-y)(a-b+c+d)$.
17. $x(x+7)(x^2+7x-25)$.
18. $5(p^2-qr)^2(3p^2-3qr-2)$.
19. $2(x-y)(3p+q)$.
20. $(x+y)(x^2+y^2-xy)$.

EXERCISE 57. (page 112)

1. $(x+y)(a+5)$.
2. $(a-b)(x-y)$.
3. $(m+3)(x-y)$.
4. $(x+2)(y+3)$.
5. $(x-3a)(x+b)$.
6. $(a^2+9)(a+3)$.
7. $(3p-5)(p^2+2)$.
8. $(2p-3q)(3p+2r)$.
9. $(p+1)(q+1)$.
10. $(a+1)^2(a-1)$.
11. $(a^2+1)(2a-5)$.
12. $(a^2+1)(5a-2)$.
13. $(x-y)(x-3)$.
14. $(2a+b)(3a-c)$.
15. $(a^3+2)(2a-1)$.
16. $(y-1)(y^2-a+1)$.
17. $(x+5)(11x^2+7)$.
18. $(ax+b)(x^2+1)$.
19. $(2x-3)(3y-2)$.
20. $(x-b)(x-2a)$.
21. $(ax-1)(x+1)$.
22. $(bx+y)(ax-1)$.
23. $(a-2b)(a-2b+3)$.
24. $(bc+a)(ac+b)$.
25. $(ax-by)(bx-ay)$.
26. $(a+1)(a^2x-ay-z)$.
27. $(x-1)(x+1)(a+bx+cx^2)$.
28. $(x-1)(x+1)(kx^2+lx+m)$.

EXERCISE 58. (page 113)

1. $(2a+7)^2$.
2. $(5p-3)^2$.
3. $(5a+4b)^2$.
4. $(7m-4n)^2$.
5. $(6a+5b)^2$.
6. $(6a-7b)^2$.
7. $(7a+3b)^2$.
8. $(5x-6y)^2$.

9. $(4x+7y)^2$.
10. $(9a-7b)^2$.
11. $(9x+5y)^2$.
12. $(3m-11n)^2$.
13. $(8x+y)^2$.
14. $(4p-6q)^2 = 4(2p-3q)^2$.
15. $9(x+3y)^2$.
16. $(5m-7n)^2$.
17. $\left(x+\frac{1}{x}\right)^2$.
18. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$.
19. $(ax+3by)^2$.
20. $(4ab-c^2)^2$.
21. $(2a+3b+c+d)^2$.
22. $(3p-q-r-s)^2$.

EXERCISE 59. (page 114)

1. $(4x-3)(4x+3)$.
2. $(9a-5b)(9a+5b)$.
3. $(1-6x)(1+6x)$.
4. $(8a-1)(8a+1)$.
5. $(3x+10)(3x-10)$.
6. $(7-5a)(7+5a)$.
7. $(11-c)(11+c)$.
8. $(1-7ac)(1+7ac)$.
9. $(a^2-5)(a^2+5)$.
10. $(3x-1)(3x+1)(9x^2+1)$.
11. $x(2x+3)(2x-3)(4x^2+9)$.
12. $(1-2x)(1+2x)(1+4x^2)$.
13. $a^2(1+3a)(1-3a)(1+9a^2)$.
14. $x^3(6x^2-5a^2)(6x^2+5a^2)$.
15. $3x^5(8x^2-9y^2)(8x^2+9y^2)$
16. $(a+5b+7c)(a+5b-7c)$.
17. $(x-2y+3z)(x+2y-3z)$.
18. $4ab$.
19. $(x-3y)(5x-y)$.
20. $(x-7y)(7x-y)$.
21. $(13a+b)(a+13b)$.
22. $4a(b-c)$.
23. $8ax(x^2+a^2)$.
24. $5n(12m-13n)$
25. $12(5a-1)(a+2)$.
26. $(19m+13n)(23n-m)$.
27. $(10a+9b-7c)(15b-25c-2a)$.

EXERCISE 60 (page 115).

1. $(m+1)(m^2-m+1)$.
2. $(p+4)(p^2-4p+16)$.
3. $(pq+6)(p^2q^2-6pq+36)$.
4. $8(2a+5)(4a^2-10a+25)$.
5. $(5x^2+4y)(25x^4-20x^2y+16y^2)$.
6. $(4mn+1)(16m^2n^2-4mn+1)$.
7. $(3m^2+5n^2)(9m^4-15m^2n^2+25n^4)$.
8. $ab(7a+4b)(49a^2-28ab+16b^2)$.
9. $(2a-3b+3c)(4a^2+9b^2+9c^2-12ab-6ac+9bc)$.
10. $(a-2b+1)(a^2+4b^2+1-4ab-a+2b)$.
11. $(x-y-1)(x^2-2xy+y^2+x-y+1)$.
12. $(18x-5y)(156x^2+240xy+277y^2)$.
13. $(3y-8x)(124x^2-138xy+39y^2)$.
14. $-3p(19p^2-45pq+27q^2)$.

EXERCISE 61. (page 116)

1. $(1-a)(1+a+a^2)$.
2. $(2-x)(4+2x+x^2)$.
3. $(5a-2)(25a^2+10a+4)$.
4. $27(3x-2)(9x^2+6x+4)$.
5. $8(x-5b)(x^2+5bx+25b^2)$.
6. $(9xy-8z)(81x^2y^2+72xyz+64z^2)$.
7. $(a^2-2b^2)(a^4+2a^2b^2+4b^4)$.
8. $(a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6)$.
9. $(a^4-2b^2c^2)(a^8+2a^4b^2c^2+4b^4c^4)$.
10. $(3x+2y-z)(9x^2+4y^2+z^2+12xy+3xz+2yz)$.
11. $(a-b)(7a^2+13ab+7b^2)$.
12. $2(b-2a)(151a^2-466ab+364b^2)$.
13. $(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
14. $(a-b)(a+b)(a^2+b^2)(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.
15. $(2a^2+1)(2a^2-1)(4a^4-2a^2+1)(4a^4+2a^2+1)$.
16. $(3a-b)(3a+b)(9a^2+3ab+b^2)(9a^2-3ab+b^2)$.
17. $a(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$.
18. $3x^7(x+2y)(x-2y)(x^2-2xy+4y^2)(x^2+2xy+4y^2)$.

EXERCISE 62. (page 119)

1. $(x+6)(x+7)$.
2. $(a-13)(a-5)$.
3. $(x+13)(x+5)$.
4. $(p-6)(p-9)$.
5. $(x+6)(x+12)$.
6. $(x+7)(x+13)$.
7. $(x-13)(x-9)$.
8. $(x+8)(x+13)$.
9. $(a+13)(a-8)$.
10. $(x+12)(x+9)$.
11. $(a+12)(a-9)$.
12. $(x+12)(x+15)$.
13. $(p+13)(p-12)$.
14. $(a-15)(a+12)$.
15. $(a-13)(a+11)$.
16. $(a-16)(a+15)$.
17. $(x+15)(x-7)$.
18. $(a-30)(a+4)$.
19. $(1-15x)(1-4x)$.
20. $(1-51a)(1+2a)$.
21. $(x+14)(x+7)$.
22. $(a-8)(a-9)$.
23. $(p-16)(p+11)$.
24. $(m+13)(m-7)$.
25. $(p-6)(p-4)$.
26. $(x+10)(x-3)$.
27. $(m+12)(m-3)$.
28. $(x-16)(x+5)$.
29. $(a-14)(a+4)$.
30. $(m+12)(m-7)$.
31. $(x-12)(x-8)$.
32. $(x-12)(x+6)$.
33. $(x-21)(x-4)$.
34. $(x+21)(x-4)$.
35. $(a-15)(a+6)$.
36. $(x-24)(x+2)$.
37. $(x-7)(x+6)$.
38. $(a+9)(a-8)$.
39. $(1-24a)(1+5a)$.
40. $(1+26x)(1-3x)$.

EXERCISE 63. (page 120)

1. $(p-10q)(p-2q)$.
2. $(m-8n)(m-4n)$.
3. $(a+6b)(a-5b)$.
4. $(a-9b)(a+5b)$.
5. $(p-16q)(p+3q)$.
6. $(a+12b)(a+8b)$.
7. $(m+7n)(m-4n)$.
8. $(x-16y)(x+5y)$.
9. $(a+2)(a-2)(a^2+3)$.
10. $(x-2)(x+2)(x^2+7)$.
11. $(x+5)(x-5)(x^2+7)$.
12. $(a^2-17)(a^2-8)$.
13. $(x+3y)(x-3y)(x^2+7y^2)$.
14. $(a-1)(a^2+a+1)(a^3+3)$.
15. $(p-1)(p+3)(p^2+p+1)(p^2-3p+9)$.
16. $(x^3-12)(x^3-15)$.
17. $(a^2-2a-6)(a^2-2a+5)$.
18. $(x^2+x-7)(x^2+x+5)$.
19. $(m^2-3m-6)(m^2-3m-2)$.
20. $(p^2-5p+9)(p^2-5p-4)$.
21. $(a-2)^2(a^2-4a+12)$.
22. $(x^2-6x-15)(x^2-6x+12)$.
23. $(m+3)(m+4)(m^2+7m-13)$.
24. $(p^2+4p+14)(p^2+4p+7)$.
25. $(x-3)(x-5)(x^2-8x-16)$.

EXERCISE 64. (page 121)

1. $2(2x-y)(5x-3y)$.
2. $(7y-4x)(4x-y)$.
3. $(a+9b)(6a+19b)$.
4. $-2(5m+3n)(m+5n)$.
5. $10a(7b-a)$.
6. $-(3m+17n)(9m+7n)$.
7. $(3a-8b)(13b-4a)$.
8. $(a-b)^2(a+2b)(a-4b)$.
9. $7b^2(9a^2-11b^2)$.
10. $(a-3b)(a-4b)^2(a-6b)$.

11. $\frac{x+2}{x}$.
12. $\frac{x+1}{x+4}$.
13. $\frac{x-4}{x-2}$.
14. $x+3$.
15. $m = 9$.
16. $p = -2$.
17. $c = 20$.

EXERCISE 65. (page 122)

1. $(2x+7)(x+2)$.
2. $(3x-1)(x+5)$.
3. $(5x+3)(x+1)$.
4. $(3x-2)(2x+3)$.
5. $2(3x+1)(x-1)$.
6. $(4x-1)(2x-3)$.
7. $(4x+3)(2x-1)$.
8. $(3m-2)(4m+5)$.
9. $(14a+15)(a+1)$.
10. $(10p-3)(2p+5)$.
11. $(1-p)(3-2p)$.
12. $(1+5m)(2-3m)$.
13. $(3a-4b)(3a-b)$.
14. $(m+3n)(2m-7n)$.
15. $(p-q)(4p+9q)$.
16. $(x-5y)(15x-2y)$.
17. $(2a+5b)(6a-b)$.
18. $(2x-7y)(5x-3y)$.

19. $\frac{x-2}{x-3}$.
20. $\frac{2x-5}{3x+5}$.

21. $(x+y-2z)(2x+2y+z)$.
22. $(2a^2+2b^2+ab)(a+b)^2$.
23. $(x-1)(x+2)(x-3)(x+4)$.
24. $(2a+3b+8)(4a-5b-6)$.
25. $(2x^2-5)(3x^2+4)$.
26. $(2x^2-3)(3x^2+1)$.
27. $(x^3-2)(5x^3+3)$.
28. $(3x-y)(3x+y)(2x^2+3y^2)$.
29. $(x-2)(x^2+2x+4)(2x-1)(4x^2+2x+1)$.
30. $(x^2-2y^2)(2x^2-y^2)(x^2+2y^2)(2x^2+y^2)$.
31. $(5x-3y)(x-2y)$.
32. $(x-3y)(5x+2y)$.
33. $(x-3)(7x-4)$.
34. $(7x+3)(x-4)$.
35. $xy(xy-5)(xy-4)$.
36. $a(2a-3b)(4a+5b)$.
37. $(a-9)(a+6)$.
38. $(x+1)(x-2)^2(x-5)$.
39. $(x^2+x-5)(x^2+x-21)$.
40. $(x+4)^2(x^2+8x+6)$.
41. $\left(\frac{x}{2}+5\right)\left(\frac{x}{2}-1\right)$.

EXERCISE 66. (page 124)

1. $(a-2b+3c)(a^2+4b^2+9c^2+2ab+6bc-3ca)$.
2. $(x-y-z)(x^2+y^2+z^2+xy-yz+zx)$.
3. $(a-b-1)(a^2+b^2+1+ab-b+a)$.
4. $(x+y+1)(x^2+y^2-xy-x-y+1)$.
5. $(a+b-6)(a^2+b^2-ab+6a+6b+36)$.
6. $(x-2y-4)(x^2+4y^2+2xy+4x-8y+16)$.
7. $(4a-3b+1)(16a^2+9b^2+12ab-4a+3b+1)$.
8. $(3x-5y-4)(9x^2+125y^2+15xy+12x-20y+16)$.
9. $(2a-3b-c)(4a^2+9b^2+c^2+6ab-3bc+2ca)$.
10. $\left(x-\frac{1}{x}-3\right)\left(x^2+\frac{1}{x^2}+3x-\frac{3}{x}+10\right)$.
11. $(a^2+2a-4)(a^4-2a^3+8a^2+8a+16)$.
12. $2y(3x^2+y^2+3z^2+3xy+3yz+3zx)$.
13. $-2(b-c)(3a^2+b^2+c^2-3ab-3ac+bc)$.
14. $9(x+2)$.
15. $4(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.

EXERCISE 67. (page 125)

1. $(x^2+5x+7)(x^2+5x+3)$.
2. $(x^2+2x-7)(x^2+2x-4)$.
3. $(x^2-3x-16)(x^2-3x-12)$.
4. $(x+5)(x+6)(x^2+11x+8)$.
5. $(x^2-11x+12)(x^2-4x+12)$.
6. $(x^2-3x-16)^2$.
7. $(x-3)(2x+3)(2x^2-3x+7)$.
8. $(x+1)(3x-7)(3x^2-4x+3)$.
9. $8(x^2+x-7)(2x^2+2x-5)$.
10. $3(3x^2+x-1)(9x^2+3x+1)$.

11. $(x+2)^2(x^2+4x-6)$. **12.** $(x^2+3x-13)(x^2+3x+5)$.
13. $4(2x-1)(x+4)(2x^2+7x+15)$.
14. $(9x^2-9x-17)(9x^2-9x-5)$.

EXERCISE 68. (page 126)

1. $(x+y+z)(x+y-z)$. **2.** $(4x+2y-3z)(4x-2y+3z)$.
3. $(2x+3y-5z)(2x-3y+5z)$.
4. $(4x+6y+5z)(4x-6y+5z)$.
5. $(6z-5x+7y)(6z-5x-7y)$ or $(5x+7y-6z)(5x-7y-6z)$.
6. $(c+d+a-b)(c+d-a+b)$.
7. $(a-2b+3c-4d)(a-2b-3c+4d)$.
8. $(4a+3b-7c+5d)(4a-3b+7c+5d)$.
9. $(1+2a-3b-5c)(1-2a-3b+5c)$.
10. $(2a+3b-5c-6)(2a-3b+5c-6)$.
11. $(5c-2+a-7b)(5c-2-a+7b)$.
12. $(7a-4b+8c-1)(7a-4b-8c+1)$.

EXERCISE 69. (page 127)

1. $(a^2-2a+3)(a^2+2a+3)$.
2. $(a^2+a+4)(a^2-a+4)$.
3. $(2a^2+2ab+3b^2)(2a^2-2ab+3b^2)$.
4. $(3a^2+5ab+4b^2)(3a^2-5ab+4b^2)$.
5. $(a-3)(a+3)(2a-1)(2a+1)$.
6. $(3a^2+3ab-4b^2)(3a^2-3ab-4b^2)$.
7. $(5a^2-7ab+3b^2)(5a^2+7ab+3b^2)$.
8. $(a^2-5ab-5b^2)(a^2+5ab-5b^2)$.
9. $(a^2-2ab-6b^2)(a^2+2ab-6b^2)$.
10. $(7a^2-2ab-4b^2)(7a^2+2ab-4b^2)$.
11. $(a^2+ab+b^2)(a^2-ab+b^2)$.
12. $(25+10a+2a^2)(25-10a+2a^2)$.
13. $(a^2+2a+2)(a^2-2a+2)$.
14. $(a^2+4a+8)(a^2-4a+8)$.
15. $9(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
16. $(8p^2-12pq+9q^2)(8p^2+12pq+9q^2)$.
17. $(a^2+a+1)(a^2-a+1)$.
18. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$.
19. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.
20. $(5a^2-5ab+6b^2)(5a^2+5ab+6b^2)$.
21. $(3p-r)(3p+r-2q)$. **22.** $(a-5c)(a+5c-6b)$.
23. $(5a-3c)(5a-4b+3c)$. **24.** $(7a-3c)(7a-4b+3c)$.

SELECTED QUESTIONS A. (page 127)

1. $(p+q)(p+q-1)$.
2. $(a+b)(a+b+2c)$.
3. $(2a+3b)(2a+3b-4)$.
4. $8ab(a^2+b^2)$.
5. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$.
6. $(2x+3)(4x^2-6x+9)$.
7. $xy(x+2y)(x^2-2xy+y^2)$.
8. $(5a^2+4b^2)(25a^4-20a^2b^2+16b^4)$.
9. $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$.
10. $(2x^2+a^2)(2x^2-a^2)(4x^4-2x^2a^2+a^4)(4x^4+2x^2a^2+a^4)$.
11. $(2x-3)(2x+3)(x-1)(4x^2+9)(x^2+x+1)$.
12. $(x+35)(x-18)$.
13. $(x+21)(x-19)$.
14. $(x+5)(2x-1)$.
15. $(4x+7)(3x-5)$.
16. $(x+3)(6x+7)$.
17. $(2x-1)(3x-5)$.
18. $(x-2)(2x+3)$.
19. $(x+2)(2x-3)$.
20. $(x+b)(2x-a)$.
21. $(2x+3a)(3x-4a)$.
22. $(5x-2a)(3x-7a)$.
23. $(x-4y)(2x+3y)$.
24. $(4x-3y)(4x-y)$.
25. $(x^2+5x+2)(x^2+5x+8)$.
26. $(x^2-3x+1)^2$.
27. $(x^2-5x-40)(x^2-5x-20)$.
28. $(x^2-14x+18)(x^2-6x+18)$.
29. $(a+b+c)(a-b+c)$.
30. $(p^2+2p+2)(p^2-2p+2)$.
31. $(x^2+4x+8)(x^2-4x+8)$.
32. $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.
33. $(2x^2-2x+1)(2x^2+2x+1)$.
34. $(x^2+3xy+4y^2)(x^2-3xy+4y^2)$.
35. $(9a^2-12ab+8b^2)(9a^2+12ab+8b^2)$.
36. $(y^2+y+1)(y^2-y+1)$.
37. $(a^2+2a+4)(a^2-2a+4)$.
38. $(x^2-3xy+y^2)(x^2+3xy+y^2)$.
39. $(a^2-2ab+3b^2)(a^2+2ab+3b^2)$.
40. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.
41. $(x^2+2x+3)(x^2-2x+3)(x^4-2x^2+9)$.
42. $(ac+ad+bd-bc)(ac-ad+bd+bc)$.
43. $(a+b-c-d)(a-b+c-d)$.

EXERCISE. 70. (page 132)

1. -1.
2. -37.
3. $4a^3-2a^2+5a-3$.
4. $-5m^3-m^2-3m-4$.
5. $-56a^4$.
11. $(x-1)(x^2+2x+3)$.
12. $(x-1)(x-2)(x-3)$.
13. $(x+1)(x+3)(x+5)$.
14. $(x-1)(3x^2+4x+5)$.
15. $(x+1)(4x^2-x+2)$.
16. $(x-1)(x+1)(2x^2+3x+4)$.

17. $(x-1)^2(x^2+4x+2)$.
18. $(x+1)^2(x^2-3x+5)$.
19. $(x-1)(x+1)(x-2)(x+3)$.
20. $(x-1)^2(2x^2+3x-4)$.

EXERCISE 71. (page 132)

1. $p = 3$.
2. $a+b+c+d = 0$.
3. $a = -2,\ b = 5$.
4. -600.
6. $p = \frac{3}{13}$.

EXERCISE 72. (page 135)

1. $(x+2)(x-3)(x+4)$.
2. $(x-4)(x-3)(x+2)$.
3. $(x+2)(2x-3)(3x-1)$.
4. $(x+2)(x+3)(x-5)$.
5. $(x-2)(x-5)(x+7)$.
6. $(x+2)(x^2-3x+6)$.
7. $(x+3)(x^2-3x+4)$.
8. $(x-3)(x-4)(x^2+5x+3)$.
9. $(a+1)(a+5)(2a-3)$.
10. $(a-3)(4a^2-3a-6)$.
11. $(a-b)(a^2+2ab+3b^2)$.
12. $(a-b)^2(a-2b)$.
13. $(a+2b)(a+5b)(a-4b)$.
14. $(a+3)(a-6)(2a-3)$.
15. $(a-1)(a-4)(2a+3)$.
16. $(2a-b)(a^2-ab+3b^2)$.
17. $(2a+b)(a-2b)(3a-4b)$.
18. $(a-b)(3a+2b)(4a+3b)$.

EXERCISE 73. (page 136)

8. $4a^2+4ax+x^2$.
9. $a^2-4ab+16b^2$.
10. $32a^5-16a^4x+8a^3x^2-4a^2x^3+2ax^4-x^5$.
11. $9m^2+3m+1$.
12. $x^3+4x^2y+16xy^2+64y^3$.
13. $1-2m+4m^2-8m^3$.
14. $16a^4-8a^3b+4a^2b^2-2ab^3+b^4$.

SELECTED QUESTIONS B. (page 137)

1. $p = 132$.
2. $p = \frac{3}{2}$.
3. $(x+2)(x+4)(x+1)$.
4. $(x+1)(x-3)(x+9)$.
5. $(a+2)(a^2-a+4)$.
6. $(x+1)(x+3)(x-4)$.
7. $(x-2)(x-3)(x+5)$.
8. $(x-1)(x-2)(x+3)$.
9. $(x-2)(x-6)(x+8)$.
10. $(x+1)^2(x-2)$.

SECTIONAL REVISION II

TEST PAPERS. (page 137)

Paper 1.

1. (*i*) The square of sum or difference of two quantities is equal to the sum of their squares plus or minus twice their product. $25x^2+16y^2 \pm 40xy$.
(*ii*) $10(a^2+b^2)$. **2.** 34.
4. (*i*) $3(a-1)(a+1)$. (*ii*) $(1-2a-b)(1+2a+b)$.
(*iii*) $4(2x-3)(3x+4)$. (*iv*) $(x-5)(x-7)(2x-3)$.
5. 0. **6.** 0·24.

Paper 2.

1. (*i*) (*a*) $49y^2$. (*b*) $132xy$. (*ii*) (*a*) 254016. (*b*) 245025.
2. 66. 3 (*i*) $3x-1$. (*ii*) $x^4+x^3y+x^2y^2+xy^3+y^4$.
4. (*i*) $-9(x^2+19x-5)$.
5. (*i*) $x(x-6)(x+7)$. (*ii*) $4(3b-2a)(2a-b)$.
6. 330300.

Paper 3.

1. The product of the sum and difference of two numbers is equal to the difference of their squares.
$p^2-4pq+4q^2-9r^2$; 3255.
2. 322. **3.** $d^2 = 3c^2-3b^2+a^2$.
4. $(3x-2)(2x-3)$.
5. (*i*) $\frac{3(a-3)}{a-5}$ (*ii*) $(a-8)(a-9)^2$. **6.** 0.

Paper 4.

1. (i) $(x+a)\,(x+b)\,(x+c) = x^3+$ (algebraic sum of 2nd terms of each factor)x^2+ x (sum of products of 2nd terms taken two at a time) + product of 2nd terms.
$(p+2)\,(p-3)\,(p+4) = p^3+(2-3+4)\,p^2$
$+\{2\times(-3)+(-3)\times4+4\times2\}p+2\times(-3)\times4.$
(ii) $x^3+4x^2-17x-60$.

2. 52. **4.** (i) 6. (ii) 18. (iii) 23.
5. (i) $(2x^2-1)(2x^2+1)(2x^2-2x+1)(2x^2+2x+1)$.
(ii) $a^2b^2(3a-2b)(9a^2+6ab+4b^2)$.
(iii) $(7a+3b)(a+9b)$. (iv) $(2x-1)(x-2)(2x^2-5x+5)$.
6. $\frac{6}{5}$.

Paper 5.

1. The sum (or difference) of two cubes is equal to the sum (difference) of their cubes increased (decreased) by three times the product of those two quantities and their sum (difference).

$$(5m-2n)^3 = (5m)^3-3(5m)^2(2n)+3(5m)(2n)^2-(2n)^3.$$

2. (i) 341. (ii) 387. (iii) 4. **3.** (ii) $v=\frac{ku}{k-u}$.

4. (i) $2x(x-3)(x^2+3)$. (ii) $x^2(x+6)(x-5)$.
(iii) $(x-y)(x+y-z)$.

5. (i) $n=\frac{r.I}{E-RI}$. (ii) -33. **6.** (i) 55. (ii) 3·030625.

Paper 6.

1. When the sum of two quantities is multiplied by the sum of their squares diminished by their product, the result is equal to the sum of their cubes.
(In the text read a^2-ab+b^2 for a^2+ab+b^2.)
$27+64a^3$; 133.
2. p^3+3p. **3.** (i) $4(x+2y)^2$. (ii) $(x+a)(x-a)(x-b)$.
4. (i) $16a^4$. (ii) $2m^3-9m^2+16m-14$.
5. (i) 0. (ii) 13·2. **6.** 1.

Paper 7.

1. The square of the sum of three numbers is equal to the sum of squares of those numbers plus twice the sum of the products of those numbers taken two at a time.
$9x^2+4y^2+16z^2-12xy-16yz+24zx$;
$2(p^2-q^2-2r^2-3pq-qr+pr)$.
2. 50. **3.** (i) 0.

4. (i) $(9a^2-12ab+8b^2)(9a^2+12ab+8b^2)$.
(ii) $(7x-7y-4)(5x-5y-3)$.
(iii) $(2x+3y)(2x-3y-3)$.
(iv) $(4x^2+16x+11)^2$.

5. 0. **6.** $x = 1$.

Paper 8.

1. $8m^3-n^3-18mn-27$.

2. 246684600000.

3. (i) $18x(28x^2-10x+1)$.
(ii) $(x+3)(2x+1)$.
(iii) $(2a+3b)(2a+3b-4)$.
(iv) $(a+1)(a+2)(2a-1)$.

5. a^2-ab+b^2. **6.** 8734.

Paper 9.

1. (i) ·414. (ii) x^8-a^8

3. $6m^2-7m-3$. **4.** 288.

5. (i) $(a^2+2a+1)(a^2-2a+2)$.
(ii) $(a+1)^2(a-1)$.
(iii) $(a^4-a^2+1)(a^2-a+1)(a^2+a+1)$.

6. $\frac{1}{4}x(x^2+3y^2)$.

Objective Type Items—II. (page 142)

1. (i) $(-3)^2-(-2)^3 = 17$.
(ii) $(3{\cdot}4)^2+(8{\cdot}6)^2+(6{\cdot}8)(8{\cdot}6) = 144$.
(iii) If $(x-2)$ is a factor of x^2-ax+8 then $a = 6$.
(iv) If $x+\frac{1}{x} = 2$, then $x^2+\frac{1}{x^2} = 2$.
(v) $5x^2-\frac{1}{5}y^2 = 5(x-\frac{1}{5}y)(x+\frac{1}{5}y)$.
(vi) $\frac{(1{\cdot}53)^2-(\cdot47)^2}{1{\cdot}06} = 2$.
(vii) $(2{\cdot}475)^3+(1{\cdot}525)^3+12(2{\cdot}475)(1{\cdot}525) = 64$.
(viii) $256a^8-b^8$.

2. (i) *X*, *D*. (ii) *X*, *B*. (iii) *X*, *C*.
(iv) *X*, *B*.

EXERCISE 74. (page 144)

1. $a-b$.
2. $a-3b$.
3. $a+2b$.
4. $a+2$.
5. $a-3$.
6. $a-1$.
7. $a+3$.
8. $a+2$.
9. $a-b-c$.
10. $x+4$.
11. $a-2$.
12. $a-2$.
13. $a+3$.
14. a^2+1.
15. $a-2$.
16. $a-3$.

EXERCISE 75. (page 145)

1. $a(a-1)^2(a+1)$.
2. $(a-2)(a+2)(a^2-2a+4)$.
3. $(a-1)^3(a^2+a+1)$.
4. $(a+1)(a+2)(a+3)$.
5. $(a+b)(a-2b)(a-3b)$.
6. $(x+a)(x+4a)(x+7a)(x+13a)$.
7. $(x+1)(x+2)(x+3)$.
8. $(a-1)(a+1)(a^2+a+1)(a^2-a+1)$
9. a^8-1.
10. $a^{12}-1$.
11. $(a^2-4b^2)^2$.
12. $36a^3b^2(a-b)^5$.
13. $72(a+b)^3(a-b)^2(a^2+ab+b^2)$.
14. $60a^3b^4(a+b)(a-b)^3(a^2+ab+b^2)$.
15. $a^3b(a+b)(a-b)^3(a^2-ab+b^2)(a^2+ab+b^2)$.
16. $(x^2-a^2)(x^2-b^2)(x^2-c^2)$.
17. $(1+2a)(1-8a^3)(1+2a-4a^2)$.
18. $(x+1)(x^2+1)(x-1)^3$.

EXERCISE 76. (page 147)

1. a^2+3a+4.
2. $a-3$.
3. $a-4$.
4. $3a-1$.
5. $2a^2+3a+7$.
6. $3a+4$.
7. $8a^4-24a^3+18a^2$.
8. $2a^2+a-1$.
9. $a^2-2ab+4b^2$.
10. a^2-a-1.
11. a^2-2.
12. a^2+3a+5.
13. a^2-3a+1.
14. a^2+5a+1.
15. a^2-3a+5.

EXERCISE 77. (page 148)

1. $x-2$.
2. $x-3$.
3. x^2-3x+2.
4. $x^2+5xy+4y^2$
5. $3x-2y$.

EXERCISE 78. (page 149)

1. $2x^2-x-3$.
2. x^2-3x+4.
3. x^2-x-2.
4. x^2+2x+3.
5. x^2-2x+1.
6. x^2+2x+3.
7. $3x^2-x+1$.
8. $\dfrac{x-2}{x^2-x-3}$.
9. $\dfrac{x-1}{x+2}$.
10. $\dfrac{x^2+x-12}{x^2+x+2}$.
11. $\dfrac{2x+3}{6x+5}$.
12. $\dfrac{x-1}{x+1}$.
13. $\dfrac{x^2+x-6}{x^2-1}$.
14. $\dfrac{3x^2+x+2}{2x^2+x+3}$.
15. $\dfrac{x^2+x-12}{x^2-x-12}$.

EXERCISE 79. (page 151)

1. H.C.F. $=x^2-2x+3$ and L.C.M. $=(x-3)(x^3+x^2-3x+9)$.
2. H.C.F. $=x^2+3x+2$ and L.C.M. $=(x-5)(x^3-x^2-10x-8)$.
3. H.C.F. $=2x^2-9$ and L.C.M. $=(2x-5)(6x^3+8x^2-27x-36)$.
4. H.C.F. $=2x^2-3x+1$ and L.C.M. $=(x^2+2x-2)(2x^4-7x^3+11x^2-8x+2)$.
5. H.C.F. $=x^2+x-6$ and L.C.M. $=(x^2-3)(x^5+6x^2-49x+42)$.
6. H.C.F. $=4x^2-3x+2$ and L.C.M. $=(2x^2-3)(16x^4-12x^3+20x^2-9x+6)$.
7. H.C.F. $=x^2-x+3$ and L.C.M. $=(x^2-x+1)(2x^4-x^3+6x^2+2x+3)$.
8. H.C.F. $=3x^2-5ax+a^2$ and L.C.M. $=(5x-2a)(6x^4-25ax^3+26a^2x^2-a^4)$.
9. $15x^3-x^2-5x-1$.
10. $4x^4-x^3+4x-1$.
11. $x^7-x^6-x^4+1$.
12. $x^4-2x^3+10x^2-12x+24$.

EXERCISE 80. (page 152)

1. $x^6+x^5+3x^4+11x^3+4x^2+12x+16$.

2. $x^7+x^6-x^5+2x^4-x^3-x^2+x-2$.

3. $x^{10}+2x^9-6x^8-6x^7+22x^6+2x^5-44x^4+42x^3+17x^2-108x+81$.

SELECTED QUESTIONS—C. (page 152)

1. $2y-x$.

2. $1+x$.

3. $4x^2-4x+1$.

4. $4x^2-4x+1$.

5. $3(a^2-1)$.

6. $2x^3+x^2+x$.

7. $x+y$.

8. x^2+5.

9. H.C.F. $=2(x-3)$ L.C.M. $=12x(x+1)(x+3)(x-3)$.

10. H.C.F. $=(a+b+c)$ L.C.M. $=(a+b+c)(a-b-c)\times(b-c-a)(c-a-b)$.

11. H.C.F. $=x^2+3x+2$ L.C.M. $=(x-1)(x+1)(x+2)(x+3)$.

12. H.C.F. $=2x+3$ L.C.M. $=(2x+3)^2(x^2-1)(6x-1)$.

13. H.C.F. $=1+x+x^2$ L.C.M. $=(1-x)(1-x+x^2)(1-x^3)$.

14. H.C.F. $=x^2-2x+3$ L.C.M. $=(x+2)(x^4+2x^2+9)$.

15. Read x^2 for x in the first expression:
H.C.F. $=x^2-x-1$ L.C.M. $=(x^2-x-1)(x^2-x+1)(x^2+x-1)$.

16. H.C.F. $=2a^2+a+3$ L.C.M. $=(a^2+a+1)(4a^4+11a^2+9)$.

EXERCISE 81. (page 155)

1. $\frac{2(a+b)}{a^2-b^2}$, $\frac{-3(a+b)}{a^2-b^2}$, $\frac{4(a-b)}{a^2-b^2}$.

2. $\frac{(a+b)^2}{a^2-b^2}$, $\frac{(a-b)^2}{a^2-b^2}$, $\frac{-2ab}{a^2-b^2}$.

3. $\frac{(4a-b)(1+4ab)}{1-16a^2b^2}$, $\frac{(4a+b)(1-4ab)}{1-16a^2b^2}$, $\frac{4b(8a^2-1)}{1-16a^2b^2}$.

4. $\frac{3(x-1)}{(x-1)(x-2)(x-3)}$, $\frac{4(x-2)}{(x-1)(x-2)(x-3)}$, $\frac{-5(x-3)}{(x-1)(x-2)(x-3)}$.

5. $\frac{a+b-c}{(a-b-c)(b-c-a)(c-a-b)}$, $\frac{b+c-a}{(a-b-c)(b-c-a)(c-a-b)}$, $\frac{c+a-b}{(a-b-c)(b-c-a)(c-a-b)}$.

6. $\frac{a^2(a-b+c)}{ab(a^2+b^2-c^2-2ab)}$, $\frac{b^2(a-b-c)}{ab(a^2+b^2-c^2-2ab)}$, $\frac{abc}{ab(a^2+b^2-c^2-2ab)}$.

7. $\frac{5x}{(x-2)(x+3)}$. **8.** $\frac{9x+1}{(x-3)(x+4)}$. **9.** $\frac{4ax}{x^2-a^2}$.

10. $\frac{(x+y)^2}{x-y}$. **11.** $\frac{2x}{(x+1)(x+2)}$. **12.** $\frac{1}{1-x}$.

13. $\frac{1}{ab}$. **14.** $\frac{1}{x-y}$. **15.** $\frac{6xy}{8x^3+27y^3}$.

16. $\frac{1}{(a-c)(b-c)}$. **17.** $\frac{2ab}{a^2-b^2}$. **18.** $\frac{2a(a+5b)}{(a^2-b^2)^2}$.

EXERCISE 82. (page 157)

1. $\frac{2(x^2-7x+13)}{(x-2)(x-3)(x-4)(x-5)}$. **2.** $\frac{1}{(x+5)(2x+3)}$.

3. $\frac{4}{(x-1)(x-5)}$. **4.** 0.

5. $\frac{3}{(x+1)(x+2)(x+3)}$. **6.** $\frac{1}{(x-1)(x+2)(x+3)}$.

7. $\frac{4}{(x+a)(x+13a)}$. **8.** $3x$. **9.** 7. **10.** $\frac{-ab(2a+b)}{(a+b)(a-b)}$.

11. $\frac{2a^2-4a-3}{(a^2-1)(a^2-4)}$. **12.** $4x^2$. **13.** $\frac{x^2+2x+3}{(x^2-1)(x^2+1)}$.

EXERCISE 83. (page 158)

1. $\frac{6(a^2-2)}{(a^2-1)(a^2-4)}$. **2.** $\frac{6ab^2}{(a^2-b^2)(4a^2-b^2)}$.

3. $\frac{8a(a^2-7b^2)}{(a^2-b^2)(a^2-9b^2)}$. **4.** $\frac{8(a^4-3)}{(a^4-1)(a^4-9)}$.

5. $\frac{1}{1-a^2}$. **6.** $\frac{1}{1-9x^2}$. **7.** $\frac{a^4+b^4}{a^4-b^4}$.

8. $\frac{8}{1-a^8}$. **9.** $\frac{6}{a^6-1}$. **10.** $\frac{-3b^2}{a(a^2-b^2)}$.

EXERCISE 84. (page 160)

1. $\frac{4x^7}{x^8-a^8}$. **2.** $\frac{1}{a^8-1}$. **3.** $-\frac{1}{x+a}$.

4. $\frac{2(a+b)}{a^2+ab+b^2}$. **5.** $\frac{1}{a^2+ab+b^2}$. **6.** 0.

7. $\frac{4}{(x-1)(x+3)}$. **8.** $\frac{6}{(x+2)(x+3)(x+4)(x+5)}$.

9. $\frac{-48}{x(x-2)(x+2)(x+4)}$. **10.** $\frac{16(x+4)}{(x+1)(x+3)(x+5)(x+7)}$.

11. $\frac{2x(a^2-b^2)}{(x^2-a^2)(x^2-b^2)}$. **12.** $\frac{10b^2(a^4-6b^4)}{(a^4-4b^4)(a^4-9b^4)}$.

EXERCISE 85. (page 162)

1. x. **2.** 1. **3.** 1.

4. $\frac{x+1}{x-1}$ **5.** $\frac{2xy}{x^2+y^2}$. **6.** 1.

7. $\frac{x^2+x-ax+a}{x-a}$. **8.** $\frac{x^4-y^4}{2xy}$. **9.** 1.

10. $\frac{x^3}{x^4-x^2+1}$. **11.** x^2. **12.** $\frac{6(1-x)}{4-x}$.

13. $\frac{a}{a-b}$. **14.** $\frac{2x^3}{y^2-x^2-2xy}$. **15.** $\frac{x(2x^2-3x+6)}{6}$.

EXERCISE 86. (page 164)

1. $1+\frac{4}{x+3}$. **2.** $1+\frac{10}{x-3}$. **3.** $2+\frac{7}{x-1}$.

4. $3-\frac{10}{x+2}$. **5.** $2-\frac{3}{3x-2}$. **6.** $x^3-x^2a+xa^2-a^3+\frac{a^4-1}{x-a}$.

7. $2x+\frac{2x+1}{2x^2+2x+3}$. **17.** 1·111.

SELECTED QUESTIONS—D. (page 165)

1. $\dfrac{3x^3-14}{(x-1)(x-2)(x-3)}$.

2. 0.

3. $\dfrac{2(a+b)}{a^2+ab+b^2}$.

4. $\dfrac{8x^7}{1-x^8}$.

5. $\dfrac{16x^{15}}{1-x^{16}}$.

6. $-\dfrac{1}{x-a}$.

7. $\dfrac{(a+b+c)^2}{2bc}$.

8. $\dfrac{2ab}{a^2+b^2}$.

EXERCISE 87. (page 168)

1. $x=\frac{6}{5}$, $y=-\frac{2}{5}$.
2. $x=1$, $y=-2$.
3. $x=3$, $y=2$.
4. $x=4$, $y=3$.
5. $x=1$, $y=1$.
6. $x=5$, $y=1$.
7. $x=7$, $y=4$.
8. $x=4$, $y=1$.
9. $x=\dfrac{b^2+ac}{a^2+b}$, $y=\dfrac{ab-c}{a^2+b}$.
10. $x=8$, $y=5$.
11. $x=\frac{29}{5}$, $y=\frac{1}{3}$.
12. $x=1$, $y=2$.
13. $x=5$, $y=6$.
14. $x=-1$, $y=-1$.
15. $x=1$, $y=-\frac{1}{2}$.
16. $x=6$, $y=-\frac{28}{3}$.
17. $x=5$, $y=3$.
18. $x=3$, $y=2$.
19. $x=6$, $y=4$.
20. $x=\dfrac{c(c-b)}{a(a-b)}$, $y=\dfrac{c(a-c)}{b(a-b)}$.

EXERCISE 88. (page 170)

1. $x=10$, $y=4$.
2. $x=19$, $y=7$.
3. $x=5$, $y=6$.
4. $x=4$, $y=2$.
5. $x=3$, $y=2$.
6. $x=14$, $y=15$.
7. $x=1$, $y=2$.
8. $x=5$, $y=3$.
9. $x=\cdot02$, $y=2\cdot9$
10. $x=18$, $y=48$.
11. $x=4$, $y=2$.
12. $x=3$, $y=2$.
13. $x=3$, $y=2$.
14. $x=8$, $y=5$.
15. $x=3$, $y=2$.
16. $x=40$, $y=60$.
17. $x=\frac{4}{3}$, $y=\frac{5}{2}$.
18. $x=5$, $y=2$.
19. $x=2$, $y=5$.
20. $x=\dfrac{ac-bd-c^2}{a^2-ac-ab+b^2}$, $y=\dfrac{bc+ad-ac}{a^2-ac-ab+b^2}$.

EXERCISE 89. (page 172)

1. $x = 1,\ y = 1.$
2. $x = 3,\ y = 4.$
3. $x = 4,\ y = 10.$
4. $x = 3,\ y = 4.$
5. $x = \frac{1}{5},\ y = \frac{1}{7}.$
6. $x = \frac{1}{2},\ y = 1.$
7. $x = \frac{mp-nq}{ap},\ y = \frac{mp-nq}{aq}.$

EXERCISE 90. (page 173)

1. $x = 2,\ y = 0.$
2. $x = 5,\ y = 3.$
3. $x = -3,\ y = 4.$
4. $x = -1,\ y = 2.$
5. $x = 10,\ y = 15.$
6. $x = 4,\ y = 6.$
7. $x = \frac{am-bn}{a^2-b^2},\ y = \frac{an-bm}{a^2-b^2}.$
8. $x = 7,\ y = 3.$

EXERCISE 91. (page 174)

1. 23, 16.
2. 15, 13.
3. 21, 12.
4. 18, 15.
5. ·32, ·24.
6. 12cm, 15cm.

EXERCISE 92. (page 175)

1. $\frac{7}{9}$.
2. $\frac{2}{3}$
3. $\frac{8}{13}$.
4. $\frac{5}{11}$.
5. $\frac{8}{12}$.

EXERCISE 93. (page 176)

1. Rs. 70, Rs. 50.
2. Rs. 1·25, Rs. 1·50.
3. Rs. 7, Rs. 4.
4. Rs. 350, Rs. 460.
5. Rs. 22, Rs. 24.
6. Rs. 535, Rs. 315.
7. Rs. 140, Rs. 180.
8. Rs. 20, 30 paise.

EXERCISE 94. (page 178)

1. 4 km per hour, $3\frac{1}{2}$ km per hour.
2. $3\frac{1}{2}$ km per hour, $1\frac{1}{2}$ km per hour.
3. 3 km per hour, 1 km per hour.
4. 8 km per hour, 3 km per hour.
5. 40 km.
6. $8\frac{1}{2}$ hours.
7. 22 km/hour, 132 km.

EXERCISE 95. (page 179)

1. 142½ kg ; 127½ kg ; 15 kg.
2. Rs. 200. **3.** 288 sq. metres. **4.** 90 kg ; 30 kg.
5. A = Rs. 4·50 ; B = Rs. 7·50 ; C = Rs. 10.
6. Men = 16 ; Sum = Rs. 48.

EXERCISE 96. (page 181)

1. $x = 7, y = 4$ or $x = 4, y = 7$.
2. $x = 12, y = 5$ or $x = 5, y = 12$.
3. $x = 15, y = 8$. **4.** $x = 12, y = 7$.
5. $x = 8, y = 7$ or $x = 7, y = 8$.
6. $x = 11, y = 7$. **7.** $x = \frac{a+\sqrt{a^2-4b^2}}{2}, y = \frac{a-\sqrt{a^2-4b^2}}{2}$.
8. $x = \frac{\sqrt{m^2+4n^2}+m}{2}, y = \frac{\sqrt{m^2+4n^2}-m}{2}$.
9. $x = 3, y = 4$ or $x = 4, y = 3$.
10. $x = 2, y = 5$. **11.** 48 metres ; 36 metres.
12. 20 metres ; 8 metres. **13.** 80 m. ; 48 m.
14. 4 cm **15.** (*i*) 36 m. (*ii*) 25 m. ; 11 m.

SELECTED QUESTIONS—E. (page 182)

1. 8 ; 5. **2.** 692. **3.** $\frac{6}{7}$.
4. Sohan : Rs. 1100 ; Mohan : Rs. 2900.
5. 40 years, 10 years.
6. 12 km per hour ; 15 km.
7. 10 km per hour ; 11 km per hour.
8. 54 km ; 6 km per hour.
9. A's time = 4 minutes, B's time = 5 minutes.
10. 8 km per hour ; 3 km per hour.
11. 100 students in Room A, 80 students in Room B.

SECTIONAL REVISION III.

TEST PAPERS. (page 183)

Paper 1.

2. (*i*) $\frac{1}{a-b}$ (*ii*) $2(a+b)$. **3.** $x = 4, y = 6$.
4. -6, 7. **5.** 52.

Paper 2.

1. $3x-2$.

2. (*i*) $x-a+\frac{a^2-1}{x+a}$ (*ii*) $\frac{2}{x^2(x+2)(x-2)}$.

3. $x = 3,\ y = 5$.

4. A = 32 years, B = 20 years.

5. $6n+44$.

Paper 3.

1. x^2-2x+1.

2. $\frac{17x}{(x-3)(x-7)(x+4)}$.

3. $x = 3,\ y = 2$.

4. 87.

5. $(2x+3)(3x-8)$.

Paper 4.

1. $\frac{2x^2+7x+3}{x^2+3x+2}$.

2. $\frac{8x^2}{1-x^8}$.

3. $x = b,\ y = 0$.

4. 15 km per hour, 90 km.

5. 27.

Paper 5

2. $\frac{34xy}{49x^2-y^2}$.

3. $x = a+b,\ y = ab$.

4. 25 km.

5. (*i*) $(x^2-3x-6)(x^2-3x+4)$ (*ii*) $(9x^2+12x+8)$ $(9x^2-12x+8)$.

Paper 6.

2. $\frac{1}{x-1}$

3. $x = 3,\ y = 4$.

4. Rs. 480, Rs. 960.

5. 143.

Paper 7.

1. $3x^2+8xy+4y^2$.

2. $\frac{2}{x}$.

4. Tea—Rs. 2.50 per kg, Coffee—Rs. 1.25 per kg.

5. -5.

Paper 8.

1. H.C.F. $= x^2+1$ L.C.M. $= x^3(x^2+1)(x^2-1)(x-2)$.

2. $\frac{(x+1)^2}{x+2}$.

3. $x = 4,\ y = 5$

4. 2 metres.

5. $(a-3)(2a^2-a+1)$.

Paper 9.

1. $(x^2+8ax-2a^2)(7x-4a)(3x-7a)$
2. 2. **3.** $a = -\frac{2}{3}$, $b = \frac{5}{2}$.
4. 48 persons ; Rs. 2.50 each **5.** 11.

Objective Type Items—III. (page 188)

1. (i) Same. (ii) Two. (iii) H.C.F.
(iv) Complex. (v) Denominator.
2. B D
3. C.
4. (i) C $A \times B = H \times L$.
(ii) C $\frac{x+4}{x}$. (iii) C $10x+y$.

EXERCISE 98. (page 199)

1. $y = -5$ (a straight line parallel to x axis at a distance of -5).
2. $x = -8$ (a straight line parallel to y axis at a distance of -8).
3. (i) $y = \frac{3}{2}x$ (a straight line passing through origin and with slope $= \frac{3}{2}$).
(ii) $y = -\frac{3}{2}x$ (a straight line passing through origin and with slope $= -\frac{3}{2}$).

EXERCISE 99. (page 202)

5. $y = \frac{1}{2}x - \frac{5}{2}$ slope $= \frac{1}{2}$.
6. $y = -2x+8$ slope $= -2$.
7. $y = \frac{3}{4}x - \frac{5}{2}$ slope $= \frac{3}{4}$.
8. $y = \frac{1}{8}x - \frac{3}{8}$ slope $= \frac{1}{8}$.
9. $y = \frac{4}{3}x+4$ slope $= \frac{4}{3}$.
10. $y = -\frac{16}{12}x + \frac{4}{6}$ slope $= -\frac{16}{12}$.
11. $y = 0x+3$ slope $= 0$.
12. $0y = -x+5$ slope $= \infty$.
13. $y = 0x+0$ slope $= 0$.
14. (i) $\frac{x}{5} + \frac{y}{-\frac{5}{2}} = 1$. (ii) $x+(-2)y-5 = 0$.

$\frac{x}{4}+\frac{y}{8}=1.$ $2x+y-8=0.$

$\frac{x}{\frac{10}{3}}+\frac{y}{-\frac{5}{2}}=1.$ $3x-4y-10=0.$

$\frac{x}{3}+\frac{y}{-\frac{3}{8}}=1.$ $x-8y-3=0.$

$\frac{x}{-3}+\frac{y}{4}=1.$ $4x-3y+12=0.$

$\frac{x}{5}+\frac{y}{4}=1.$ $16x+25y-20=0.$

34. 34·225 sq. units.

EXERCISE 100. (page 206)

1. $x=3,\ y=4.$
2. $x=5,\ y=4.$
3. $x=4,\ y=3.$
4. $x=4,\ y=5.$
5. $x=2,\ y=3.$
6. $x=3,\ y=2.$
7. $x=-5,\ y=-2.$
8. $x=2,\ y=-5.$
9. $x=4,\ y=3.$
10. $x=4,\ y=-3.$
11. $x=-3,\ y=4.$
12. $x=3,\ y=4.$
13. $x=5,\ y=6.$
14. $x=4,\ y=5.$
15. $x=2,\ y=3.$
16. $(-5,\ -5)$; $(-5,\ 9)$; $(2,\ 2)$; $7\sqrt{2},\ 7\sqrt{2},\ 14.$

SELECTED QUESTIONS—F. (page 206)

1. $x=-\frac{10}{3},\ y=-\frac{2}{9}$
2. $x=-1,\ y=-1.$
3. $x=-1,\ y=-2.$
4. $x=2,\ y=3.$
5. $x=13,\ y=11.$
6. $x=6, y=-4$; $(8,\ 0)$; $(0,\ -16)$
7. $x=2,\ y=1.$
8. $x=\frac{1}{3},\ y=3.$

EXERCISE 101. (page 214)

1. $y=\frac{3}{4}x.$
4. 72·6 km.
6. 102·7°F, 10-20 A.M., 7-20 P.M.
7. From 22nd June to 5 July.
8. 13, 14·4, 21·3.
9. 65 cm, 51·5 cm.
10. 38·6°.
11. (i) 99·2 m. (ii) 24·8 m.

EXERCISE 102. (page 220)

1. $\sqrt[5]{a^3}$. **2.** $\sqrt[5]{\frac{1}{x^3}}$. **3.** $\frac{6}{\sqrt[3]{x^2}.\sqrt{a}}$.

4. $\frac{\cdot 9}{\sqrt[4]{p^{11}}}$ **5.** $\frac{\sqrt{a^3}}{\sqrt[3]{m^2}}$. **6.** $4.\sqrt[3]{x^2}$.

7. $\sqrt{m^3.x^2}$. **8.** $\sqrt{\frac{a^5}{x^7}}$. **9.** $\sqrt{x^{17}}$.

10. $x^{\frac{7}{12}}$. **11.** $\frac{1}{m^{\frac{9}{12}}}$. **12.** m. **13.** $a^{\frac{4}{3}}$.

14. x. **15.** $m^{\frac{1}{2}}$ **16.** $\frac{1}{2}$. **17.** 4. **18.** 8.
19. 16. **20.** $\frac{1}{4}$. **21.** $\frac{1}{9}$. **22.** 25. **23.** $\frac{1}{27}$.
24. $\frac{1}{8}$. **25.** $\frac{1}{49}$. **26.** 9. **27.** 81. **28.** 36.
29. $\frac{1}{8}$. **30.** $\frac{1}{50}$. **31.** 11. **32.** 5.

33. 1562·5. **34.** $\frac{x^2+1}{x(x-1)}$. **35.** a^{m-2n}.

EXERCISE 103. (page 223)

1. $\frac{1}{x^4}$. **2.** $\sqrt[3]{\frac{y^{15}}{x^{18}}}$ **3.** $x.b^6$.

4. $x^{-8}y^{-\frac{5}{3}}$. **5.** m^3n^4. **6.** $a^{-6}b^3$.

7. $m^{-\frac{21}{8}}$. **8.** $\frac{y}{x^2}$. **9.** $\frac{1}{\sqrt[4]{a}}$.

10. $\sqrt[8]{\frac{b^{13}}{a^{12}}}$. **11.** $\frac{27}{64}.a^3b^3$. **12.** $\frac{9}{25a^2b^2}$.

13. $\sqrt[12]{\frac{a^5}{b^{10}}}$. **14.** $\frac{x^{\frac{3}{2}}z^{\frac{1}{2}}}{y^{\frac{1}{3}}}$. **15.** $x^{2bc\,(a+c)}$.

16. 1. **17.** 2^{3^2} is greater by 448.

EXERCISE 104. (Page 225)

1. (i) $5x^{-2}+3x^{-1}+1-7x+4x^2$.
(ii) $6x^{-3}+4x^{-2}+7x^{-1}+3x+5x^3$.
(iii) $-x^{-\frac{1}{n}}+6x^{-\frac{1}{2n}}+4+3x^{\frac{1}{2n}}+5x^{\frac{1}{n}}+x$.

2. (i) $x^{-2}y^2+2+x^2y^{-2}$.

(ii) $x^{-\frac{2}{3}}y^{\frac{2}{3}}-1+x^{\frac{2}{3}}y^{-\frac{2}{3}}$.

(iii) $1+7x^3-5x^2y^{-1}-xy^{-2}+3y^{-3}$.

3. $a+a^{-1}$.

4. $a^{\frac{3}{2}}+b^{\frac{3}{2}}$.

5. $a^2+16c^{\frac{2}{3}}-9b+8ac^{\frac{1}{3}}$.

6. $a-2a^{\frac{1}{2}}+1$.

7. $x^{-2}+x^{-1}y^{-1}+y^{-2}$.

8. $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$.

9. $1-b^{-\frac{1}{3}}-a^{\frac{1}{3}}+a^{\frac{1}{3}}b^{-\frac{1}{3}}$.

10. $x^{\frac{2}{3}}-a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$.

11. $x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}}$.

12. $1-x^{-\frac{1}{4}}+x^{\frac{3}{4}}$.

EXERCISE 105. (Page 226)

1. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

2. a^2+2+a^{-2}.

3. $a^2+a+a^{\frac{2}{3}}+2a^{\frac{3}{2}}-2a^{\frac{4}{3}}-2a^{\frac{5}{6}}$.

4. $a^{\frac{2}{3}}+4a+a^{\frac{5}{3}}-4a^{\frac{5}{6}}+2a^{\frac{7}{6}}-4a^{\frac{4}{3}}$.

5. $1+a^{-\frac{2}{3}}+a^{\frac{2}{3}}+2a^{-\frac{1}{3}}+2a^{\frac{1}{3}}+2$.

6. $a^{\frac{3}{2}}+ab^{-\frac{1}{2}}+b-2a^{\frac{5}{4}}b^{-\frac{1}{4}}+2a^{\frac{3}{4}}b^{\frac{1}{2}}-2a^{\frac{1}{2}}b^{\frac{1}{4}}$.

7. $a+a^{-1}$.

8. $a^{\frac{3}{2}}+b^{\frac{3}{2}}$.

9. $x^{-2}+y^{-2}+x^{-1}y^{-1}$.

10. $x^{-\frac{2}{3}}-x^{-\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$.

11. $4a+b^{-1}+c^{\frac{2}{3}}-2a^{\frac{1}{2}}b^{-\frac{1}{2}}+2a^{\frac{1}{2}}c^{\frac{1}{3}}+b^{-\frac{1}{2}}c^{\frac{1}{3}}$.

12. $a^{3n}-b^{-3n}$.

13. $a^n+a^{\frac{n}{2}}b^{\frac{n}{2}}+b^n$.

14. a^3-b^3.

15. $\left(x^{2^{n+3}}\right)$.

16. $x^{3^{(n+3)}}$.

17. $\frac{1}{3}$.

18. a^{m-1}.

20. $1:2^8$.

19. $x^{2^n}+2.x^{2^{n-1}}.a^{2^{n-1}}+a^{2^n}$.

EXERCISE 106. (Page 228)

1. $\frac{2}{3}$. **2.** 7. **3.** $\frac{1}{2^{n-1}}$.

4. $\frac{8}{b}$. **5.** $\frac{1}{3}$. **6.** 3^{-2n-2}.

9. $\left(\frac{p}{q}\right)^{m+n}$. **10.** $1+x^{-1}$. **11.** $x^2+xy^{-1}+y^{-2}$.

12. $\frac{1}{(1-x)^n}$. **13.** 1. **14.** 8. **15.** 1.

16. $-\{(b^{-1}-a^{-1})(a^{-1}-c^{-1})(c^{-1}-b^{-1})(a+b+c)\}$.

19. 1.

SELECTED QUESTIONS—G. (page 230)

1. 1. **2.** 3^{3n} **3.** $\frac{8}{9}$. **4.** $x^{-\frac{11}{8}}y^{-\frac{9}{2}}$.

5. $x^{-p^2-q^2}$ **6.** $-\frac{2}{3}a^{\frac{1}{6}}b^2$. **7.** 1. **8.** 1.

9. 1. **10.** $x^{2(a^3+b^3+c^3)}$ **11.** $x^{2(a^3+b^3+c^3)}$ **12.** x^{-2a^3}.

EXERCISE 107. (page 234)

1. (i) 5 : 12. (ii) $a : d$. (iii) $x^3+1 : x^3-1$.

2. (i) 9 : 49. (ii) $a^2b^2 : c^2d^2$. (iii) $a^2+x^2+2ax : a^2+x^2-2ax$.
(iv) $x^2+2x+1 : x^2-2x+1$.

3. (i) 64 : 125. (ii) $m^3n^3 : p^3q^3$. (iii) $(a-b)^3 : (a+b)^3$.
(iv) $x^6 : 2^{3x}$.

4. (i) 4 : 5. (ii) $x^3 : y^4$. (iii) $x+2a : x-2a$.

5. (i) 4 : 3. (ii) $x^3 : y^2$. (iii) $5a^2b^4 : 7xy^2$.

6. (i) $5 : 6 > 2 : 3$. (ii) $8 : 5 > 14 : 11$.

7. -57. **11.** 23. **12.** $-\frac{ab}{a+b}$.

13. $-\frac{ab(a+b)}{a^2+ab+b^2}$. **14.** 28 and 32. **15.** 15 and 20.

16. $49\frac{1}{2}$ and $38\frac{1}{2}$. **17.** 21 and 35. **18.** $\frac{b}{a}x$.

19. $\frac{7}{18}$. **20.** $\frac{38}{39}$. **21.** $\frac{3}{4}$.

22. $\frac{5}{4}$. **23.** $\frac{6}{5}$; $-\frac{2}{11}$. **24.** $\frac{y+x}{y-x}$.

25. $\frac{x^3+y^3}{x^2+y^2}$. **26.** 2. **27.** $-3, -12$.

28. 45 yrs. and 36 yrs. **29.** 9 : 10.

30. 28000 and 8400.

EXERCISE 108. (page 237)

1. $-\frac{9}{2}$. **2.** $\frac{13}{12}$. **3.** $-\frac{3}{2}$. **4.** $-\frac{2}{7a}$.

5. $\frac{3}{2b}$. **6.** 2.

EXERCISE 109. (page 239)

1. 21. **2.** 20. **3.** 40. **4.** $\frac{1}{6}$.

5. $4a^2b^2$. **6.** 12. **7.** 35. **8.** 24.

9. 132. **10.** 3. **11.** xy. **12.** $(a^2-b^2)^2$.

13. 8. **14.** $\frac{64}{5}$. **15.** $\frac{12}{5}$. **16.** $\frac{(x+y)^3}{x-y}$.

17. 3.

EXERCISE 110. (page 241)

1. 5. **2.** 6. **4.** 4, 36. **5.** 22. **7.** 36, 63.

8. 60 kg, 120 kg, 160 kg, 200 kg. **9.** 13 : 35.

10. 3 : 4.

EXERCISE 112. (page 246)

1. 2. **2.** 2.

EXERCISE 113. (page 248)

17. 1.

EXERCISE 114. (page 249)

11. $\frac{x}{8}=\frac{y}{3}=\frac{x+y}{11}=\frac{x-y}{5}=\frac{15x-12y}{84}$.

12. $\frac{x}{7}=\frac{y}{3}=\frac{x+y}{10}=\frac{x-y}{4}$.

EXERCISE 117.

1. $\frac{1}{2}$. **2.** $\frac{25}{12}$. **3.** 4.2. **4.** $\frac{1}{50}$.

5. $\frac{1}{ab}$. **6.** 3. **7.** 7. **8.** 6.

9. 12. **10.** $\frac{5}{13}$. **11.** $-7\frac{6}{11}$. **12.** $\frac{2ab}{a+b}$.

13. $-\frac{2ab}{a^2+b^2}$. **14.** 14. **15.** $-\frac{1}{a+b}$. **16.** 1.

EXERCISE 118.

1. $-2\frac{1}{2}$. **2.** -6. **3.** $-1\frac{1}{2}$. **4.** 4.

5. 6. **6.** $-5\frac{1}{3}$. **7.** $\frac{a^2+b^2}{a+b}$. **8.** $\frac{1}{12}$.

9. 7. **10.** 7. **11.** 13. **12.** 0.

EXERCISE 119.

1. $-1\frac{1}{2}$. **2.** $-1\frac{3}{5}$. **3.** $2\frac{21}{29}$. **4.** $3\frac{11}{38}$.

5. $-\frac{2ab}{a+b}$ **6.** $\frac{2(a+b)}{a+2b}$. **7.** $\frac{ab(a+b-2c)}{(a^2+b^2-ac-bc)}$.

8. $\frac{ab}{c-a-b}$. **9.** $2\frac{3}{4}$. **10.** $3\frac{4}{7}$. **11.** $-\frac{2}{13}$.

12. $\frac{3}{4}$. **13.** $2\frac{29}{41}$. **14.** $-\frac{50}{263}$. **15.** $\frac{bc(b-c)}{2a-b-c}$.

16. $-2\frac{1}{2}$. **17.** $3a$. **18.** $-\frac{7}{26}$.

EXERCISE 120.

1. 3. **2.** 2. **3.** 3. **4.** $\frac{1}{2}$.

5. 1. **6.** $\frac{ac-bd}{ad-bc}$. **7.** $-\frac{7}{12}$. **8.** $-\frac{5}{16}$.

9. $-\frac{1}{20}$. **10.** $-1\frac{1}{3}$. **11.** 0. **12.** $\frac{b^3-ac^2}{a^2-b^2+c^2-ab}$.

13. 3. **14.** 5. **15.** 6. **16.** 7.

17. 8. **18.** 7.

EXERCISE 121.

1. $-\frac{13}{18}$. **2.** $-\frac{44}{105}$. **3.** $\frac{2}{19}$. **4.** $-\frac{b^3}{(a-b)^2}$.

5. 4. **6.** 4. **7.** $-\frac{3}{5}$. **8.** $1\frac{4}{23}$.

9. $2\frac{1}{5}$. **10.** $1\frac{1}{3}$. **11.** $-1\frac{2}{5}$. **12.** -4.

13. $\frac{7}{9}$. **14.** $-1\frac{2}{3}$. **15.** $\frac{2}{5}$. **16.** $-\frac{10}{11}$.

17. $\frac{8}{17}$. **18.** $-1\frac{1}{7}$. **19.** $-2\frac{3}{5}$. **20.** $-1\frac{4}{7}$.

21. $1\frac{5}{7}$. **22.** $\frac{1}{5}$. **23.** $\frac{ab}{a+b}$. **24.** $-\frac{3}{4}$.

EXERCISE 122.

1. -4. **2.** $\frac{1}{2}$. **3.** $8\frac{1}{2}$. **4.** 8.

5. $2\frac{1}{2}$. **6.** $-1\frac{1}{4}$. **7.** 4. **8.** $4\frac{1}{2}$.

9. $-\frac{5}{6}$. **10.** $-3\frac{1}{2}$. **11.** 1.

EXERCISE 123.

1. $\frac{8}{29}$. **2.** $\frac{13}{22}$. **3.** $\frac{13}{37}$.

4. $\frac{1}{5}$. **5.** $-\frac{5}{74}$. **6.** $\frac{9}{38}$.

7. $\frac{q^3-pr^2}{p^2-q^2+r^2-pq}$. **8.** $\frac{1}{ab+3a-b-2}$.

EXERCISE 124.

1. $-3\frac{2}{7}$. **2.** $-4\frac{1}{3}$. **3.** $-3\frac{5}{7}$. **4.** $-4\frac{6}{7}$.

5. $-3\frac{10}{11}$. **6.** -1. **7.** $-2\frac{2}{5}$. **8.** $-3\frac{3}{4}$.

9. $4\frac{4}{9}$. **10.** $1\frac{7}{8}$. **11.** $\frac{20}{21}$. **12.** -4.

13. $\frac{2bc}{a(b+c)}$. **14.** $\frac{b-c}{2a}$.

EXERCISE 125.

1. $\frac{3}{7}$. **2.** -1. **3.** $2\frac{2}{3}$. **4.** $1\frac{3}{5}$ (Read x^2+2x-5 in R.H.S.)

5. $-\frac{1}{4}$. **6.** $2\frac{1}{2}$. **7.** 1. **8.** 4.

EXERCISE 126.

1. $a+b$.

2. $a+b$.

3. $\frac{2ab}{a+b}$.

4. $\frac{a+b}{2}$.

5. $\frac{a^2+b^2+4ab}{2(a+b)}$.

6. $\frac{6a}{7}$.

7. $\frac{1}{ab}$.

8. 1.

9. ab.

10. $\frac{a+b+c}{3}$.

11. either a or b.

12. $a+b+1$.

13. $ab+bc+ca$.

14. $\frac{ab+bc+ca}{abc}$.

15. $a(a+b+c)$.

16. $a^3+b^3+c^3$.

17. $\frac{a+b}{2}$.

18. $\frac{2}{a+b}$.

19. $-\frac{a+b}{2}$.

20. $\frac{b}{a}(a-b+c)$.

21. $\frac{a+b-2c}{bc+ca-2ab}$.

22. 1.

23. $\frac{a+b}{2}$.

24. $\frac{2ab}{b-a}$.

25. $\frac{cq-br}{ar-cp}$.

26. $-b$.

27. 25.

28. $-\frac{a+b}{2}$.

29. $\frac{a}{3}$.

EXERCISE 127.

1. $x = 6,\ y = 11,\ z = 6$.

2. $x = 1,\ y = 2,\ z = 3$.

3. $x = -11,\ y = -8,\ z = -6$.

4. $x = 2,\ y = 3,\ z = 1$.

5. $x = 3,\ y = 2,\ z = 5$.

6. $x = 2,\ y = 3,\ z = 4$.

7. $x = 1,\ y = \frac{1}{2},\ z = \frac{1}{3}$.

8. $x = -\frac{4}{5},\ y = \frac{2}{35},\ z = \frac{4}{11}$.

9. $x = 11,\ y = 9,\ z = 7$.

10. $x = c+a-b,\ y = a+b-c,\ z = b+c-a$.

11. $x = \frac{2}{3},\ y = -2,\ z = \frac{2}{5}$.

12. $x = \frac{2}{c+a-b},\quad y = \frac{2}{a+b-c},\quad z = \frac{2}{b+c-a}$.

13. $x = 14,\ y = 12,\ z = 10$.

14. $x = 4,\ y = 3,\ z = 2.$

15. $x = \dfrac{b^2+c^2-a^2}{2bc},\ y = \dfrac{c^2+a^2-b^2}{2ca},\ z = \dfrac{a^2+b^2-c^2}{2ab}.$

16. $x = 5,\ y = 5,\ z = 5.$

17. $x = 1,\ y = 1,\ z = 1.$

18. $x = 2,\ y = 4,\ z = 6.$

19. $x = \dfrac{2}{c+a-b},\quad y = \dfrac{2}{a+b-c}\quad z = \dfrac{2}{b+c-a}.$

20. $x = abc,\ y = ab+bc+ca,\ z = a+b+c.$

21. $x = -(ab+bc+ca),\ y = a+b+c,\ z = 1.$

22. $x = 5,\ y = 4,\ z = 6.$

23. $x = 4,\ y = 5,\ z = 6.$

24. $x = 7,\ y = 5,\ z = 3.$

EXERCISE 128.

1. 15, 22, 40.

2. 18, 12.

3. 20, 12, 8.

4. $x = 160,\ y = 180,\ z = 210.$

5. 79.

6. 426.

7. 648.

8. 253.

EXERCISE 129.

1. 1 : 3 : 5.

2. 3 : 4 : 2.

3. 3 : 4 : 5.

4. $ab-c^2 :\ bc-a^2 : ac-b^2.$

5. $x = 7,\ y = 10,\ z = 8.$

6. $x = 2,\ y = -3,\ z = 1.$

7. $x = 2, y = -4, z = 6.$

8. $x = 8, y = 12, z = 20.$

9. $x = 3,\ y = 4,\ z = 5.$

10. $x = 3,\ y = 2,\ z = 1;$
or $x = -3, y = -2, z = -1.$

11. $x = 1,\ y = 2,\ z = 3.$

12. $x = b-c,\ y = c-a,\ z = a-b.$

13. $x = 8,\ y = 6,\ z = 4.$

14. $x = 3,\ y = 9,\ z = 15.$

EXERCISE 130.

1. $x = 3,\ y = \frac{3}{2}.$

2. $x = 7,\ y = 9.$

3. $x = 1,\ y = -\frac{1}{2}.$

4. $x = 3,\ y = 2.$

5. $x = \dfrac{2p^2-3}{3p+5},\quad y = \dfrac{10p+9}{3p+5}.$

6. $x = \dfrac{lp}{l^2+m^2},\quad y = \dfrac{mp}{l^2+m^2}.$

7. $x = \dfrac{1-b}{1+a},\quad y = \dfrac{b-1}{1+a}.$

8. 5.

9. $x = -2,\ y = 1.$

EXERCISE 131.

1. $x = \frac{1}{2}$, $y = \frac{1}{3}$.
2. $x = \frac{1}{2}$, $y = \frac{1}{5}$.
3. $x = -1$, $y = -\frac{1}{2}$.
4. $x = 3$, $y = \frac{1}{2}$.
5. $x = \frac{1}{3}$, $y = \frac{1}{5}$.
6. $x = 1$, $y = 1$.
7. $x = \dfrac{a^2-b^2}{am-bn}$, $y = \dfrac{a^2-b^2}{an-bm}$.
8. $x = \dfrac{1}{a}$, $y = \dfrac{1}{b}$.
9. $x = 1\frac{17}{45}$, $y = -1\frac{1}{45}$.
10. $x = -2$, $y = 1$.
11. $x = \frac{1}{5}$, $y = \frac{1}{6}$.
12. $x = \frac{2}{7}$, $y = 4$.

EXERCISE 132.

1. $x = -3$, $y = 3$, $z = 1$.
2. $x = 3$, $y = 2$, $z = 1$.
3. $x = -7$, $y = 3$, $z = 8$.
4. $x = 4$, $y = 5$, $z = 6$.
5. $x = 1$, $y = 1$, $z = 1$.
6. $x = 1$, $y = \frac{1}{2}$, $z = \frac{1}{3}$.
7. $x = 6\frac{1}{3}$, $y = 3$, $z = -\frac{3}{20}$.

SECTIONAL REVISION IV

TEST PAPERS

Paper 1.

1. (*i*) (*a*) $3^3 \times 10^4$ (*b*) $(23)^3 \times 10^3$.
2. (*ii*) (*a*) 1. (*b*) x.
6. (*i*) $\dfrac{a+b-2c}{bc+ca-2ab}$. (*ii*) $x = a, \dfrac{a}{2}, 2a$.
7. $x = 3$, $y = -2$.

Paper 2.

1. $c : 8a$.
2. -2.
5. (*i*) $x = \frac{3}{4}$ or 0. (*ii*) $x = 1$.
6. $\dfrac{2^n-1}{2^n}$.
7. $\frac{4}{5}$.

Paper 3.

1. 22.
2. $\frac{39}{88}$.
3. 2 : 1
4. $-\dfrac{2(a^2+ab+b^2)}{ab}$.

5. (*i*) $\frac{1}{8}$. (*ii*) $-3\frac{2}{7}$.

6. $y = -\frac{5}{4}x + \frac{3}{4}$;
slope $= -\frac{5}{4}$;
intercept on y-axis $= \frac{3}{4}$.

7. $x = 3,\ y = 2$.

Paper 4.

1. (*i*) $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ (*ii*) $x^2 - y^2$

2. 4 : 1.

3. 20 : 27.

4. (*i*) $1\frac{7}{8}$. (*ii*) $-4\frac{3}{11}$.

7. $x = 4,\ y = 3,\ z = 2$.

Paper 5.

2. 4.

5. (*i*) $\frac{4a}{5}$ (*ii*) 1.

6. x^{-2nr}.

Paper 6.

1. (*i*) 27, (*ii*) 243, (*iii*) 16, (*iv*) ·001.

2. 72 and 27.

5. (*i*) 7, (*ii*) 7.

6. $x = \frac{3}{10}$, $y = -\frac{2}{5}$.

Paper 7.

1. $\frac{6}{5}$.

2. 0.

5. (*i*) -4, (*ii*) $1\frac{3}{4}$.

6. $x = 10$, $y = 5$.

Paper 8.

1. (*i*) $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$. (*ii*) $x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}}$.

2. 30,000 ; 18,000.

3. (*i*) 28, (*ii*) 22·5, (*iii*) 37·5

5. (*i*) $-2\frac{3}{5}$. (*ii*) 8.

6. 20 km per hour.

7. $(-4,\ 5)$, $(3,\ 2)$, $(1, -3)$.

Objective Type Items—IV.

1. (*i*) Zero, (*ii*) straight line, (*iii*) duplicate ratio, (*iv*) the product, (*v*) increased, (*vi*) two of them are of the form $ax + by + cz = 0$.

2. (*i*) *E*, (*ii*) *C*, (*iii*) *D*, (*iv*) *D*, (*v*) *D*.

3. $\frac{1}{x-2} - \frac{1}{x-3} = \frac{1}{x-6} - \frac{1}{x-7}$.

4. (2, 3).

5. *D* (*i*), *C* (*ii*), *B* (*iii*), *A* (*iv*).

EXERCISE 138.

1. $x^3+x^2-10x+8$.
2. $x^3+5x^2-2x-24$.
3. $x^3+9x^2+26x+24$.
4. $x^3-9x^2+26x-24$.
5. $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$.
6. $x^4-6x^3-5x^2+42x+40$.
7. $x^4+4x^3-19x^2-46x+120$.
8. $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$
$-(abc+abd+acd+bcd)x+abcd$.
9. $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$.
10. $x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6$.
11. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.
12. $x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5$.

EXERCISE 139.

1. $x^5+5x^4+10x^3+10x^2+5x+1$.
2. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
3. $p^7+7p^6q+21p^5q^2+35p^4q^3+35p^3q^4+21p^2q^5+7pq^6+q^7$.
4. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
5. $32+80m+80m^2+40m^3+10m^4+m^5$.
6. $81-108a+54a^2-12a^3+a^4$.
7. $32x^5-80x^4+80x^3-40x^2+10x-1$.
8. $1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$.
9. $1-10x+45x^2-120x^3+210x^4-252x^5+210x^6-120x^7+45x^8$
$-10x^9+x^{10}$.
10. $64+576x+2160x^2+4320x^3+4860x^4+2916x^5+729x^6$.
11. $243a^5-810a^4b+1080a^3b^2-720a^2b^3+240ab^4-32b^5$.
12. $a^4+2a^3b+\frac{3}{2}a^2b^2+\frac{1}{2}ab^3+\frac{1}{16}b^4$.
13. $a^5+5a^3+10a+\frac{10}{a}+\frac{5}{a^3}+\frac{1}{a^5}$.
14. $x^7-7x^5+21x^3-35x+\frac{35}{x}-\frac{21}{x^3}+\frac{7}{x^5}-\frac{1}{x^7}$.
15. $8x^3+8x$.
16. $2x^5+20x^3+10x$.
17. $12x^5a+40x^3a^3+12xa^5$.
18. $864p^3q+1536pq^3$.

EXERCISE 140.

1. 114.
2. 11.
3. 249.
4. 126.
5. 1·0253.
6. 1·218.
7. 1·0075.
8. Rs. 1·47

EXERCISE 141

1. $\pm a^2b^8c$.

2. $\pm 8a^{\frac{5}{2}}b^4c^5$.

3. $\pm\dfrac{9a^3bc^2}{x^5y^6z^8}$.

4. $\pm 1{\cdot}1\ \dfrac{a^7b^9}{c^3d^5}$.

5. $\pm\dfrac{2a^3(x-y)^2}{b^4(x+y)^3}$.

6. $5a^2bc^3$.

7. $-4xb^2a^5$.

8. $\dfrac{7ab^2}{x^3y^4}$.

9. $-\dfrac{\frac{3}{4}a^2b^3}{x^4y^6}$.

10. $\pm 2xy^2z^3$.

11. $-2a^2b^3$.

12. $\dfrac{2}{xy^2}$.

13. $-\dfrac{x^2y^5}{3}$.

EXERCISE 142.

1. $\pm(2a-3b)$.

2. $\pm(4x^2+5)$.

3. $\pm(3x^2+5y^2)$.

4. $\pm(9a-b^2)$.

5. $\pm(a^3-1)$.

6. $\pm(1-6x)$.

7. $\pm(x-\frac{1}{2})$.

8. $\pm(x^2y+\frac{1}{2})$.

9. $\pm(m^2-\frac{1}{3})$.

10. $\pm\left(\dfrac{a}{2b}-\dfrac{2b}{a}\right)$.

11. $\pm\left(\dfrac{x^2}{y^2}-\dfrac{y^2}{2x^2}\right)$.

12. $\pm\left(x^2-\dfrac{y^2}{4}\right)$.

13. $\pm(\frac{1}{3}a-\frac{1}{4}b)$.

14. $\pm\left(\dfrac{1}{2x}+\dfrac{x}{3y}\right)$.

15. $\pm(a^2-\frac{1}{2}ab)$.

EXERCISE 143.

1. $\pm\left(x+4+\dfrac{1}{x}\right)$.

2. $\pm\left(x+2+\dfrac{1}{x}\right)$.

3. $\pm\left(x^2+1+\dfrac{1}{x^2}\right)$.

4. $\pm\left(x+1-\dfrac{1}{x}\right)$.

5. $\pm\left(x-2-\dfrac{1}{x}\right)$.

6. $\pm\left(x^2-2+\dfrac{1}{x^2}\right)$.

7. $\pm\left(x+2+\frac{1}{x}\right)$. **8.** $\pm\left(x^2-2+\frac{1}{x^2}\right)$.

EXERCISE 144.

1. $\pm(2x^2+3)$. **2.** $\pm(3a^2+8)$.

3. $\pm\frac{x}{y}$. **4.** $\pm(5b-a)$.

5. $\pm 2a^2$. **6.** $\mp\frac{4ab}{a^2-b^2}$.

7. $\pm(3a+3b+x+y)$. **8.** $\mp(x^2-2x+1)$.

9. $\pm(2x^2+3x-5)$. **10.** $\pm(x^2+xy+y^2)$.

11. $\pm(3x^2-5xy+4y^2)$. **12.** $\pm\left(\frac{x}{y}+1+\frac{y}{x}\right)$.

13. $\pm\left(x+\frac{y}{4}-z\right)$. **14.** $\pm\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$.

15. $\pm(ab+bc-ca)$. **16.** $\pm(x+2y)(2x+y)(3x+y)$.

17. $\pm\left(\frac{y+z-2x}{6}\right)$. **18.** $\pm\left(x+1+\frac{1}{x^2-1}\right)$.

19. $\pm(a^2+b^2)$. **20.** $\pm(x-1)$. **21.** $\pm(x-1)$.

EXERCISE 145.

1. $\pm(x^m-2x^n)$. **2.** $\pm(2^{2m}+3^n)$.

3. $\pm(5x^m-2x^n)$.

EXERCISE 146.

1. $\pm(4x-7y)$. **2.** $\pm(x^2+3x+3)$.

3. $\pm(4x^2-3x+2)$. **4.** $\pm(x^2-ax+2a^2)$.

5. $\pm(x^3-6x^2+12x-8)$. **6.** $\pm(ax^2+2x+3a)$.

7. $\pm\left(\frac{x^2}{8}+\frac{x}{2}-1\right)$. **8.** $\pm 3(x^2-x-\frac{1}{2})$.

9. $\pm(\frac{2}{3}x^2+\frac{3}{4}x+\frac{1}{2})$. **10.** $\pm(3x-4+\frac{1}{2}x)$.

11. $\pm(x^2-x+\frac{1}{4})$. **12.** $\pm(\frac{5}{2}-ab+3a^2b^2)$.

EXERCISE 147.

1. $\pm\left(\frac{a}{2b}-1-\frac{2b}{a}\right)$.

2. $\pm\left(\frac{x}{y}-\frac{1}{2}-\frac{y}{x}\right)$.

3. $\pm\left(\frac{3x}{2y}-1+\frac{y}{2x}\right)$.

4. $\pm\left(2x^2+3x+\frac{3}{x}\right)$.

5. $\pm\left(a^3-1+\frac{1}{2a^3}\right)$.

6. $\pm\left(\frac{x^2}{y^2}-2+\frac{y^2}{x^2}\right)$.

7. $\pm\left(\frac{x}{y}+1+\frac{y}{x}\right)$.

8. $\pm(1-x)$.

9. $\pm(2x-3y)$.

EXERCISE 148.

1. $\pm(2x^2+2x+1)$; $+221$.

2. $\pm(x^2+x+3)$; ± 113.

3. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^4}$.

4. $1+\frac{x}{2}-\frac{5}{8}x^2+\frac{5}{16}x^3$.

5. $a-\frac{1}{4}$ should be subtracted or $\frac{1}{4}-a$ should be added.

6. $x = 2$.

7. $p^2 = 4q$.

8. $a = 12,\ b = 9$.

SELECTED QUESTIONS—I.

1. $\pm\left(a+1-\frac{1}{2a}\right)$.

2. $\pm\left(x-2-\frac{1}{x}\right)$.

3. $\pm\left(x^2+2-\frac{1}{x^2}\right)$.

4. $\pm\left(x^2+4-\frac{1}{x^2}\right)$.

5. $\pm\left(\frac{x^2}{2}-2x+\frac{a}{3}\right)$.

6. $\pm\left(\frac{3x^2}{2}-\frac{ax}{3}-\frac{b}{5}\right)$.

7. $\pm\left(x^2+2+\frac{1}{x}\right)$.

8. $\pm\left(x^3-2+\frac{1}{x^3}\right)$.

9. $\pm\left(x+1-\frac{1}{3x}\right)$.

10. $\pm\left(2x-\frac{3}{2}+\frac{1}{3x}\right)$.

11. $\pm\left(x^2-\frac{1}{2}+\frac{2}{x}\right)$.

12. $\pm\left(2x^2+\frac{1}{2}x+1-\frac{1}{x}\right)$.

13. $\pm\left(\frac{x}{y}+3-\frac{y}{x}\right)$. **14.** $\pm\left(\frac{2}{2y^2}+1+\frac{2y^2}{x^2}\right)$.

15. $\pm\left(\frac{y}{x}-\frac{1}{2}-\frac{x}{2y}\right)$: (Read $-\frac{3}{4}$ for $-\frac{5}{4}$).

16. 4. **17.** 2. **18.** 16.

EXERCISE 149.

1. $\sqrt{96}$. **2.** $\sqrt[3]{500}$.

3. $\sqrt[4]{96}$. **4.** $\sqrt[4]{405}$.

5. $\sqrt[5]{96}$. **6.** $\sqrt[5]{972}$.

7. $\sqrt[n]{x^n y}$. **8.** $\sqrt[p]{x^{2p}y}$.

9. $\sqrt[5]{x^{15}y^2}$. **10.** $4\sqrt{3}$.

11. $3\sqrt{7}$. **12.** $2\sqrt{11}$.

13. $5\sqrt[3]{2}$. **14.** $3\sqrt[4]{7}$.

15. $2\sqrt[5]{\frac{5}{2}}$. **16.** $x^2\sqrt[3]{y}$.

17. $x^3\sqrt[n]{a}$. **18.** $x^4\sqrt[a]{y^2}$.

19. $-8\sqrt[3]{10}$. **20.** $5x^2y\sqrt[3]{2xy}$.

EXERCISE 150.

1. $7\sqrt{5}$. **2.** 0.

3. $4\sqrt{3}$. **4.** $9\sqrt[3]{4}$.

5. $3\sqrt[3]{5}$. **6.** -34.

7. $3x(2\sqrt{5x}-15\sqrt{y})$. **8.** $a(x-4a)\sqrt{x}$.

9. $4a\sqrt[3]{a}$.

EXERCISE 151.

1. $\sqrt[12]{46656}$, $\sqrt[12]{27}$, $\sqrt[12]{16}$. **2.** $\sqrt[6]{4}$, $\sqrt[6]{16}$, $\sqrt[6]{8}$.

3. $\sqrt[12]{x^{18}}$, $\sqrt[12]{x^8}$, $\sqrt[12]{x^3}$. **4.** $\sqrt[3]{5}$.

5. $\sqrt[3]{3}$. **6.** $\sqrt[3]{4}$. **7.** $\sqrt[3]{6}$.

8. $\sqrt[3]{4}$, $\sqrt[4]{5}$, $\sqrt{2}$. **9.** $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{6}$.

10. $3\sqrt{2}$, $2\sqrt{3}$, $\sqrt[5]{7}$. **11.** $\sqrt[3]{6}$, $\sqrt[4]{8}$, $\sqrt[6]{12}$.

EXERCISE 152.

1. $\sqrt[3]{63}$. **2.** $\sqrt[4]{60}$. **3.** $\sqrt[5]{144}$.

4. $\sqrt[5]{\frac{3}{32}}$. **5.** $\sqrt[6]{\frac{4}{3}}$. **6.** $\sqrt[4]{\frac{4}{3}}$.

7. $\sqrt[12]{944784}$. **8.** $\sqrt[6]{486000}$. **9.** $\sqrt[15]{882165816}$.

10. $\sqrt[12]{\frac{1280}{27}}$. **11.** $\sqrt[30]{\frac{2187}{512}}$. **12.** $\sqrt[8]{32}$.

13. $18\sqrt{6}$. **14.** $16\sqrt{33}$. **15.** $12\sqrt[3]{4}$.

16. $35\sqrt[3]{2}$. **17.** $\frac{3}{10}$. **18.** $\frac{5}{8}\sqrt[6]{\frac{9}{256}}$.

19. $\sqrt[6]{432}$. **20.** $2\sqrt[12]{442368}$. **21.** $\sqrt[6]{360}$.

22. $\frac{1}{3}$. **23.** $\sqrt[3]{\frac{7}{6}}$. **24.** $x^2y\sqrt[3]{xy^2}$.

EXERCISE 153.

1. $\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}$.

2. $\sqrt{15}-\sqrt{10}+3\sqrt{2}-2\sqrt{3}$.

3. $2\sqrt{6}-29$.

4. $12\sqrt{35}+24\sqrt{21}-12\sqrt{15}-30$.

5. $54-22\sqrt{6}$. **6.** $62-20\sqrt{6}$.

7. $3-\sqrt{2}$. **8.** $9\sqrt{2}+2\sqrt{3}+\sqrt{6}-6$.

9. $69+12\sqrt{30}$. **10.** $159-24\sqrt{42}$.

11. $43-24\sqrt{3}$. **12.** $2x^2-1-2x\sqrt{x^2-1}$.

13. $a^4+a^2-b^2+2a^2\sqrt{a^2-b^2}$. **14.** $2a^2-2\sqrt{a^4-4b^4}$.

15. $5-4x+12\sqrt{12x-1-32x^2}$.

16. $25a^2-7b^2+24\sqrt{a^4-b^4}$.

EXERCISE 154.

1. $\sqrt{2}, \sqrt{5}, \sqrt{6}, \sqrt{ab}, \sqrt[3]{a^2}, \sqrt{x-1}, \sqrt[4]{a^2b}$.

2. $\sqrt{6}$. **3.** $\sqrt[3]{2}.\sqrt{3}$. **4.** $\sqrt{6}.\sqrt[3]{36}$.

5. $\sqrt{3}-\sqrt{2}$. **6.** $\sqrt{5}+\sqrt{3}$. **7.** $\sqrt{c}+\sqrt{b}$.

8. $a-\sqrt{b}$. **9.** $\sqrt{5}.\sqrt[3]{36}.\sqrt[4]{7}$.

10. $\sqrt[3]{50}.\sqrt[4]{7}$. **11.** $a\sqrt{x}+b\sqrt{y}$. **12.** $4\sqrt{3}-2\sqrt{5}$.

13. $3\sqrt{2}+2\sqrt{3}$. **14.** $\sqrt{a+x}+\sqrt{a-x}$ **15.** $x-\sqrt{x^3-1}$.

EXERCISE 155.

1. $\frac{\sqrt{15}}{7}$. **2.** $\frac{7}{5}\sqrt[3]{2}$. **3.** $\frac{\sqrt[3]{20}}{2}$.

4. $\frac{\sqrt[4]{135}}{9}$. **5.** 0·462. **6.** 1·060.

7. 1·010. **8.** ·149. **9.** ·671.

10. ·693.

EXERCISE 156.

1. 5·828. **2.** −3·732. **3.** 17·944.

4. 9·898. **5.** 2·956. **6.** 5·585.

7. 12. **8.** 1154. **9.** $18+13\sqrt{2}$.

10. $\frac{1}{3}(4\sqrt{6}+\sqrt{15})$. **11.** $\frac{1}{b^2}\left(2a^2-b^2+2a\sqrt{a^2-b^2}\right)$.

12. $x+\sqrt{x^2-1}$. **13.** $x^2+\sqrt{x^4-1}$.

EXERCISE 157.

1. $\sqrt[3]{25}+\sqrt[3]{4}+\sqrt[3]{10}$. **2.** $\sqrt[3]{16}+\sqrt[3]{9}-\sqrt[3]{12}$.

3. $\sqrt[3]{36}+\sqrt[3]{25}+\sqrt[3]{30}$. **4.** $\sqrt[3]{36}+\sqrt[3]{9}-\sqrt[3]{18}$.

5. $\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{xy}$. **6.** $\sqrt[3]{x^2}+\sqrt[3]{y^2}-\sqrt[3]{xy}$.

7. $\sqrt[4]{27}+\sqrt{3}.\ \sqrt[4]{2}+\sqrt[4]{3}.\ \sqrt{2}+\sqrt[4]{8}$.

8. $\sqrt[5]{81}+\sqrt[5]{54}+\sqrt[5]{36}+\sqrt[5]{24}+\sqrt[5]{16}$.

9. $\sqrt[6]{3125}+\sqrt[6]{1875}+\sqrt[6]{1125}+\sqrt[6]{675}+\sqrt[6]{405}+\sqrt[6]{243}$.

10. $\sqrt[4]{27}-\sqrt[4]{18}+\sqrt[4]{12}-\sqrt[4]{8}$.

11. $\sqrt[5]{81}-\sqrt[5]{54}+\sqrt[5]{36}-\sqrt[5]{24}+\sqrt[5]{16}$.

12. $\sqrt[6]{3125}-\sqrt[6]{1875}+\sqrt[6]{1125}-\sqrt[6]{675}+\sqrt[6]{405}-\sqrt[6]{243}$.

EXERCISE 158.

1. $\sqrt{2}(1-\sqrt{2}-\sqrt{3})$. **2.** $\sqrt{2}(\sqrt{3}-\sqrt{2}+1)$.

3. $\sqrt{6}(\sqrt{3}-\sqrt{2}-\sqrt{5})$. **4.** $\sqrt{6}(\sqrt{2}+\sqrt{3}+\sqrt{5})$.

5. $(8-\sqrt{3})(3\sqrt{5}+2\sqrt{3}-1)$.

6. $(\sqrt{6}+2)(3\sqrt{2}+2\sqrt{3}+3\sqrt{6})$.

EXERCISE 159.

1. $2a$. **2.** $\frac{2\sqrt{a^2-b^2}}{b^2}$. **3.** $2\sqrt{2}+\sqrt{3}$.

4. $\frac{\sqrt{5}+\sqrt{3}.}{4}$ **5.** $3\sqrt{2}-2\sqrt{3}$. **6.** $4a\sqrt{a^2-1}$.

7. $2x^2$. **8.** $-30\sqrt{3}$.

EXERCISE 160.

1. $7+\sqrt{3}$. **2.** $3-2\sqrt{2}$. **3.** $6+\sqrt{5}$.

4. $4+\sqrt{3}$: Read $8\sqrt{3}$ for $\sqrt{83}$. **5.** $3-\sqrt{5}$.

6. $7+3\sqrt{5}$. **7.** $5-3\sqrt{2}$. **8.** $5-\sqrt{2}$.

9. $3\sqrt{7}-2\sqrt{6}$. **10.** $\sqrt{7}+\sqrt{5}$. **11.** $\sqrt{5}+\sqrt{3}$.

12. $\sqrt{3}+\sqrt{2}$. **13.** $2+\sqrt{3}$. **14.** $4-\sqrt{5}$.

15. $\sqrt{7}-1$. **16.** $x = 15,\ y = 6$.

EXERCISE 161.

1. $\sqrt[4]{2}(\sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}})$. **2.** $\sqrt[4]{3}(\sqrt{2}-1)$.

3. $\sqrt[4]{3}(\sqrt{3}+\sqrt{2})$. **4.** $\sqrt[4]{2}(\sqrt{5}-\sqrt{3})$.

5. $\sqrt[4]{3}(\sqrt{3}+\sqrt{2})$. **6.** $\sqrt[4]{5}(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})$.

7. $2+\sqrt{2}$. **8.** $\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}$.

9. $\sqrt{3}-1$. **10.** $\sqrt{5}-\sqrt{3}$.

EXERCISE 162.

1. $\sqrt{2}-1$. **2.** 1. **3.** $-(\sqrt{3}+1)$.

4. $\sqrt{\frac{5}{2}}$. **5.** $\frac{5}{6}\sqrt{6}$. **6.** $2\sqrt{5}-\sqrt{2}+\sqrt{3}-\sqrt{6}+1$.

7. 2. **8.** 1. **9.** $1+\sqrt{2}+\sqrt{3}$.

10. $\sqrt{2}-\sqrt{3}+\sqrt{5}$. **11.** $1+\sqrt{3}-\sqrt{5}$.

12. $2-\sqrt{5}-\sqrt{6}$: (Read $+2\sqrt{30}$).

SELECTED QUESTIONS—J

1. 0. **2.** 98. **3.** $-144\sqrt{5}$.
4. 15. **6.** 568. **7.** 76.
8. 1154. **9.** 39202.

EXERCISE 163.

1. $2m = n$. **2.** $a^2m^2-b^2 = 0$.
3. $ab'-a'b = 0$. **4.** $am^2-bm+c = 0$.
5. $pb^2-qab+ra^2 = 0$. **6.** $aq^3-bq^2p+cqp^2-dp^3 = 0$.
7. $y^2 = 4ax$. **8.** $v^2 = u^2+2fs$.
9. $l(a-c) = m(b-d)$. **10.** $lb^2-mab+na^2 = 0$.

EXERCISE 164.

1. $aq-bp = 0$. **2.** $8a^2n-m^3 = 0$.
3. $a^3d^2+b^3c^2 = 0$. **4.** $p^4n^3-q^4m^3 = 0$.
5. $a^5d^4+b^5c^4 = 0$.

EXERCISE 165.

1. $p^2-q^2 = 2$. **2.** $q^3-p^3 = 3q$.
3. $p^3-q^3 = 3q$. **4.** $q^4-p^4 = 2$.
5. $p^4 = q^4-4q^2+2$. **6.** $p^4 = q^4+4q^2+2$.

16458.

EXERCISE 166.

1. $a^2-b^2 = 4$. **2.** $m^2-n^2 = 24$.
3. $p^2-q^2 = 4$. **4.** $10ab+3 = 0$.
5. $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. **6.** $a^2-b^2 = 1$.
7. $a^2x^2-b^2y^2 = 1$. **8.** $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.
9. $\frac{a^2}{x^2}-\frac{b^2}{y^2} = 1$. **10.** $2a^2-1 = 0$.

EXERCISE 167.

1. $m^2+n^2 = a^2+1$.
2. $a^2+b^2 = r^2$.
3. $a^3+2c^3 = 3ab^2$.
4. $b^6+3a^4b^2 = 4a^3c^3$.
5. $a^6-2b^6+9a^2c^4-8a^3b^3 = 0$.
6. $m^3-am^2+b^2m-c^3 = 0$.

EXERCISE 168.

1. $a(q-r)+b(r-p)+c(p-q) = 0$.
2. $abc+2fgh-af^2-bg^2-ch^2 = 0$.
3. $(fy-gx)(g-f) = (x-y)^2$.
4. $(yz'-y'z)(xy'-x'y) = (zx-z x)^2$.
5. $(c-ad)^2+b^2d = 0$.
6. $(bc-b'c)(ab'-a'b)^2 = (ca'-c'a)^3$.
7. $(bc'-b'c)^2(ab'-a'b) = (ca'-c'a)^3$.
8. $p^3r = (r+pq)^2$.
9. $(c^2+4ab)(b^2-3ab-ac) = (4a^2-3ac+bc)^2$.

EXERCISE 169.

1. $a'(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b) = 0$.
2. $2abc+ab+bc+ca = 1$.
3. $x^2+y^2+z^2+2xyz = 1$.
4. $ab+bc+ca+1 = 0$.
5. $ab+bc+ca+2abc = 1$.

EXERCISE 170.

1. $a+b+c = 0$.
2. $(p+q)(q+r)(r+p) = 1$.
3. $ab+bc+ca+1 = 0$.
4. $a^2 = b^2+2c^2$.
5. $2a = b(1+c)$.
6. $p+r = 4q$.
7. $m(p+q)^3 = q(p+m)^3$
8. $x+a = 0$.
9. $p^{b-c}\times q^{c-a}\times r^{a-b} = 1$.
10. $a^3+b^3+c^3+abc = 0$.

SELECTED QUESTIONS—K.

1. $am^3-blm^2+cl^2m-dl^3 = 0$.
2. $a^2-b^2 = 1$.
3. $b^3-a^3+3a+3b = 0$.
4. $m^2-n^2 = 4$.
5. $y^2-x^2 = 4x^2y^2$.
6. $(bc'-b'c)^2+(ca'-c'a)^2 = (ab'-a'b)^2$.
7. $(bc'-b'c)^2+(ca'-c'a)(bc'-b'c)+(ca'-c'a)^2 = 0$.
8. $(x-a)\left(\frac{1}{z}-b\right) = 1$.
9. $\frac{a^2}{(b+d)^2}-\frac{c^2}{(b-d)^2} = -1$.

EXERCISE 171.

1. Fractional and irrational.
2. Fractional and rational.
3. Integral and irrational.
4. Fractional and rational.
5. Integral and rational.
6. Fractional and irrational.
7. (*i*) 4. (*ii*) 9. (*iii*) 12.
(*iv*) 5. (*v*) $6x^2-8x+10$
(*vi*) $9k^2+12k+15$. (*vii*) $3m^2+2m+4$.
8. (*i*) 14. (*ii*) 6. (*iii*) $2(1-p^2)$.
9. $2(b+2ax)$. **10.** 7.
11. x. **12.** x.

EXERCISE 172.

1. $a_0x^8+a_1x^7+a_2x^6+a_3x^5+a_4x^4+a_5x^3+a_6x^2+a_7x+a_8$.
2. $a_0x^4+a_1x^3y+a_2x^2y^2+a_3xy^3+a_4y^4+a_5x^3+a_6x^2y+a_7xy^2$
$+a_8y^3+a_9x^2+a_{10}xy+a_{11}y^2+a_{12}x+a_{13}y+a_{14}$.
3. $2x^4-5x^3-2x^2+11x-6$.
4. $2x^5-10x^4+3x^3+16x^2-14x+3$.
5. $15x^6+5x^5-6x^4-x^2-2x-1$.
6. $12x^8+2x^7-18x^6-3x^5+22x^4+4x^3+3x^2-4$.
7. $2x^8+5x^7-2x^6-9x^5-x^4-5x^3-14x^2-3x+3$.
8. $6x^3-x^2(5+6a)+x(5a-6)+6a$; $-\frac{5}{6}$.
9. x^2-x. **10.** $2x-1$.
11. x^3-x^2+x-1. **12.** x^4-x^2+1.

EXERCISE 173.

1. $a=2,\ b=8,\ c=-3$.
2. $a=3,\ b=-9,\ c=0,\ d=15$.
3. $l=1,\ m=1,\ n=-11,\ k=-23$.
4. $a=2,\ b=0,\ c=12,\ d=0,\ e=2$.
5. 18. **6.** $a=2, k=-8$.
7. $A=1,\ B=2$. **8.** $A=2,\ B=-3$.
9. $A=1,\ B=2,\ C=3$.
10. $A=1,\ B=-1,\ C=0,\ D=2$.
11. $3(x-1)^2+2(x+2)+1$.
12. $3(x+1)^3-2(x+1)+1$.
13. $(x+1)^3-4(x+1)^2+(x+1)-4$.

14. $\dfrac{-1}{x+1} - \dfrac{1}{5(x-3)} + \dfrac{6}{5(x+2)}$.

15. $\dfrac{3}{2x+1} + \dfrac{7-6x}{4x^2-2x+1}$.

16. $\dfrac{1}{x-2} - \dfrac{x+4}{(x+1)^2}$.

17. $A = 3,\ B = -1,\ C = 2$.

18. $A = 1,\ B = 2,\ C = -1,\ D = -1$.

EXERCISE 174.

1. Homogeneous and symmetrical, absolute symmetry.
2. Homogeneous but not symmetrical.
3. Neither homogeneous nor symmetrical.
4. Homogeneous but not symmetrical.
5. Homogeneous but not symmetrical.
6. Homogeneous but not symmetrical.
7. Homogeneous and symmetrical, absolute symmetry.
8. Symmetrical but not homogeneous, absolute symmetry.
9. Homogeneous but not symmetrical.
10. Homogeneous and symmetrical, absolute symmetry.
11. Not homogeneous but symmetrical, absolute symmetry.
12. Homogeneous and symmetrical, cyclic symmetry.
13. $x^5,\ x^4y,\ x^3y^2$.
14. $x^4,\ x^3y,\ x^2yz,\ x^2y^2$.
15. $a(x^4+y^4+z^4)+b(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y)+$
$c(x^2y^2+y^2z^2+z^2x^2)+d(x^2yz+y^2zx+z^2xy)$.
16. $2x^2+3xy+2y^2$.
17. $2x^3+3x^2y+3xy^2+2y^3$.
18. $(x+a)(x+b)+(x+b)(x+c)+(x+c)(x+a)$.
19. $(x-a)(b-a)(c-a)+(x-b)(c-b)(a-b)+$
$(x-c)(a-c)(b-c)$.
20. $(x+a)(x+b)(x-c)+(x+b)(x+c)(x-a)$
$+(x+c)(x+a)(x-b)$.
21. $\dfrac{x^2(a+b)}{ab} + \dfrac{x^2(b+c)}{bc} + \dfrac{x^2(c+a)}{ca}$.
22. $\dfrac{bc}{(a-b)(a-c)} + \dfrac{ca}{(b-c)(b-a)} + \dfrac{ab}{(c-a)(c-b)}$.

23. $\dfrac{1}{(a-b)(a-c)(x-a)} + \dfrac{1}{(b-c)(b-a)(x-b)} + \dfrac{1}{(c-a)(c-b)(x-c)}$

24. $3x^2+2x(a+b+c)+(ab+bc+ca)$.
25. $2(a^2+b^2+c^2-ab-bc-ca)$.
26. $3x^2+2x(a+b+c)$.
27. $ab(a-b)+bc(b-c)+ca(c-a)$.
28. $\Sigma x^2+3\Sigma xy$.
29. $\Sigma x^2-\Sigma xy$.
30. $x^3y+y^3z+z^3x-x^2y^2-y^2z^2-z^2x^2$.
31. $3\Sigma a^2-2\Sigma ab$.
32. (*i*) $a=b=7$ and $c=d=-3$.
(*ii*) $p=4$ and $q=-2$.
(*iii*) $m=-n=2$ and $-p=q=7$.

EXERCISE 175.

1. $A=1,\ B=1,\ C=0$.
2. $3\Sigma x^2y+6xyz$.
3. $24xyz$.
4. 0.
5. $(x-1)(x^2+2x+3)$.
6. $(x-1)(x-2)(x-3)$.
7. $(x-1)(x-3)(x+5)$.
8. $(x+2)(x-3)(x+4)$.
9. $(x+2)(x+3)(x+5)$.
10. $(x-1)(x+1)\ (x+5)$.
11. $(x+2)(x+3)(x-5)$.
12. $(x+2)(x^2-3x+6)$.
13. $(x+1)(x-1)(x^2-2x-2)$.
14. $(x-3)(x-4)(x^2+5x+3)$.

EXERCISE 176.

3. $(a-b)\ (b-c)\ (c-a)$.
4. $-(a-b)\ (b-c)\ (c-a)$.
5. $(a+b)\ (b+c)\ (c+a)$.
6. $3\ (a-b)\ (b-c)\ (c-a)$.
7. $(a+b)\ (b+c)\ (c+a)$.
8. $(a+b)\ (b+c)\ (c+a)$.
9. $24abc$.
10. $-(a-b)\ (b-c)\ (c-a)\ (a+b+c)$.
11. $(a-b)\ (b-c)\ (c-a)\ (a+b+c)$.
12. $(a+b+c)\ (ab+bc+ca)$.
13. $2(a-b)\ (b-c)\ (c-a)\ (a+b+c)$.
14. $-(a-b)\ (b-c)\ (c-a)\ (a^2+b^2+c^2+ab+bc+ca)$.
15. $-(a-b)\ (b-c)\ (c-a)\ (ab+bc+ca)$.
16. $5(x-y)\ (y-z)\ (z-x)\ (x^2+y^2+z^2-xy-yz-zx)$.
17. $-(a-b)\ (b-c)\ (c-a)\ (a^2+b^2+c^2+ab+bc+ca)$.
18. $5xy\ (x+y)\ (x^2+xy+y^2)$.

19. $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.
20. $-(x+y)(y+z)(z+x)(x-y)(y-z)(z-x)$.
21. $-(a-b)(b-c)(c-a)(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+abc)$.

EXERCISE 179.

1. 0.
2. 1.
3. 1.
4. $\frac{1}{abc}$.
5. 0.
6. $\frac{1}{abc}$.
7. 0.
8. 0.
9. 2.
10. $ab+bc+ca$.
11. $a^2+b^2+c^2+ab+bc+ca$.
12. $a+b+c$.
13. 0.
14. $\frac{1}{abc}$.
15. $a+b+c$.
16. 1.
17. $-\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.
18. 1.
19. 2.
20. -3.
21. x^2.

EXERCISE 180.

1. $\frac{1}{(x-a)(x-b)(x-c)}$.
2. $\frac{x^2}{(x-a)(x-b)(x-c)}$.
3. $\frac{1}{(x+a)(x+b)(x+c)}$.
4. $\frac{x^2}{(x+a)(x+b)(x+c)}$.
5. $\frac{1+x}{(x-a)(x-b)(x-c)}$.
6. $\frac{1+px+qx^2}{(x-a)(x-b)(x-c)}$.

EXERCISE 181.

1. $(a+b)(b+c)(c+a)$.
2. $ab+bc+ca$.
3. 3.
4. $\frac{3}{2}(a+b+c+d)$.
5. -1.
6. $x+y+z$.

SELECTED QUESTIONS — L.

1. 0.

2. $\dfrac{a+b+c}{(a+b-c)\ (b+c-a)\ (c+a-b)}$.

3. 0.

4. -1.

5. x.

6. x^2.

EXERCISE 182.

1. $(3x-5y)(6x-7y)$.

2. $(3a-5b)(9a+4b)$.

3. $(11x+4y)(5x-3y)$.

4. $x^2y^2(15x-4y)(8x+7y)$.

5. $(4x-5y)(11x-4y)$.

6. $(7x-3y)(3x-7y)$.

7. $(4x+y)(7x-20y)$.

EXERCISE 183.

1. $(a+x-2)(a+y+2)$.

2. $(x+1)(ax-x-a)$.

3. $(x+a+2)(x-a-1)$.

4. $(x-a)(x-b-1)$.

5. $(bx-x+1)(bx+x+b)$.

6. $(a-b-3)(a-b+2)$.

7. $(x+p+1)(x+p-1)$.

8. $(a+x)(a+b+c)$.

EXERCISE 184.

1. $(x-1)(x-3)(x+4)$.

2. $(x+2)(x-4)^2$.

3. $(x-3)(x^2+3x+6)$.

4. $(x+3)(x^2-3x+4)$.

5. $(x+2y)(x^2-5xy+10y^2)$.

6. $(2x-1)(4x^2+2x+3)$.

7. $(a-b)(2a^2+ab+b^2)$.

8. $(3x-1)(9x^2+3x+5)$.

EXERCISE 185.

1. $(x^2-3x-1)(x^2+2x-1)$.

2. $(x-1)^2(x^2+6x+1)$.
Read $+1$ in the question.

3. $(x^2-x+1)(x^2-4x+1)$.

4. $(2x^2+x+2)(x^2+x+1)$.

5. $(x+y)^2(3x^2-7xy+3y^2)$.
Read $-xy^3$ in the question.

6. $(2x+y)(x+2y)(2x^2-xy+2y^2)$

7. $(x^2-y^2-xy)(6x^2-6y^2-19xy)$.

EXERCISE 186.

1. $(a-2b+3c)(2a+3b-5c)$.

2. $(x+2y-3z)(2x-y+z)$.

3. $(x+y+z)(x-2y+3z)$.

4. $(a+2x-y)(a-x+2y)$.

5. $(x+2a+b)(x-a+2b)$.
6. $(x+y+3z)(x-2y+z)$.
7. $(x+3y-2z)(2x-y+z)$.

EXERCISE 187.

1. $3(2a-b)(b-3c)(3c-2a)$.
2. $3(a-2b+1)(a+b-1)(b-2a)$.
3. $3(a-1)(a-2)(3-2a)$.
4. $-6a(a+b+1)(a-b-1)$.
5. $3(2x+y)(x-y)(x+2y)$.
6. $3abc(b-c)(c-a)(a-b)$.
7. $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.
8. $-5(x-1)(x-2)(2x-3)(3x^2-9x+7)$.

SECTIONAL REVISION V

TEST PAPERS

Paper 1.

1. (*i*) $15x^7-x^5+13x^4-14x^3-3x^2-2x-8$. (*ii*) $\frac{5}{4}$.
2. (*i*) 1·0004. (*ii*) 1·02016.
3. (*i*) $81p^4+216p^3q+216p^2q^2+96pq^3+16q^4$.
(*ii*) $81p^4-216p^3q+216p^2q^2-96pq^3+16q^4$.
4. $a=12,\ b=9$. **5.** $\pm(7-2\sqrt{2})$.
6. $A=-1,\ B=2$.

Paper 2.

1. (*i*) $-1\ ;\ 1\ ;\ -13$. (*ii*) $x+1$.
2. $\frac{ab}{x^2(a-b)}+\frac{bc}{x^2(b-c)}+\frac{ca}{x^2(c-a)}$.
4. $12x+40x^3+12x^5$. **5.** $\pm(3\sqrt{3}-2)$.
6. $\frac{x^2}{(x-a)(x-b)(x-c)}$.

Paper 3.

2. $\frac{x^2-y^2}{x(y-z)}+\frac{y^2-z^2}{y(z-x)}+\frac{z^2-x^2}{z(x-y)}$.
3. Product is not symmetrical.

4. $A = \frac{15}{4}$, $B = -\frac{14}{5}$, $C = \frac{1}{20}$. **5.** -48.

6. $\pm\sqrt[4]{5}(\sqrt{3}+\sqrt{2})$.

Paper 4.

1. $-a^3(b-c)-b^3(c-a)-c^3(a-b)$.

2. $5x^2+3x-1+\frac{7}{x-1}-\frac{16}{x-2}$.

3. $\frac{1}{2}(x^2+y^2+z^2)+\frac{1}{3}(xy+yz+zx)$.

4. 1·797. **5.** $\sqrt{2}$. **6.** $a^2+b^2+c^2-4 = abc$.

Paper 5.

1. (*i*) $\sqrt{2}$. (*ii*) $9+4\sqrt{5}$.

2. $\left(\pm x+3-\frac{1}{x}\right)$. **3.** $A = -2$, $B = 3$, $P = 2$.

5. $(a+b+c-4)^2 = abc$.

Paper 6.

2. $\frac{1}{abc}$. **3.** $\pm(\frac{1}{2}x+\frac{1}{3}y+3)$.

5. (*i*) $\frac{4\sqrt{5}+3\sqrt{15}-10\sqrt{3}-6}{11}$. (*ii*) 0.

6. $n = x^{m^{\frac{1}{x}}}$.

Paper 7.

1. (*i*) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.
(*ii*) $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

2. $x^2+\frac{x}{2}+\frac{1}{x}$. **3.** $\pm 0{\cdot}414$.

4. $a^3 = b^3-3b$. **5.** $-\frac{1}{abc}$.

Paper 8.

1. (*i*) $x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$.
(*ii*) $x^5-5ax^4+10a^2x^3-10a^3x^2+5a^4x-a^5$.

2. $2x^2-3x+5$. **3.** $a^2b^2(a^2+b^2+3)=1$.

4. 1. **5.** 1, 2 or 5.

6. $A=3, \quad B=4, \quad C=2$.

OBJECTIVE TYPE ITEMS—V

1. (*i*) Evolution. (*ii*) 4.
(*iii*) Homogeneous. (*iv*) Two or more simple.
(*v*) Conjugate. (*vi*) Equidistant.

2. (*i*) T. (*ii*) T. (*iii*) F.
(*iv*) T. (*v*) T. (*vi*) T.

3. (*i*) B. (*ii*) C. (*iii*) D.
(*iv*) D.

4. (*i*) (√). (*ii*) ×. (*iii*) ×.
(*iv*) (√). (*v*) ×.

MISCELLANEOUS QUESTIONS

(A)

1. x^2+x+2. **3.** (*i*) $(a-b)(a-ab+b)$.
(*ii*) $(a+b+c)(2ab+2bc+2ca-a^2-b^2-c^2)$.
(*iii*) $(a+b)(a-b)\left(1+\frac{1}{ab}\right)\left(1-\frac{1}{ab}\right)$.

4. 1. **5.** (*i*) $x=5, y=3, z=7$.
(*ii*) $a+b+c$. **7.** $\frac{5}{x+2}-\frac{2}{x+3}$.

8. $21m$, $15m$.

(B)

1. x^2+x-4. **3.** (*i*) $(a+1)(a-1)(b+1)(b-1)$.
(*ii*) $(y-x)(xy+a)$.
(*iii*) $(18x^2+6x+1)(18x^2-6x+1)$; 1861×1741.

4. x^2. **5.** (*i*) 5. (*ii*) 5.

6. $abc=1$. **7.** $\frac{3}{x-2}-\frac{3x+1}{(x+1)^2}$.

8. (*i*) $46m$. (*ii*) $32m$; $14m$.

(C)

1. (*i*) 16 ; 27, 81. (*ii*) $2a+2\sqrt{a^2-b^2}$.
4. (*i*) $(x-37)(x+36)$. (*ii*) $(x^2-yz)(z-y)$.
5. $5+\sqrt{6}$. **6.** $x^4-y^4=4x^2$.
7. $A=\frac{1}{2},\ B=-4,\ C=4\frac{1}{2}$. **8.** Rs 27 ; 24 men.

(D)

1. (*a*) 27 ; $\frac{1}{216}$ **2.** (*b*) $\frac{5}{x}$.
3. $2x^2+3x+2$. **5.** (*i*) $(ax-a+1)(ax+x+a)$.
(*ii*) $(x^2+y^2-1)^2$. **6.** $\frac{x^2+x-3}{x^2-2}$.
7. $A=6,\ B=5,\ C=-6,\ D=-4$. **8.** Rs 2725.

(E)

1. (*a*) $8x^3$. (*b*) 8.
3. (*i*) $(x+4)(x^2-4x+16)$. (*ii*) $(x+y)(y+z)(z+x)$.
4. $\frac{1}{8}$. **5.** (*i*) 4. (*ii*) $x=y=2$ or $-\frac{1}{2}$.
6. $abc=1$. **7.** $A=4,\ B=-6,\ C=-3$.
8. 30 km per hour.

(F)

2. $x+2-\frac{1}{x}$. **3.** $x-\sqrt{x^2-1}$.
4. (*i*) $(x^2+4x+8)(x^2-4x+8)$. (*ii*) $(x+a)(x+b+c)$.
5. $\frac{1}{(x-a)(x-b)(x-c)}$. **6.** $\frac{7}{12}$; $4\frac{8}{16}$.
7. $p=6,\ q=11,\ r=6$. **8.** 72 of each kind.

(G)

1. (*i*) 83. (*ii*) 21.
2. x^2+x-6 ; x^2+2x-3.
4. (*i*) $(2x^2+1)(2x^2-1)(2x^2+2x+1)(2x^2-2x+1)$.
(*ii*) $6(x-y)(3x+5y)(5x+3y)$.
5. $\frac{x(x+1)}{x-1}$. **6.** k^3+3k.
7. $k=3,\ l=0,\ m=-1$. **8.** 63.

(H)

1. (*a*) -1. (*b*) (*i*) $4-x$. (*ii*) $x-4$. (*iii*) 4.
2. x^3-7x-6.
5. 1.
6. (*i*) 5 or $\frac{5}{4}$. (*ii*) $x = 4\frac{1}{2}$, $y = 1\frac{1}{2}$.
7. -1 ; 0 ; $6x^2-12x+8$.
8. $32\frac{4}{13}$ minutes past 5.

(I)

1. (*a*) $\frac{1}{\sqrt{a^2+b^2}}$. (*b*) $\frac{x^2-y^2}{\sqrt{a^2-4b^2}}$.
2. (*a*) 0. (*b*) x^8+x^4+1.
3. 3. **4.** $a^4-4a^2b+2b^2$.
5. $\frac{x}{x-3}$. **6.** 0.
8. $12abc(a+b+c)$.

(J)

1. (*a*) 1. **2.** (*b*) $1+x+x^2+x^3+x^4+x^5+x^6+x^7$.
3. $(3a+2b)(2a-3b)(a-b)(4a^2+6ab+9b^2)$.
4. (*i*) $(a-2b)(a+5b)(a-3b)$. (*ii*) $(x-2)(x^2+2x-13)$.
5. 2. **6.** $3\sqrt{2}-2$.

EXERCISE 188.

1. ± 4. **2.** ± 2. **3.** 0 or 10.
4. ± 2. **5.** $\pm\sqrt{5}$. **6.** $\pm\frac{3}{2}\sqrt{2}$.
7. $\pm\sqrt{a}$. **8.** ± 2. **9.** ± 1.
10. ± 3. **11.** ± 2. **12.** ± 9.
13. $\pm\sqrt{\frac{1}{11}}$. **14.** $\pm\sqrt{\frac{1}{2}}$. **15.** $\pm\sqrt{a(a-2)}$.
16. 4 or -8.

EXERCISE 189.

1. 4 or 6. **2.** -4 or -6.
3. 5 or -7. **4.** -8 or 9.
5. 1 or 3. **6.** 2 or -5.
7. a or $-b$. **8.** $\frac{3}{2}$ or $\frac{4}{5}$.

9. $\frac{1}{6}$ or $-\frac{2}{5}$.

10. $\frac{b}{a}$ or $-\frac{c}{b}$

11. -6 or $\frac{2}{3}$.

12. 0 or -5.

13. -4.

14. $\frac{7}{8}$.

15. $-\frac{a}{3}$.

16. $\frac{b}{a}$.

17. $p+q$.

18. $p+q$ or $q-p$.

19. 2 or 7.

20. -6 or -4.

21. 4 or -3.

22. $-\frac{1}{3}$ or $-\frac{1}{4}$.

23. -3 or -9.

24. $\frac{1}{5}$ or -1.

25. $\frac{2}{3}$ or $\frac{3}{2}$.

26. 3 or $\frac{1}{3}$.

27. 9 or -4.

28. $-\frac{1}{16}$ or 10.

29. $\frac{3}{2}$ or $\frac{1}{4}$.

30. $\frac{10}{3}$ or $-\frac{2}{5}$.

EXERCISE 190.

1. 9 or -5.

2. $3\pm\sqrt{74}$.

3. -6 or $\frac{5}{3}$.

4. $\frac{2}{3}$ or $-\frac{3}{2}$.

5. $\frac{9}{4}$ or $-\frac{1}{2}$.

6. $\frac{4}{3}$ or $-\frac{3}{5}$.

7. 1 or $-\frac{3}{5}$.

8. $\frac{25}{2}$ or $-\frac{1}{2}$.

9. 8 or 9.

10. $b\pm\frac{\sqrt{b^2+4ac}}{2a}$.

EXERCISE 191.

1. 2 or $-\frac{1}{3}$.

2. 1 or $-\frac{1}{3}$.

3. 2 or $\frac{1}{3}$.

4. $\frac{2}{3}$ or $\frac{3}{4}$.

5. 14 or $\frac{2}{4}$.

6. $-\frac{3}{2}$ or $\frac{2}{5}$.

7. 2 or $\frac{20}{3}$.

8. -2 or $-\frac{3}{7}$.

9. 4 or $\frac{3}{5}$.

10. $\frac{2p}{3}$ or $-\frac{4p}{5}$.

11. 2 or $-p$.

12. $\frac{p}{q}$ or $-\frac{q}{p}$.

EXERCISE 192.

1. $\frac{1}{3}$.

2. $\frac{3}{4}$ or $\frac{4}{3}$.

3. 13, 14.

4. 12 & 14 or -12 & -10.

5. 11, 13.

6. 5, 12.

7. $17m$; $15m$.

8. (*i*) $(\pm\sqrt{3}-1)a$. (*ii*) $(2\pm\sqrt{2})a$.

(*iii*) $\frac{3}{4}a$ or a. (*iv*) $\frac{2\pm\sqrt{2}}{4}a$.

9. $24m$; $10m$. **10.** $30m$; $16m$.
11. $50m$; $18m$. **12.** $4m$.
13. $24m$; $10m$. **14.** 11 hours.
15. 15 km per hour. **16.** 6 hours 40 minutes.
17. $3m$; $2m$. **18.** $6m$; $9m$.
19. $24m$; $18m$, $15m$. **20.** 1024.
21. Rs 80. **22.** Rs 70 or Rs 30 :
Read "a pen for Rs 21·00".

EXERCISE 193.

1. (*a*) $13i$. (*b*) $10-\sqrt{15}+5\sqrt{5i}+2\sqrt{3i}$.
(*c*) 34.
2. (*a*) $-39+80i$. (*b*) $-33-56i$.
(*c*) p^2-q^2-2pqi. (*d*) $+6\sqrt{5i}$.
(*e*) $24-10i$. (*f*) $c^2-d+2c\sqrt{di}$.
3. (*a*) $\frac{13+8\sqrt{-3}}{19}$. (*b*) $\frac{26-65\sqrt{-3}}{79}$.
(*c*) $\frac{-4-2\sqrt{-5}}{9}$.
4. (*a*) $-\frac{22}{9}+\frac{7}{9}\sqrt{5i}$. (*b*) $+1+i$.
5. (*b*) i : read $\frac{1}{(1-i)^2}-\frac{1}{(1+i)^2}$.
6. (*a*) $\frac{2(p^2-q^2)}{p^2+q^2}$. (*b*) Read "by $\frac{a-bi}{a+bi}$."
$\frac{a^4+b^4-6a^2b^2+4abi\,(a^2-b^2)}{a^4+b^4+2a^2b^2}$.

EXERCISE 194.

1. (*i*) -7, 9. (*ii*) 13, 7. (*iii*) $-\frac{5}{6}$, -2. (*iv*) $\frac{7}{8}$, 3.
2. (*i*) $\frac{8}{\text{१३}}$, $\frac{7}{\text{१३}}$. (*ii*) $\frac{l}{m^2}$, $-\frac{k}{m^2}$. (*iii*) $\frac{l}{m}-\frac{n}{m}$. (*iv*) 0, $-\frac{c}{a}$.

EXERCISE 195

1. (*i*) Imaginary and unequal.
(*ii*) Real, rational and unequal.
(*iii*) Real, rational and unequal.
(*iv*) Real, rational and unequal.
(*v*) Real, rational and unequal.

4. $q \geqslant r$. **5.** 1 or 9. **6.** 3, 5.

7. $m = 0$ or 3; 1, $\frac{5}{2}$.

EXERCISE 196.

1. (*i*) -2. (*ii*) 6. (*iii*) -10. (*iv*) -6. (*v*) $-3\frac{1}{3}$. (*vi*) $\frac{2\sqrt{3}}{3}$.

2. -1. **3.** $-6\frac{5}{9}$. **4.** -27.

5. ± 1. **6.** (*a*) $\frac{p}{q}$. (*b*) $\frac{p^3-3pq}{q^2}$. (*c*) $\frac{p^2-2q}{q^2}$.

7. (*i*) $\frac{a-b+c}{a}$. (*ii*) $\frac{c(b^2-2ac)}{a^3}$. (*iii*) $\frac{b^2-3ac}{a^2}$.
(*iv*) $\frac{b^2-2ac}{ac}$. (*v*) $\frac{b^4-4ab^2c}{a^2c^2}$. (*vi*) $\frac{3abc-b^3}{c^3}$.

8. $\frac{3a^3bc-a^2b^3+c^5}{a^2c^3}$. [Read $ax^2+bx+c=0$]

EXERCISE 197.

1. (*i*) $x^2-16x+55=0$. (*ii*) $x^2-9x-112=0$.
(*iii*) $x^2-(3a+7b)x+21ab=0$.
(*iv*) $x^2-2mx+m^2-n^2=0$.
(*v*) $x^2-4x+1=0$. (*vi*) $pq^2x^2-(p^2+q^4)x+pq^2=0$.
(*vii*) $(p^2-q^2)x^2-2(p^2+q^2)x+(p^2-q^2)=0$.

2. (*i*) $x^2-6x+27=0$. (*ii*) $3x^2+2x+3=0$.
(*iii*) $x^2-4x+6=0$.

3. (*i*) $cx^2+bx+a=0$. (*ii*) $a^2x^2-(b^2-2ac)x+c^2=0$.
(*iii*) $bcx^2-a(b-c)x^2-a^2=0$.
(*iv*) $acx^2+b(a+c)x+(a+c)^2=0$.
(*v*) $(a-b+c)x^2-2(a-c)x+(a+b+c)=0$.

4. $2x^2-11x+19=0$.

EXERCISE 198.

1. $p = r$. **4.** (*i*) $3b^2 = 16ac$. (*ii*) $4b^2 = 25ac$.
(*iii*) $kb^2 = (k+1)^2ac$.
6. $b^3+a^2c+ac^2-3abc = 0$.

EXERCISE 199.

1. 9.
2. $ax^2-4bx+c = 0$.
4. $12b^2 = 49ac$.
5. $\frac{a^4-4a^2b+2b^2}{b^4}$.

EXERCISE 200.

1. (*i*) *A*. (*ii*) *B*. (*iii*) *C*. (*iv*) *B*. (*v*) *A*.
2. (*a*) *A.P.*; -6 (*b*) *A.P.*; $\frac{-p-2q}{7}$. (*c*) Sequence.
(*d*) *A.P.*; 1·1. (*e*) *A.P.*; $\frac{2a^2-2}{a}$.

EXERCISE 201.

1. $1+3+5+7+9$.
2. $-4-1+2+5+\ldots\ldots\ldots\ldots$
3. $46\frac{1}{4}$.
4. $\frac{n^2-n+1}{n}$.
5. (*i*) 24·1.
(*ii*) $5-4n$.

EXERCISE 202.

1. 19th term.
2. Yes; 21st term.
3. No term.
4. 9th term.
5. 10.
6. 8.

EXERCISE 203.

1. 0.
2. 14, 86.
3. $5+8+11+14+\ldots\ldots\ldots\ldots$
4. 19, 18, 17,.....; yes; 20th term.

EXERCISE 204.

1. (*i*) -1221. (*ii*) 89.5. (*iii*) $\frac{n}{2}\{a(n+1)-2bn\}$ (*iv*) 952.
2. 1280.
3. 7650; 7500.
4. $4p+1$.
5. 4; 8.

EXERCISE 205.

1. 1068. **2.** 10. **3.** $2\frac{6}{7}$. **4.** 15.
5. 17. **6.** $\frac{1}{2}$. **7.** 6 ; 2. **8.** 12.
9. 8 ; 4.

EXERCISE 206.

1. (*i*) x^2+y^2. (*ii*) 6·6. **2.** 11, 17, 23, 29.
3. 6, 11, 16, 21, 26, 31, 36.
4. p, $2p-1$, $3p-2$, ... $p(p-1)+1$.

EXERCISE 207.

3. 13, 17, 21. **4.** $\frac{7}{2}$, $\frac{11}{2}$, $\frac{15}{2}$, $\frac{19}{2}$.
5. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5.

EXERCISE 208.

3. $\frac{1}{2}(1+mn)$. **4.** 53 : 39.
6. 19. **7.** 9 ; Rs 4·50. **8.** Rs 33,000.

EXERCISE 209.

1. *G. P.* ; a. **2.** *G. P.* ; $\frac{1}{4p^2}$.
3. *G. P.* ; $-\frac{2}{81}$. **4.** Not a *G. P.*
5. *G. P.* ; $\frac{1}{m+n}$. **6.** *G. P.* ; $\sqrt{3}$.

EXERCISE 210.

1. $2+6+18+54+162+$.....
2. $8+4+2+1+$.....
3. $-\frac{128}{2187}$. **4.** n.
5. (*i*) $\frac{1}{1953}$. (*ii*) $(-1)^{p-1}.\frac{n^{2p-3}}{m^{2p-3}}$.

EXERCISE 211.

1. yes ; 9th term. **2.** 8th term.
3. 9 terms. **4.** 7 terms.
5. $\frac{128}{729}$ is not a term of the *C. P.* $2\frac{1}{4}+1\frac{1}{2}+1+$.....

EXERCISE 212.

1. $16+24+36+54+\ldots\ldots$ **2.** $9+6+4+\frac{8}{3}+\ldots\ldots$ or $9-6+4-\frac{8}{3}+\ldots\ldots$

3. $2+6+18+54+\ldots\ldots$ **4.** 4, 8, 16, 32,.....

5. $\pm\sqrt{pq}$.

EXERCISE 213.

1. (*i*) 12285. (*ii*) $16\ \frac{161}{384}$.

(*iii*) $\frac{192}{7}\{1+(\frac{3}{4})^9\}$. (*iv*) $121(\sqrt{3}+1)$.

(*v*) $\frac{3}{2}\left(1-\frac{1}{3n}\right)$. **2.** $21\frac{341}{1024}$.

3. (i) $\frac{3}{2}(3^n-1)$. (ii) 2^n-1

4. $(2n-1).2^{n+1}+2$. **5.** 1093.

6. $\frac{9}{2}\left\{\frac{3^{10}-1}{3^{10}}\right\}$.

EXERCISE 214.

1. 5 terms. **2.** 12 terms.

3. 2. **4.** 7 terms.

EXERCISE 215.

1. (*i*) $\pm\frac{1}{2}$. (*ii*) $\pm\cdot06$. **2.** 12, 36, 108.

3. $\frac{1}{3}$, 1, 3. **4.** 8, 12, 18, 27.

EXERCISE 216.

1. 2. **2.** $\frac{5}{6}$.

3. 5. **4.** $\frac{23}{48}$.

5. $2\frac{1}{3}$. **6.** (*i*) $\frac{44}{45}$. (*ii*) $3\frac{3}{77}$.

7. $\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots\ldots$ to ∞. **8.** 4.

9. $16+8+4+2+\ldots\ldots$ to ∞.

EXERCISE 217.

2. 4, 6, 9. **3.** 1, 2, 4, 8.

5. 2, 5, 8 or 26, 5, −16.

EXERCISE 218.

2. (*i*) $\frac{10}{27}(10^n-1)-\frac{1}{3}.n$. (*ii*) $\frac{80}{81}(10^n-1)-\frac{8}{9}.n$.

(*iii*) $\frac{1}{9}n-\frac{1}{81}\left(\frac{10^n-1}{10^n}\right)$ (*iv*) $\frac{2}{3}n-\frac{2}{27}\left(\frac{10^n-1}{10^n}\right)$.

3. 3, 27. **4.** $\frac{1}{x-y}\left\{\frac{x^2(x^n-1)}{x-1}-\frac{y^2(y^n-1)}{y-1}\right\}$

EXERCISE 219.

1. $1\frac{1}{3}$. **2.** $\frac{3}{4}$.

3. 4, 2, $1\frac{1}{3}$, 1, $\frac{4}{5}$, $\frac{2}{3}$.....

4. (*i*) $\frac{14}{6+n}$. (*ii*) $\frac{3}{8+n}$.

5. $-\frac{176}{95}$. **6.** $\frac{1}{46}$.

7. $\frac{3}{4}+\frac{12}{17}+\frac{2}{3}+$.....

EXERCISE 220.

1. (*i*) 12. (*ii*) $\frac{1}{4}$. (*iii*) $\frac{m^2-n^2}{m^2+n^2}$. (*iv*) $\frac{a^2-b^2}{a}$.

2. $\frac{1}{2}$, $\frac{2}{7}$. **3.** $\frac{16}{7}$, $\frac{8}{5}$, $\frac{16}{5}$.

4. $\frac{6}{11}$, $\frac{6}{15}$, $\frac{2}{5}$, $\frac{6}{17}$. **5.** $\frac{6}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.

EXERCISE 221.

1. 2, 6; $2\sqrt{3}$.

SECTIONAL REVISION—VI

TEST PAPERS

Paper 1.

1. $-2\frac{1}{2}$ or 3. **2.** ± 2.

3. $2\frac{1}{2}$. **4.** 64; $1\frac{1}{4}$.

5. $-\frac{2}{7}$, -1, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{2}{13}$, $\frac{1}{9}$, $\frac{2}{23}$,.....

6. $1\frac{1}{15}$.

Paper 2

1. 2 or $-1\frac{1}{2}$.

2. 20.

3. $\frac{48}{b+1}$; 36 or 16.

4. $\frac{40}{81}(10^n-1)-\frac{4}{9}n$.

5. $7x^2-30x+31=0$.

6. $\frac{52}{17}+\frac{47}{17}i$.

Paper 3

1. 0·791 or $-3\cdot791$.

2. 1, 7, 13.

3. 5 or $-\frac{5}{7}$.

Paper 4

2. $\frac{3}{2}$; $\frac{6}{7}$.

3. $qx^2+p(q+1)x+(p^2+q^2-2q+1)=0$.

4. $\frac{60}{16-n}$.

6. 0·42 or 3·58.

Paper 5

1. 3; $\frac{3}{2}$.

2. $p=3$, $q=-20$.

3. 2.

4. $1\frac{29}{121}$, $1\frac{29}{46}$, $2\frac{8}{21}$, $4\frac{7}{17}$.

6. $2\frac{1}{2}$ or -1.

OBJECTIVE TYPE ITEMS—VI.

1. (*i*) two
(*ii*) zero.
(*iii*) an *A.P.*
(*iv*) $\pm\sqrt{mn}$.
(*v*) less than one.
(*vi*) *G.P.*

2. (*i*) *B.*
(*ii*) *B.*
(*iii*) *C.*
(*iv*) *B.*

3. (*a*) (*iv*).
(*b*) (*iv*).
(*c*) (*i*).
(*d*) (*iii*).
(*e*) (*iii*).
(*f*) (*iii*).
(*g*) (*iii*).